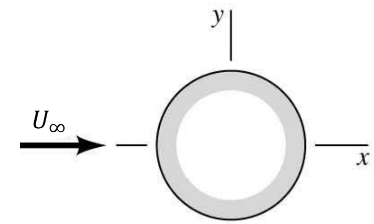
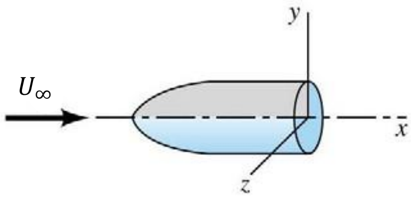
# Chapter 5:  Boundary layers

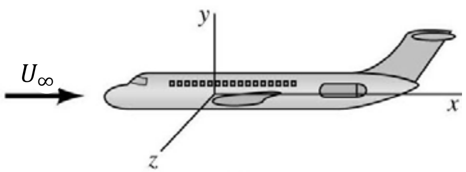
# External flows




**Two-Dimensional:** infinitely long in $z$ and of constant cross-sectional size and shape.



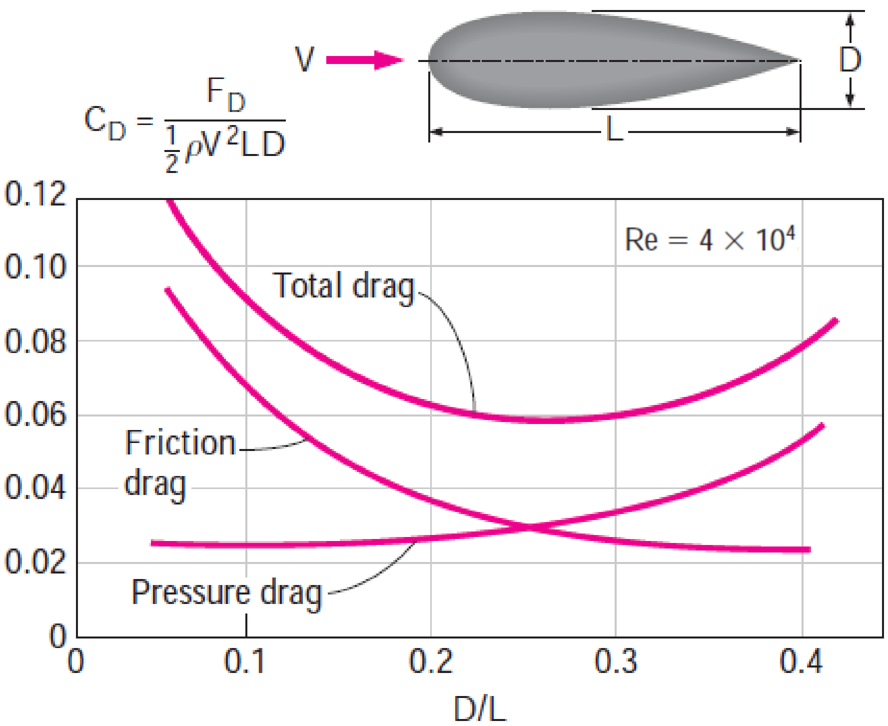

**Axisymmetric:** formed by rotating their cross-sectional shape about the axis of symmetry.

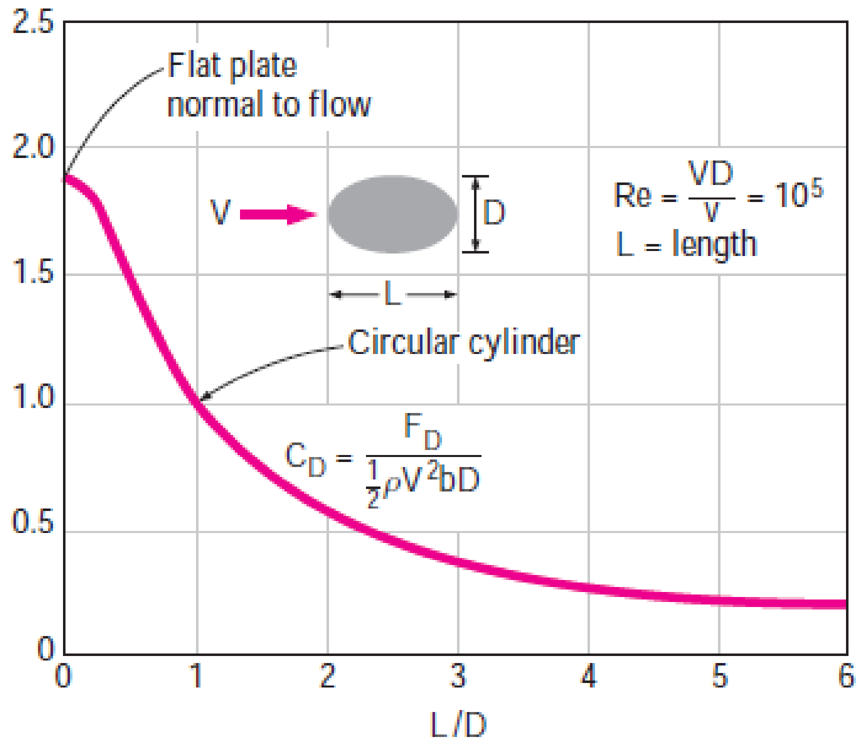



**Three-Dimensional:** may or may not possess a line of symmetry.

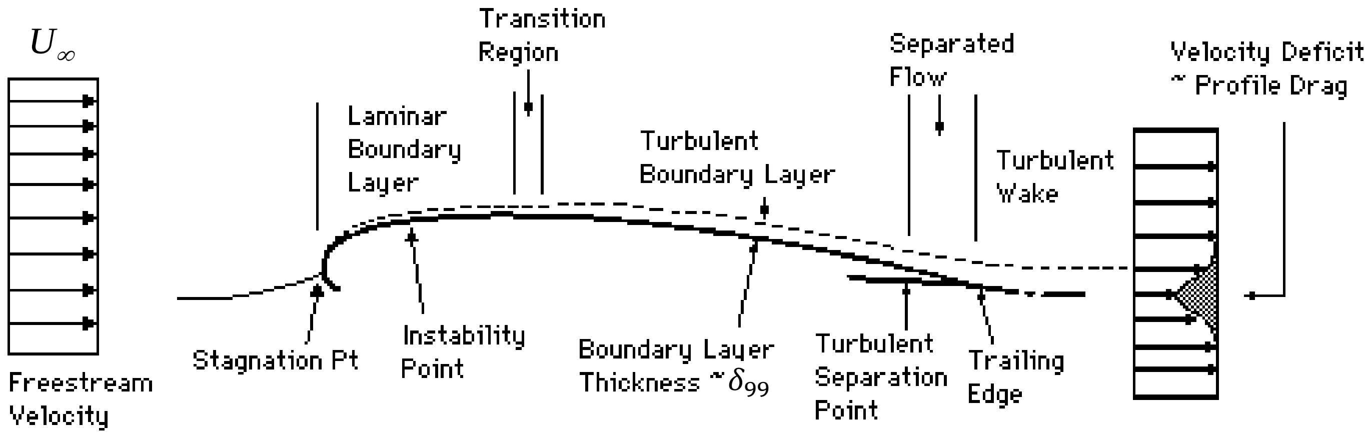
The bodies can be classified as streamlined or blunt. The flow characteristics depend strongly on the amount of streamlining present. *Streamlined* object typically display a lower drag force

# External flows



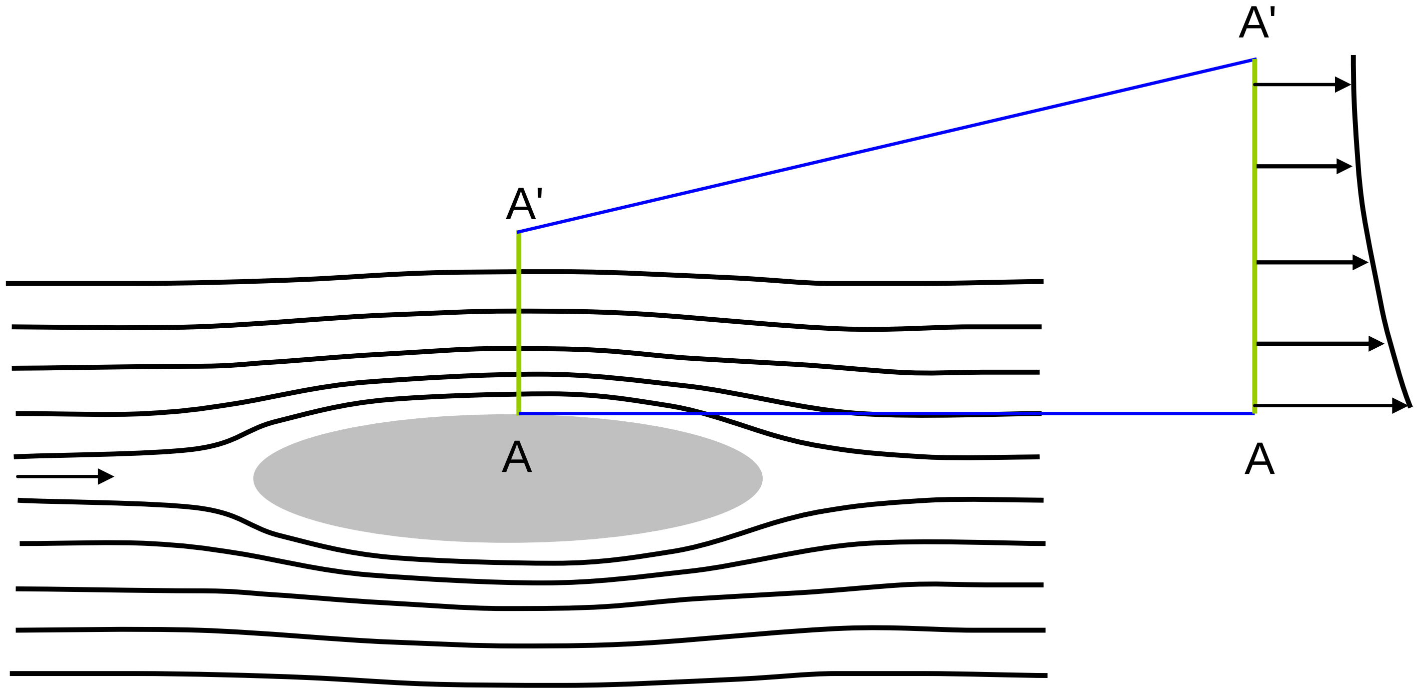
# Qualitative description of b.l. (2D)




*Streamlining* a body reduces the adverse pressure gradient; if the angle of attack $\alpha$ is sufficiently small, drag is mostly due to wall shear stress. The potential flow approximation used so far must be supplemented, including near-wall viscous effects

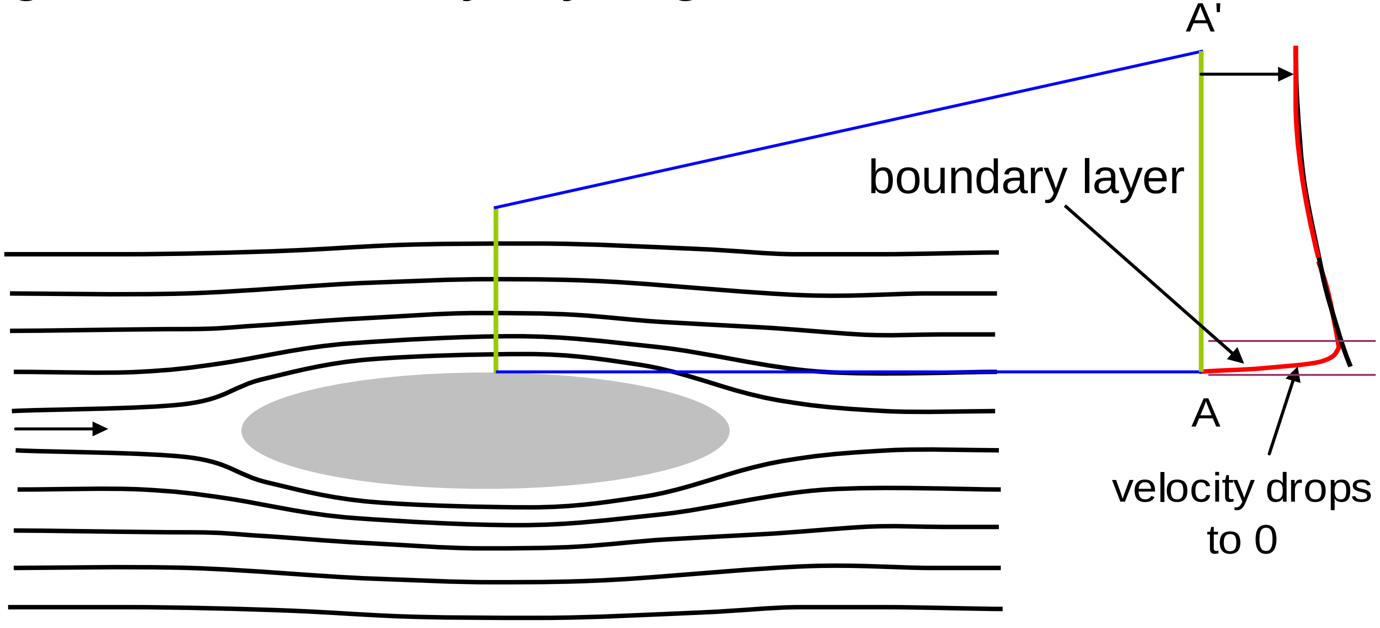
# Qualitative description of b.l. (2D)

The potential flow is forced to accelerate around the body. Thus $u\,(\partial u/\partial s) > 0$, along a streamline near the body from near the upstream stagnation point to section A – A'. The velocity profile along the line A – A' reflects this.



# Qualitative description of b.l. (2D)

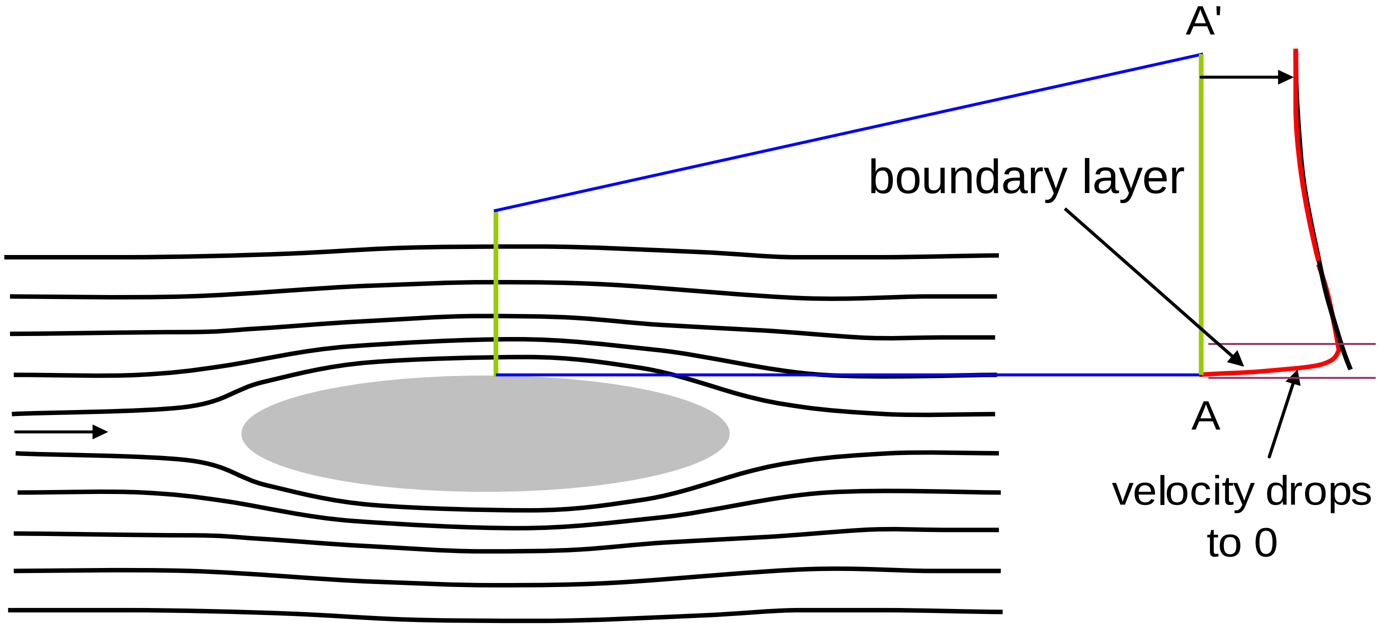
If the body is not moving, the region where the flow velocity drops to zero is called a **boundary layer**. We will find that boundary layer thickness depends on an appropriately defined **Reynolds number** of the flow, and as this number gets larger the boundary layer gets thinner



# Qualitative description of b.l. (2D)

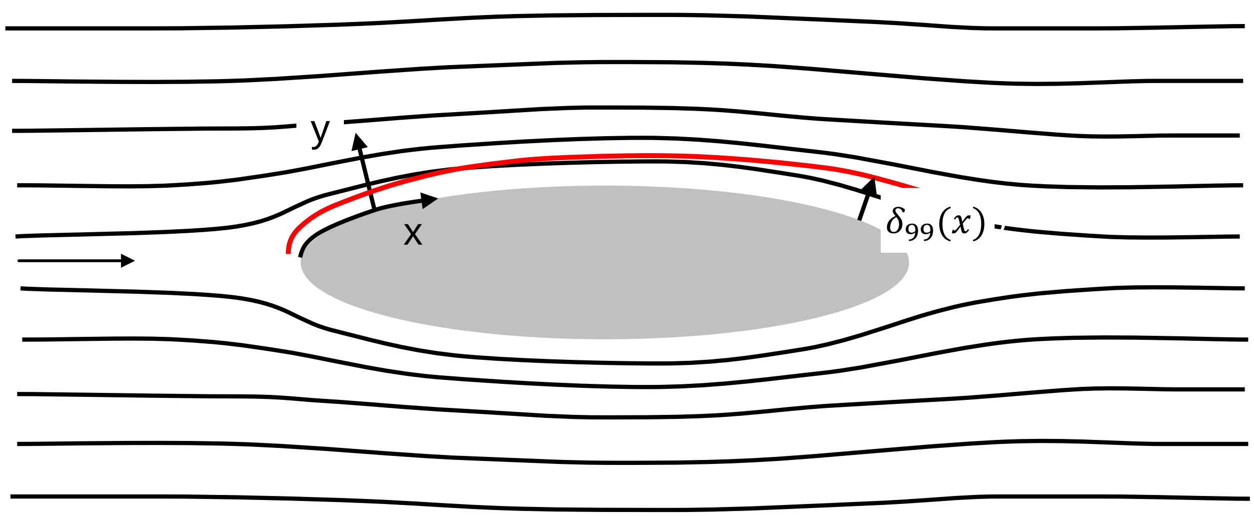
Here we are interested in the case of **large Reynolds numbers**.

For a thin boundary layer, the pressure within is equal to the pressure at the wall (*inviscid approximation*)

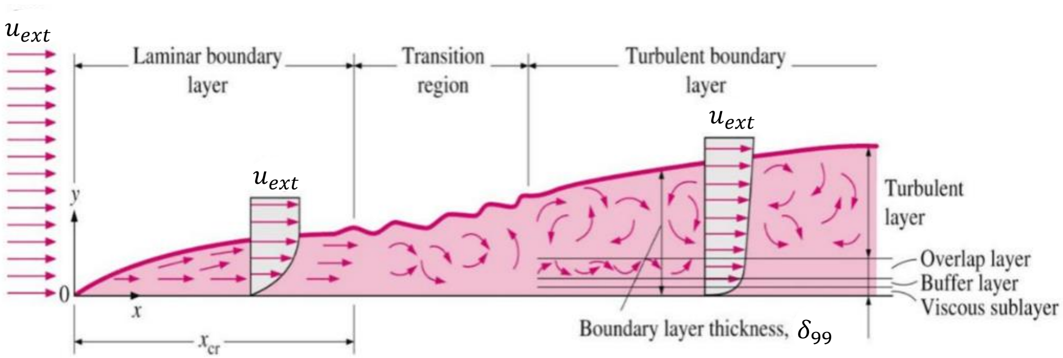


# Qualitative description of b.l. (2D)

Let $x$ denotes a boundary-attached streamwise coordinate, and $y$ denote a boundary-attached normal coordinate, as noted below. Furthermore, let $\delta(x)$ be *some measure* of the **boundary layer thickness**, i.e. the thickness of the zone of retarded flow near the boundary (better definition later). Then the boundary layer is illustrated by the red line



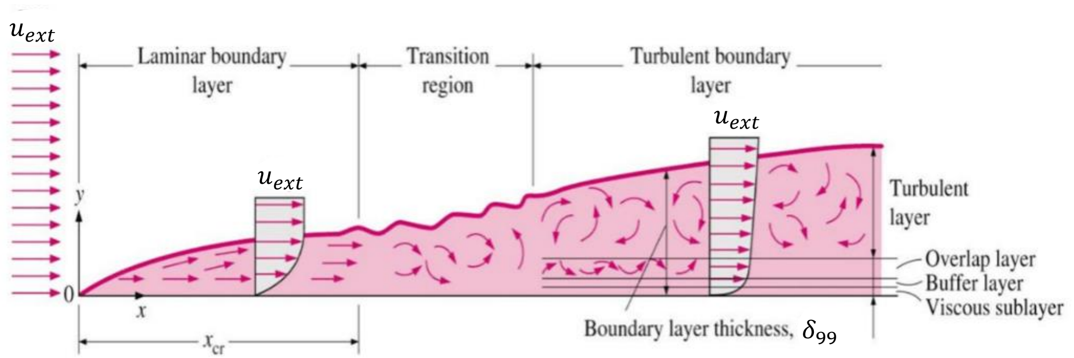
# Qualitative description of b.l. (2D)




The boundary layer is laminar near the leading edge. After a distance $x_{cr}$ instability waves appear and laminar-to-turbulent transition occurs. The transition onset $x_{cr}$ depends on

- Free-stream velocity
- Viscosity
- Pressure gradient
- Wall roughness
- Free-stream fluctuation level
- Wall rigidity

# Qualitative description of b.l. (2D)



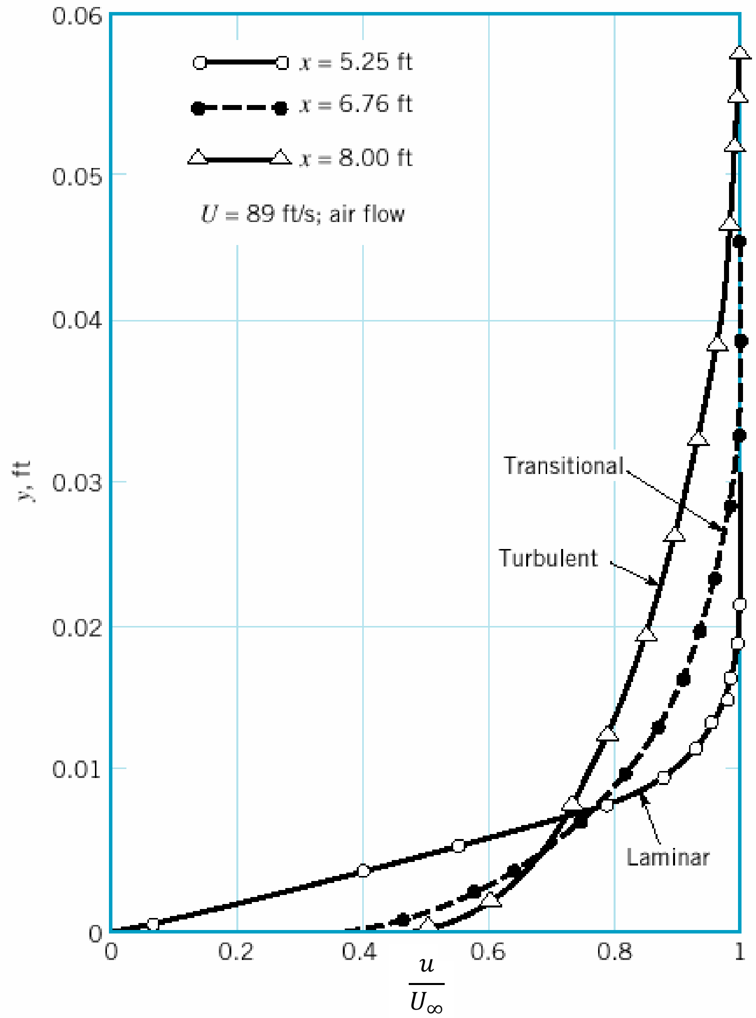

At $x_{cr}$ the local Reynolds number is

$$Re_{x_{cr}} = \frac{\rho\, u_{ext}\, x_{cr}}{\mu}$$

and $Re_{cr}$ can be as large as $10^6$, depending on *environmental conditions*, i.e. depending on the **receptivity** of the boundary layer

# Qualitative description of b.l. (2D)

*Turbulent spot* in transitional flow



# Similarity solutions

Lots of *self-similar* solutions exist for boundary layers and other flows like jets and wakes (cf. H. Schlichting, BOUNDARY LAYER THEORY, 1975) or even supersonic blast waves induced by strong explosions (Sedov, 1946): they are typically found by *appropriately scaling dependent and independent variables*.

In the problems of interest here we build a characteristic length (or time) scale using the kinematic viscosity $\nu$.

Transforming the equations using the new space/time variables permits to transform the PDE's into one (or more) ODE(s)

# Falkner-Skan boundary layer (2D)

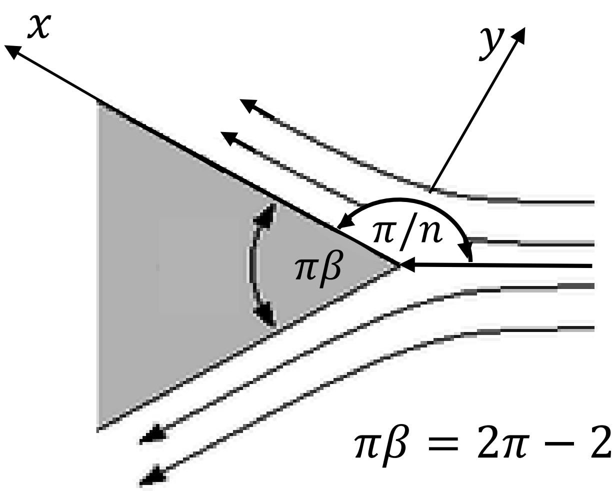
Let us start with the two-dimensional flow over a wedge (cf. chapter 3, slide 13, *flow in a sector*). The potential flow is $F(z) = cz^n$ and the velocity components at the wall ($\theta = \pi/n$) are:

$$v_r = n\,c\,r^{n-1}\cos n\theta = -n\,c\,r^{n-1}$$
$$v_\theta = -n\,c\,r^{n-1}\sin n\theta = 0$$

Now, let's move to Cartesian coordinates $(x, y)$, on the wall

$$u_{wall} = \mathcal{U}\,x^{n-1} = \mathcal{U}\,x^m$$
$$v_{wall} = 0$$

with $m = n - 1$, so that $\beta = \frac{2\,m}{m+1}$




$$\pi\beta = 2\pi - 2\left(\frac{\pi}{n}\right)$$
$$\beta = 2\frac{(n-1)}{n}$$

# Falkner-Skan boundary layer

The irrotational streamwise velocity at the wall is taken as the external flow for a boundary layer solution (careful: $\mathcal{U}$ has dimensions of velocity only when $m = 0$):

$$u_{ext} = \mathcal{U}\, x^m$$

so that the steady, 2D, laminar, incompressible, dimensional b.l. equations (**after order of magnitude estimates**) are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial p}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_{ext}\frac{du_{ext}}{dx} + \nu\frac{\partial^2 u}{\partial y^2} = m\,\mathcal{U}^2\, x^{2m-1} + \nu\frac{\partial^2 u}{\partial y^2}$$

$u = v = 0$ for $y = 0$, $\quad u = u_{ext}$ for $y \to \infty$

# Falkner-Skan boundary layer

**Important note**: in the boundary layer equations the dynamic pressure is constant along $y$ in the boundary layer. This stems from the order of magnitude estimates!

This means that the pressure in the boundary layer can be obtained from the outer, inviscid solution, i.e.

$$p = p_{inviscid}(x, 0).$$

Thus the solution of any boundary layer problem also requires the inviscid (potential flow) solution. In the Falkner-Skan case the $x$-pressure gradient comes from application of Bernoulli's equation

# Falkner-Skan boundary layer

Streamfunction $\psi$, such that: $u = \psi_y, \; v = -\psi_x$

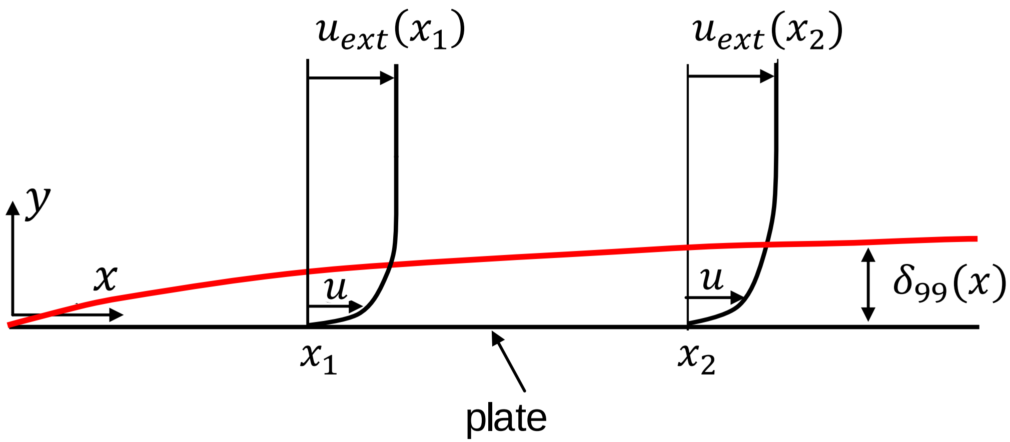
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = m\,\mathcal{U}^2\,x^{2m-1} + \nu\frac{\partial^2 u}{\partial y^2}$$

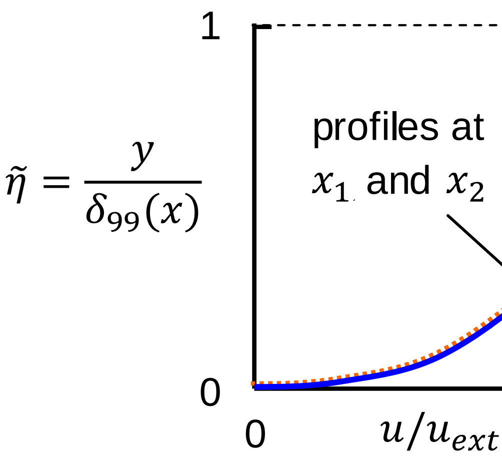
$$\psi_y\,\psi_{xy} - \psi_x\psi_{yy} = m\,\mathcal{U}^2 x^{2m-1} + \nu\,\psi_{yyy}$$

together with $\psi_y = \psi_x = 0$ for $y = 0, \; \psi_y = u_{ext}$ for $y \to \infty$

Now, we introduce the **similarity variable** $\eta = y/\delta(x)$ to transform the PDE above into an ODE. Basically, we *locally* normalize the vertical (physical) coordinate $y$ by a quantity proportional to the local boundary layer thickness, in order for the b.l. to be **invariant along $x$** in the new coordinates $(x, \eta)$

# Falkner-Skan boundary layer




$$\delta_{99}(x) \propto \delta(x)$$

**Note**: $Re_x = \frac{u_{ext}\, x}{\nu} \gg 1,$ but $Re_x$ is not so large for turbulence to be present

# Falkner-Skan boundary layer

A simple dimensional estimate suggests: $\delta(x) = \sqrt{\nu x / u_{ext}}$

**(diffusion thickness)**

$$\eta = \frac{y}{\delta(x)} = \frac{y}{\sqrt{\frac{\nu x}{u_{ext}}}} = y x^{(m-1)/2} \left(\frac{\mathcal{U}}{\nu}\right)^{1/2}$$

Let us introduce also the dimensionless streamfunction $f$

$$f = \frac{\psi}{u_{ext}(x)\,\delta(x)} = \frac{\psi}{x^{(m+1)/2}(\mathcal{U}\nu)^{1/2}}$$

and plug into the $x$-momentum equation

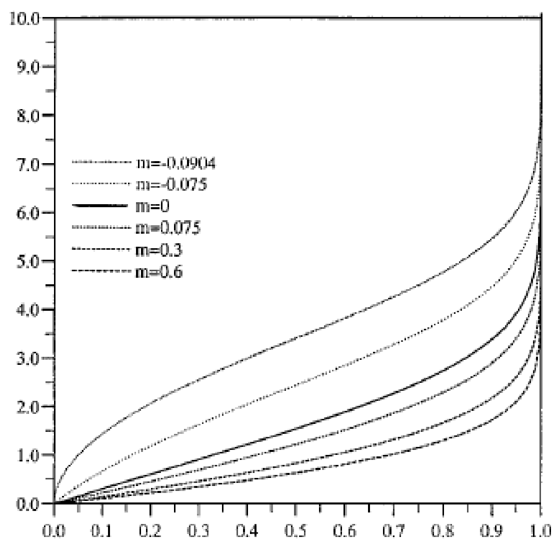
# Falkner-Skan boundary layer

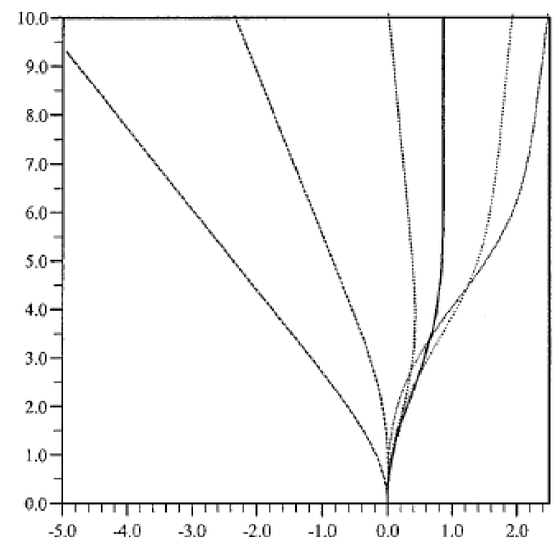
Falkner-Skan equation: $f''' + \frac{m+1}{2}\, f\, f'' + m\left(1 - f'^2\right) = 0$

with b.c.: $f(0) = f'(0) = 0,\ f'(\eta \to \infty) = 1$

$$\boxed{m = \frac{\beta}{2 - \beta}}$$

$\eta = y\sqrt{\frac{u_{ext}}{\nu x}}$




$\frac{u}{u_{ext}} = f'$

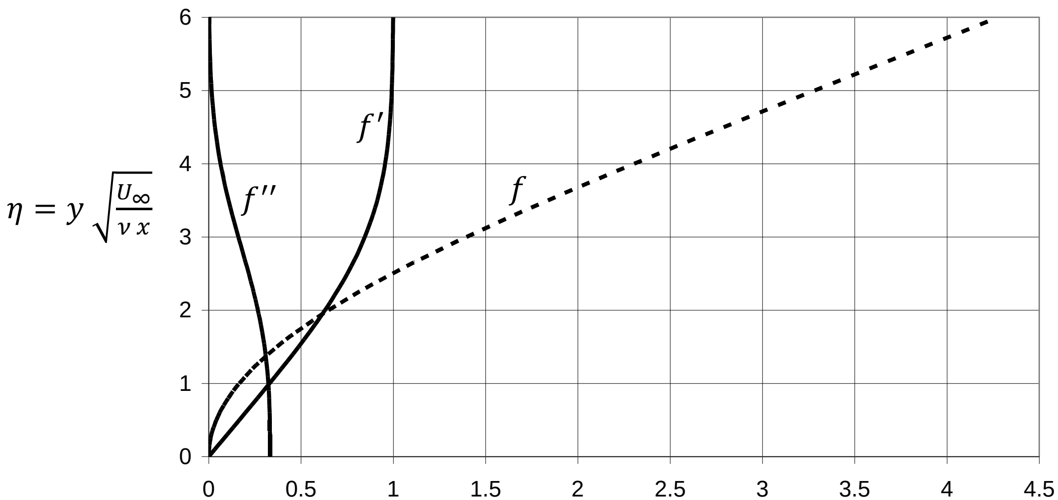
$\frac{v}{u_{ext}}\sqrt{\frac{u_{ext}\, x}{\nu}} = \frac{1}{2}\left[(1-m)\,\eta\, f' - (m+1)\, f\right]$

# Falkner-Skan boundary layer

Special case: Blasius $$f''' + \frac{1}{2} f\, f'' = 0$$

with b.c.: $f(0) = f'(0) = 0,\ f'(\eta \to \infty) = 1$

Blasius solution $(m = 0)$




when $\eta = 4.91$

$$\Downarrow$$

$y = \delta_{99}$

# Falkner-Skan boundary layer

Special cases:

$m = \beta = 0$ Blasius solution, $u_{ext} = U_\infty = \text{constant}$

$m, \beta > 0$ accelerated flows

$m, \beta < 0$ retarded flows

$$\boxed{\begin{array}{c} -0.090429 \leq m \leq 2 \\ -0.198838 \leq \beta \leq 4/3 \end{array}}$$

$m = -0.090429,\ \beta = -0.198838$ incipient separation, below these values of $m$ and $\beta$ we cannot employ the b.l. (*parabolic*) equations

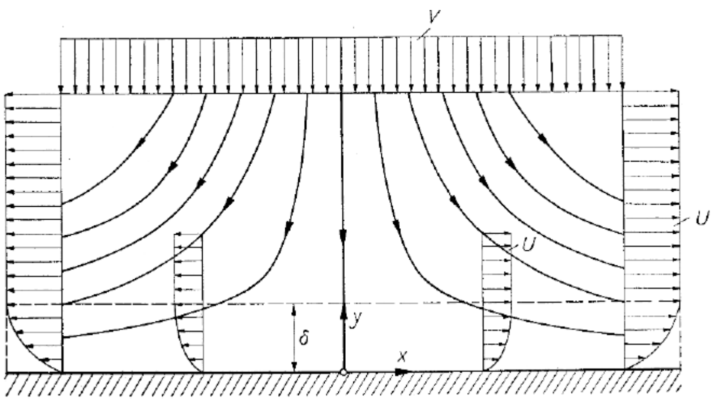
$m = \beta = 1$ Hiemenz stagnation point flow

# Hiemenz stagnation point flow ($m = 1$)

Hiemenz flow: $$f''' + f\,f'' + 1 - f'^2 = 0 \qquad (m = 1)$$

with b.c.: $f(0) = f'(0) = 0, \;\; f'(\eta \to \infty) = 1$

$$\eta = \frac{y}{\delta(x)} = \frac{y}{\sqrt{\frac{\nu x}{u_{ext}}}} = y\left(\frac{\mathcal{U}}{\nu}\right)^{1/2}$$ independent of $x$!

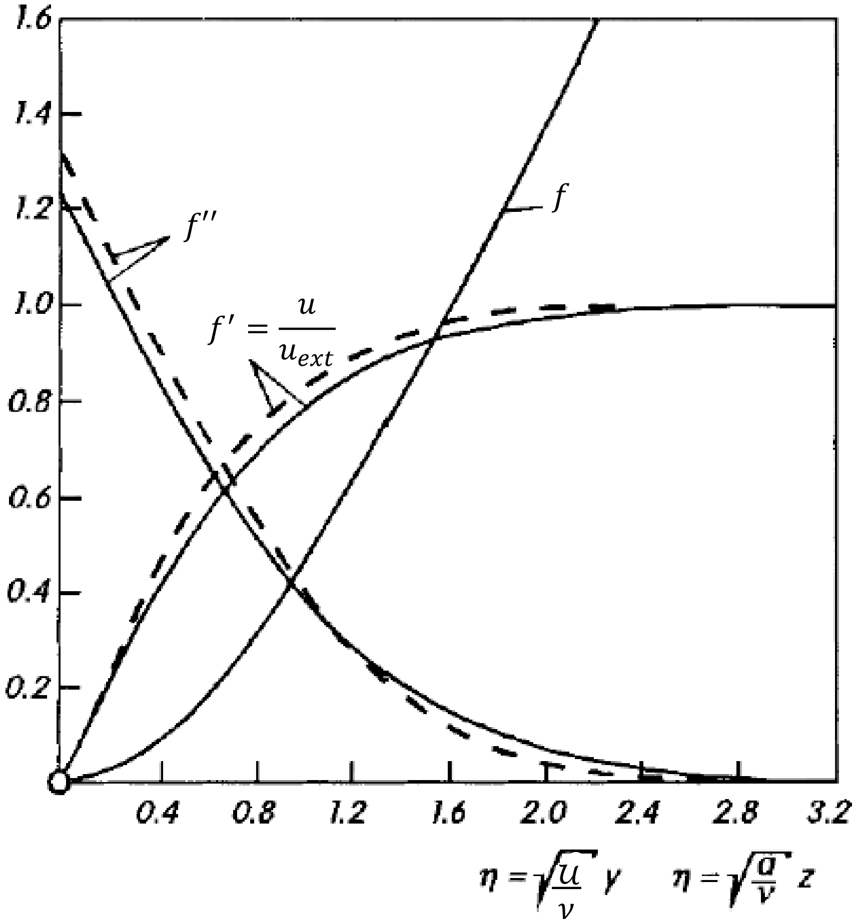



*Outer* potential flow:

$$u_{ext} = \mathcal{U}\,x$$
$$v_{ext} = -\,\mathcal{U}\,y$$

$$p_{ext} = p_0 - \frac{\rho}{2}(x^2 + y^2)\,\mathcal{U}^2$$

# Hiemenz stagnation point flow ($m = 1$)




*Parallel* boundary layer flow, a solution of the Navier-Stokes equations, not just Prandtl's equations!

Velocity distribution of plane (——) and axisymmetric (- - -) stagnation point flows

# Shear stress at the wall and drag

The shear stress at the wall is $\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=0} = \ldots$

$$\tau_w = \rho\, u_{ext} \sqrt{\frac{\nu\, u_{ext}}{x}}\, f''(0)$$

**Local** friction coef: $c_f = \dfrac{\tau_w}{\frac{1}{2}\rho\, u_{ext}^2} = 2\sqrt{\dfrac{\nu}{x\, u_{ext}}}\, f''(0) = \dfrac{2\, f''(0)}{Re_x^{1/2}}$

For the Blasius case: $u_{ext} = U_\infty = \text{const} \;\rightarrow\; f''(0) = 0.332$

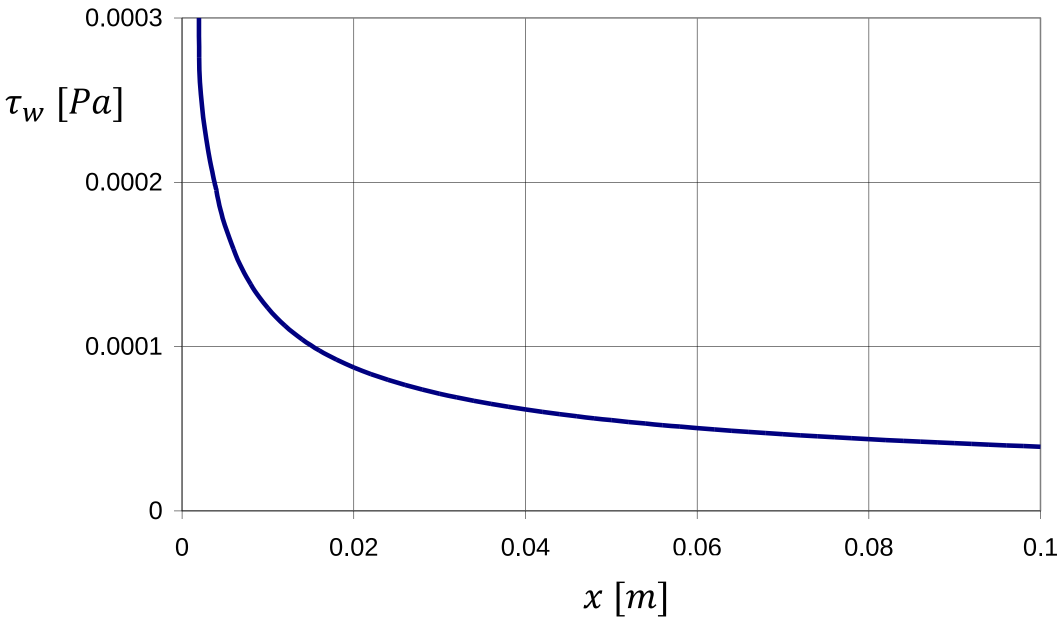
# Shear stress at the wall and drag

## Blasius solution

| $\eta$ | $f$ | $f'$ | $f''$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.33206 |
| 0.1 | 0.00166 | 0.033206 | 0.332051 |
| 0.2 | 0.006641 | 0.066408 | 0.331987 |
| 0.3 | 0.014942 | 0.099599 | 0.331812 |
| 0.4 | 0.02656 | 0.132765 | 0.331473 |
| 0.5 | 0.041493 | 0.165887 | 0.330914 |
| 0.6 | 0.059735 | 0.198939 | 0.330082 |
| 0.7 | 0.081278 | 0.231892 | 0.328925 |
| 0.8 | 0.106109 | 0.264711 | 0.327392 |
| 0.9 | 0.134214 | 0.297356 | 0.325435 |
| 1 | 0.165573 | 0.329783 | 0.32301 |
| 1.1 | 0.200162 | 0.361941 | 0.320074 |
| 1.2 | 0.237951 | 0.393779 | 0.316592 |
| 1.3 | 0.278905 | 0.42524 | 0.312531 |
| 1.4 | 0.322984 | 0.456265 | 0.307868 |
| 1.5 | 0.370142 | 0.486793 | 0.302583 |
| 1.6 | 0.420324 | 0.516761 | 0.296666 |
| 1.7 | 0.473473 | 0.546105 | 0.290114 |
| 1.8 | 0.529522 | 0.574763 | 0.282933 |
| 1.9 | 0.5884 | 0.602671 | 0.275138 |
| 2 | 0.65003 | 0.62977 | 0.266753 |
| 2.1 | 0.714326 | 0.656003 | 0.257811 |

Ex. $U_\infty = 0.04\ \dfrac{m}{s},\ L = 0.1\, m\,,$

$$\nu_{air} = 1.5\ \times 10^{-5}\ \frac{m^2}{s}, \rho_{air} = 1.2\ \frac{kg}{m^3}$$



# Shear stress at the wall and drag

Taking a spanwise length equal to $b$, the **drag force due to friction** on a *single* side of a flat plate of length $L$ is

$$D_f = b \int_0^L \tau_w \, dx = 0.332 \, \rho \, b \, U_\infty \sqrt{\nu \, U_\infty} \int_0^L x^{-1/2} dx$$

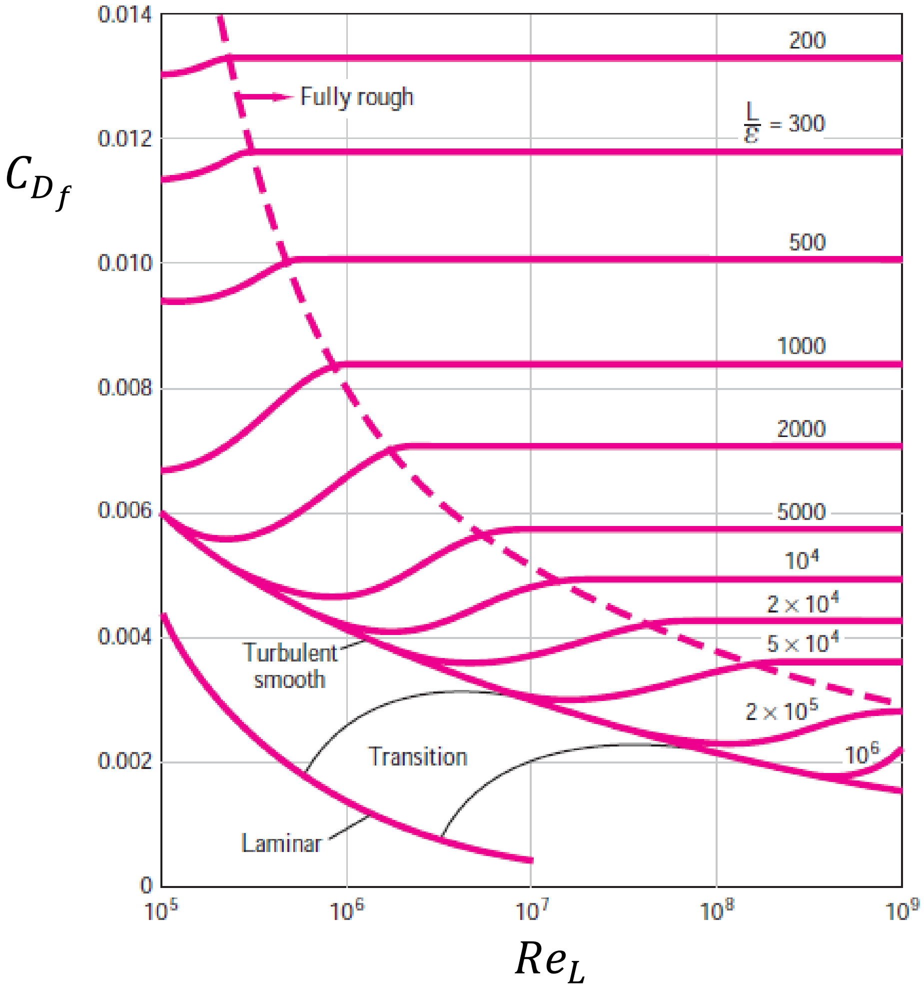
$$\int_0^L x^{-1/2} \, dx = 2 \, L^{1/2}$$

$$D_f = 0.664 \, \rho \, b \, U_\infty \sqrt{\nu \, L \, U_\infty}$$

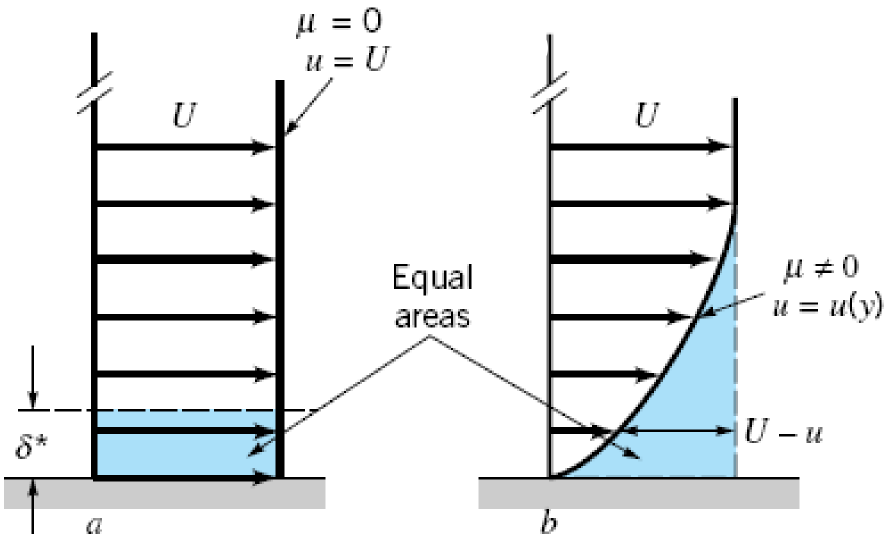
Drag coefficient for the Blasius flow on one side of the plate:

$$\boxed{C_{D_f} = \frac{2 \, D_f}{\rho \, U_\infty^2 \, L \, b} = \frac{1.328}{Re_L^{1/2}}} \qquad \left( Re_L = \frac{L \, U_\infty}{\nu} \right)$$

# Drag coefficient on a flat plate b.l.



# B.l. thicknesses




Displacement thickness $\delta^*$:

by how much must the wall be displaced for a potential (inviscid) flow to have the **same mass flux** of the boundary layer flow

$$\int_{\delta^*}^{\infty} U\,dy = \int_0^{\infty} u\,dy \longrightarrow \int_0^{\infty} U\,dy - \int_0^{\delta^*} U\,dy = \int_0^{\infty} u\,dy$$

$$\int_0^{\infty} U\,dy - U\,\delta^* = \int_0^{\infty} u\,dy$$

$$\boxed{\delta^* = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy}$$

# B.l. thicknesses

Momentum thickness $\vartheta$:

by how much must the wall be displaced for a potential (inviscid) flow to have the **same momentum flux** of the boundary layer flow

$$\vartheta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

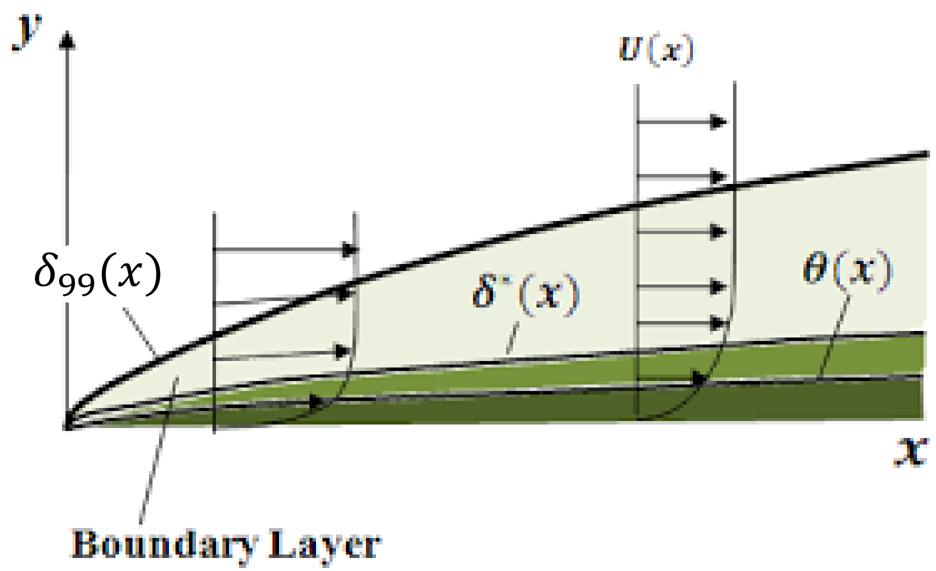
(a measure of the friction drag!)

Ex. laminar Blasius flow

$$\delta_{99}(x) = 4.91\,\delta(x)$$

$$\delta^*(x) = 1.72\,\delta(x)$$

$$\vartheta\,(x) = 0.664\,\delta(x)$$

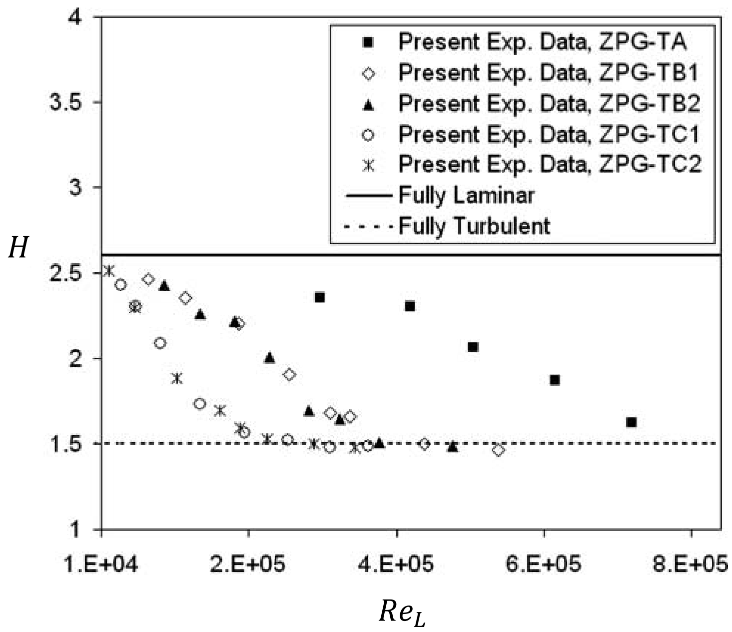


# B.l. thicknesses

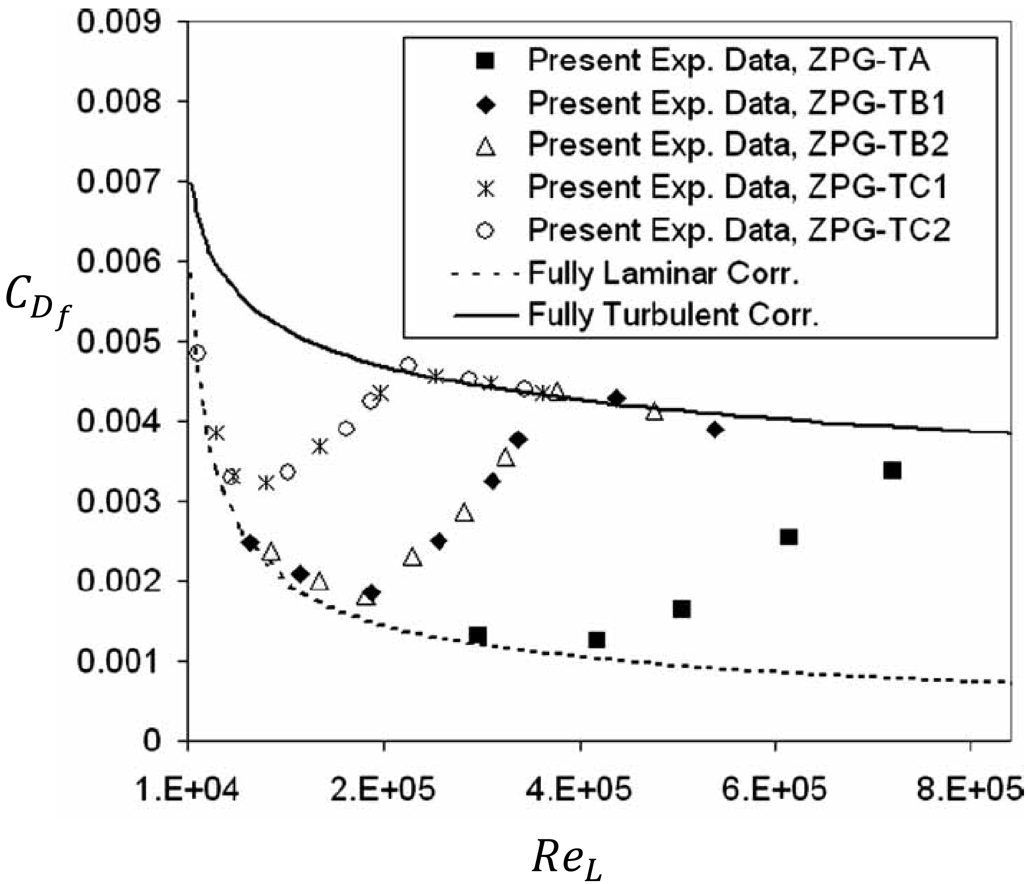
Shape factor: $H = \frac{\delta^*}{\vartheta}$

Laminar Blasius flow: $H = 2.59$. Turb. flow: $H$ decreases.
If $H$ reaches a value close to 2.7 flow separation is incipient

Flat plate b.l. measurements for different FST intensities: $H$ is a reliable measure of transition onset



# Transition on flat plate b.l.



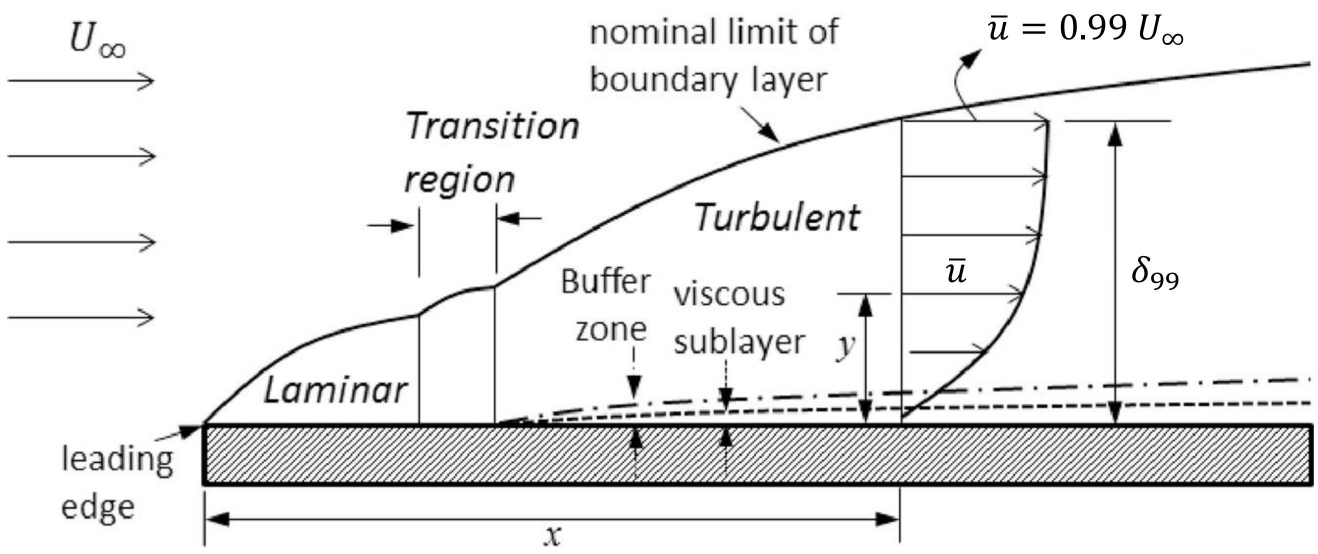

How does transition take place?
What factors affect it?

**Receptivity**

# Turbulence (empirical correlations)

Empirical laws for turbulent flow over a flat plate (with turbulence assumed to be present *right from the start* of the boundary layer!), valid for $5 \times 10^5 \leq Re_L \leq 10^7$:

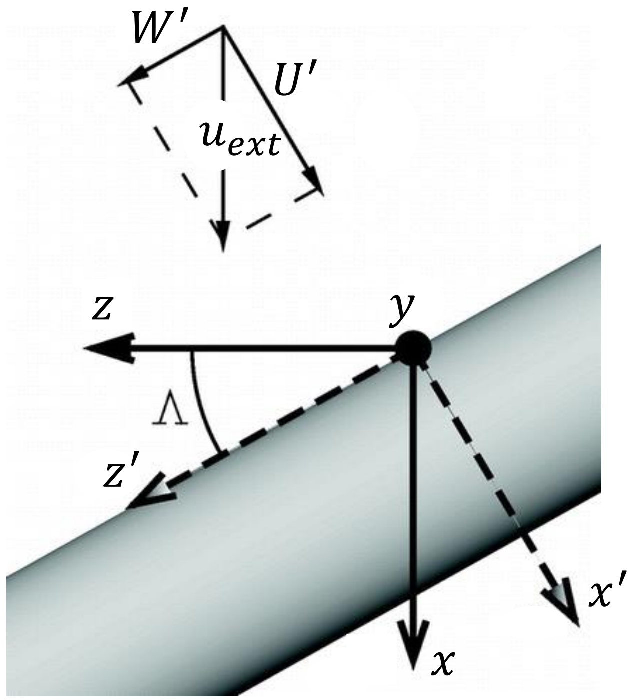
$$\delta_{99}(x) \approx \frac{0.38\,x}{Re_x^{1/5}} \qquad c_f(x) \approx \frac{0.059}{Re_x^{1/5}} \qquad C_{D_f} \approx \frac{0.074}{Re_L^{1/5}}$$



Before giving a qualitative description of some aspects of the transition phenomenon, plus some elements on turbulent boundary layers, let us go back to laminar flows and examine a significant 3D *self-similar* boundary layer solution

# 3D laminar boundary layer (F-S-C)

Often wings are swept with respect to the outer flow $u_{ext}$. A model of the flow over an *infinite* swept wing is the Falkner-Skan-Cooke flow




Let the wall-normal direction be $y = y$'.

$x$' and $z$' are wing-fixed coordinates

# 3D laminar boundary layer (F-S-C)

The steady, 3D boundary layer equations (in the wing-fixed coordinate system) are

$$
\begin{cases}
\dfrac{\partial u'}{\partial x'} + \dfrac{\partial v'}{\partial y'} + \dfrac{\partial w'}{\partial z'} = 0 \\
u'\dfrac{\partial u'}{\partial x'} + v'\dfrac{\partial u'}{\partial y'} + w'\dfrac{\partial u'}{\partial z'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x'} + \nu\dfrac{\partial^2 u'}{\partial y'^2} \\
u'\dfrac{\partial w'}{\partial x'} + v'\dfrac{\partial w'}{\partial y'} + w'\dfrac{\partial w'}{\partial z'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z'} + \nu\dfrac{\partial^2 w'}{\partial y'^2} \\
\dfrac{\partial p}{\partial y'} = 0
\end{cases}
$$

with $u' = v' = w' = 0$ at $y' = 0$

$u' = U',\ w' = W'$ at $y' \to \infty$

# 3D laminar boundary layer (F-S-C)

Assume: $U' = \mathcal{U}\, x'^m,\ W' = W_\infty$ = const, plus the fields are invariant along $z'$

$$
\begin{cases}
\dfrac{\partial u'}{\partial x'} + \dfrac{\partial v'}{\partial y'} + \cancel{\dfrac{\partial w'}{\partial z'}} = 0 \\
u'\dfrac{\partial u'}{\partial x'} + v'\dfrac{\partial u'}{\partial y'} + \cancel{w'\dfrac{\partial u'}{\partial z'}} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x'} + \nu\dfrac{\partial^2 u'}{\partial y'^2} \\
u'\dfrac{\partial w'}{\partial x'} + v'\dfrac{\partial w'}{\partial y'} + \cancel{w'\dfrac{\partial w'}{\partial z'}} = \cancel{-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z'}} + \nu\dfrac{\partial^2 w'}{\partial y'^2} \\
\dfrac{\partial p}{\partial y'} = 0
\end{cases}
$$

with $u' = v' = w' = 0$ at $y' = 0$

$u' = \mathcal{U}\, x'^m,\ w' = W_\infty$ at $y' \to \infty$

# 3D laminar boundary layer (F-S-C)

Falkner-Skan system plus an independent equation for $w'$

$$\begin{cases} \dfrac{\partial u'}{\partial x'} + \dfrac{\partial v'}{\partial y'} = 0 \\ u'\dfrac{\partial u'}{\partial x'} + v'\dfrac{\partial u'}{\partial y'} = m\,\mathcal{U}^2\, x^{2m-1} + \nu\dfrac{\partial^2 u'}{\partial y'^2} \end{cases}$$

with $u' = v' = 0$ at $y' = 0$
$u' = \mathcal{U}\, x'^m$ at $y' \to \infty$

F-S system

$$u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} = \nu\frac{\partial^2 w'}{\partial y'^2}$$

with $w' = 0$ at $y' = 0$
$w' = W_\infty$ at $y' \to \infty$

Cooke equation (linear)

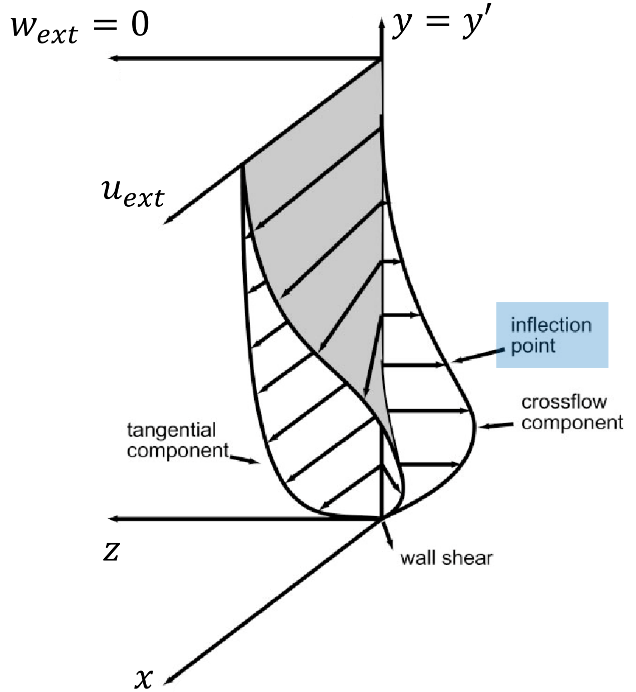
Similarity variable: $\eta = y'\sqrt{\dfrac{U'(x')}{\nu\, x'}}$

**assume:** $w' = W_\infty\, g(\eta)$

# 3D laminar boundary layer (F-S-C)

Cooke's equation is: $$\frac{m+1}{2} f\, g' + g'' = 0$$

subject to $g(0) = 0,\;\; g(\eta \to \infty) = 1$



# Instability of boundary layers

The manner through which different boundary layers become unstable (and progressively become turbulent) is very much dependent on the type/shape of boundary layer itself (*the basic flow state*) and on the *environmental conditions*.

For example, the first instability of the Blasius boundary layer may consist of the onset and amplification of 2D streamwise travelling waves (Tollmien-Schlichting waves); instability in the F-S-C flow may appear in the form of steady or travelling cross-flow vortices …

# Instability of boundary layers

## Flat plate

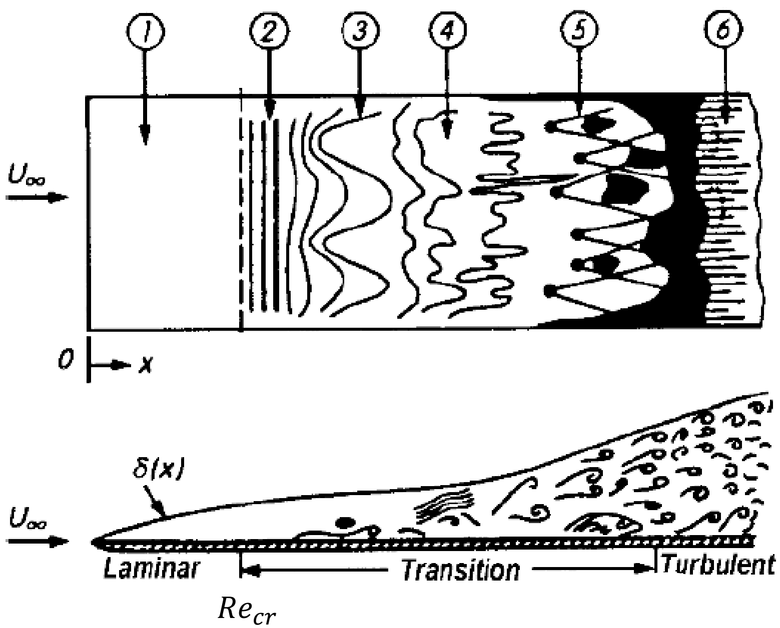



**Fig. 15.5.** Sketch of laminar–turbulent transition in the boundary layer on a flat plate at zero incidence, after F.M. White (1974)

(*1*) stable laminar flow

(*2*) unstable Tollmien–Schlichting waves

(*3*) three–dimensional waves and vortex formation ($\Lambda$-structures)

(*4*) vortex decay

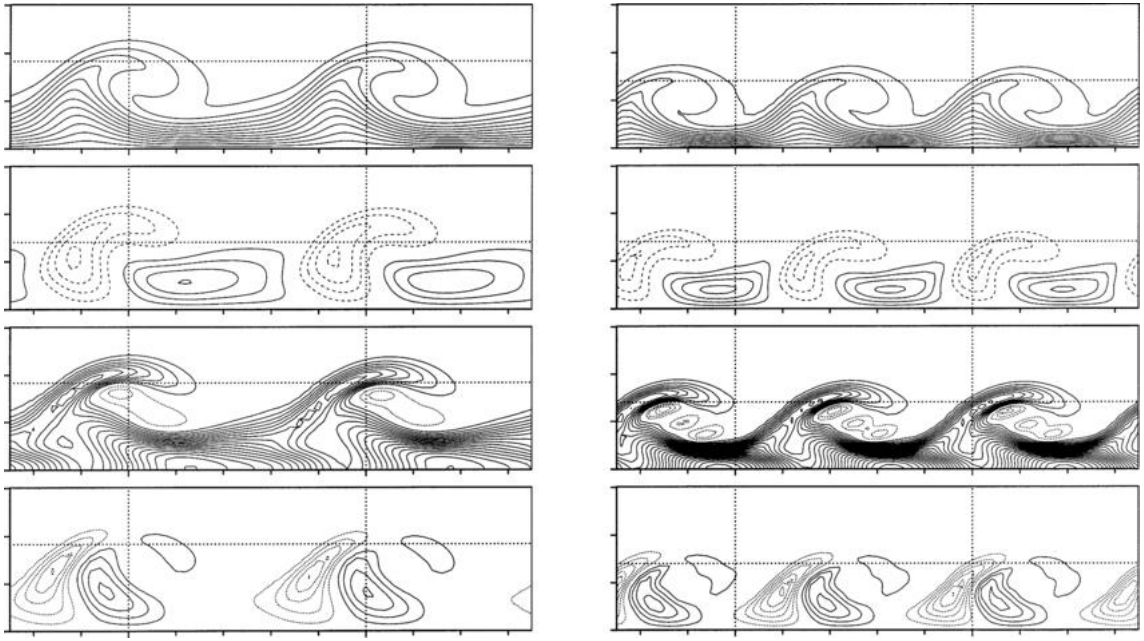
(*5*) formation of turbulent spots

(*6*) fully turbulent flow

((after H. Schlichting, 1975)

# Instability of boundary layers

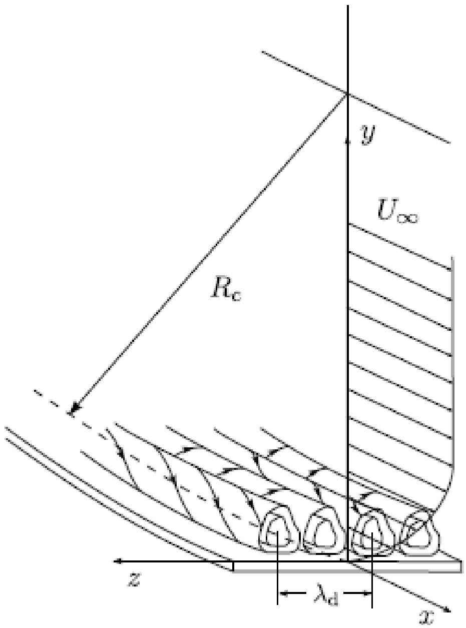
## Infinite plate with a sweep angle



Steady, co-rotating cross-flow vortices, induced by inflectional velocity profile; wavelength determined by spanwise roughness spacing

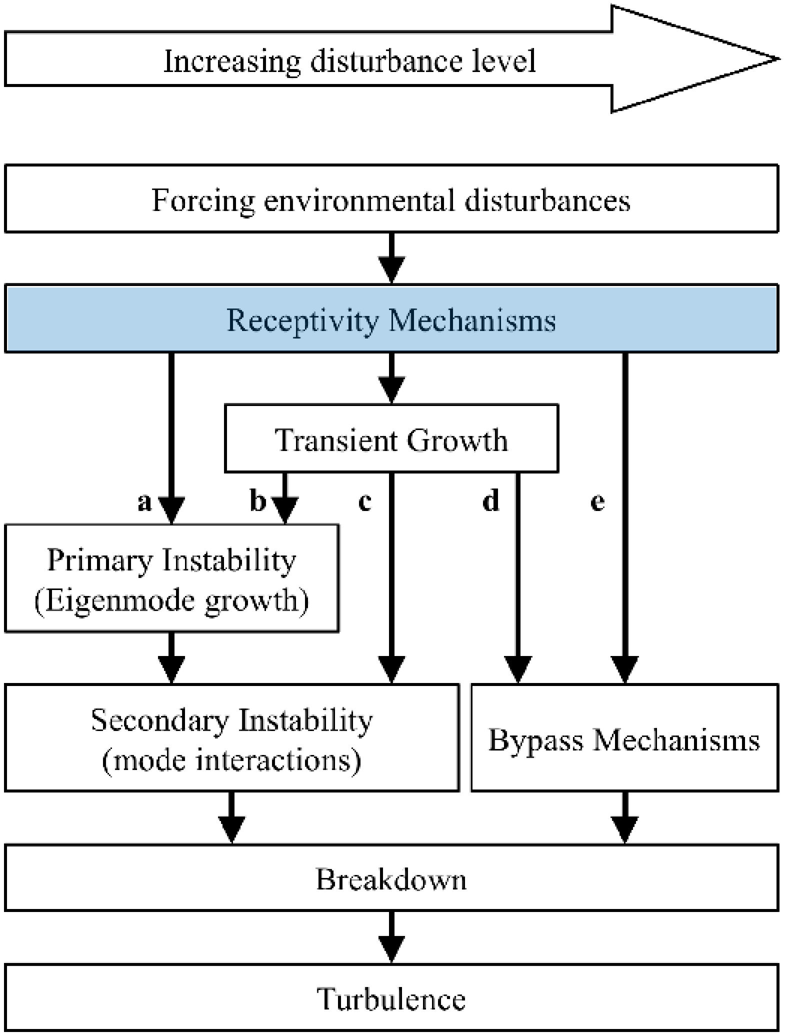
# Instability of boundary layers

## Curved plate, concave curvature




Steady, counter-rotating Görtler vortices, induced by an imbalance between the wall-normal pressure gradient and the centrifugal force

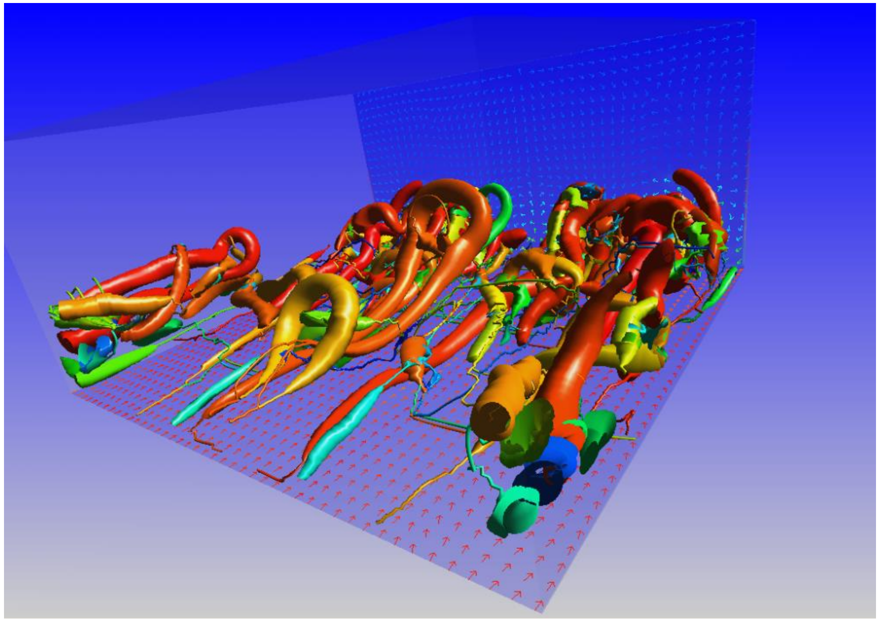
# Receptivity




https://www.youtube.com/watch?v=wXsl4eyupUY

# Turbulence

After breakdown, fully turbulent conditions are encountered, with flows characterized by large unsteady fluctuations ($u'$, etc.)

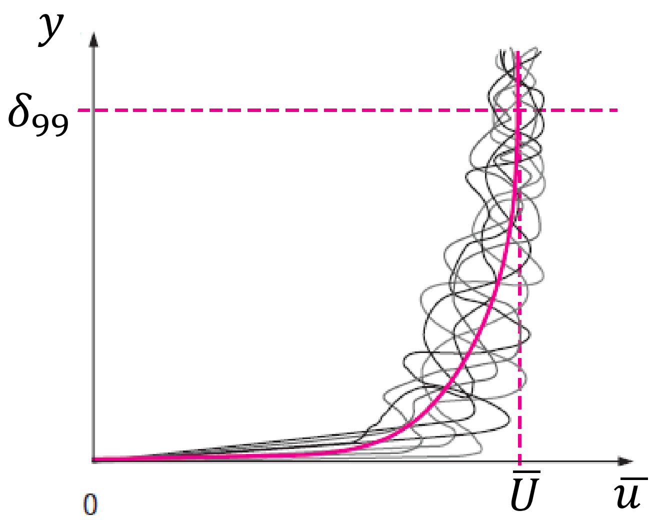


https://www.youtube.com/watch?v=DX_wPZJYfAQ&feature=emb_logo&ab_channel=oanamovies

# Turbulent flat plate boundary layer

The turbulent boundary layer over a flat plate is often modelled with empirical laws, such as the 1/7 power law

$$\frac{\bar{u}}{\overline{U}} \approx \left(\frac{y}{\delta_{99}}\right)^{1/7} \quad \text{for} \quad y \leq \delta_{99} \qquad\qquad \frac{\bar{u}}{\overline{U}} \approx 1 \quad \text{for} \quad y > \delta_{99}$$

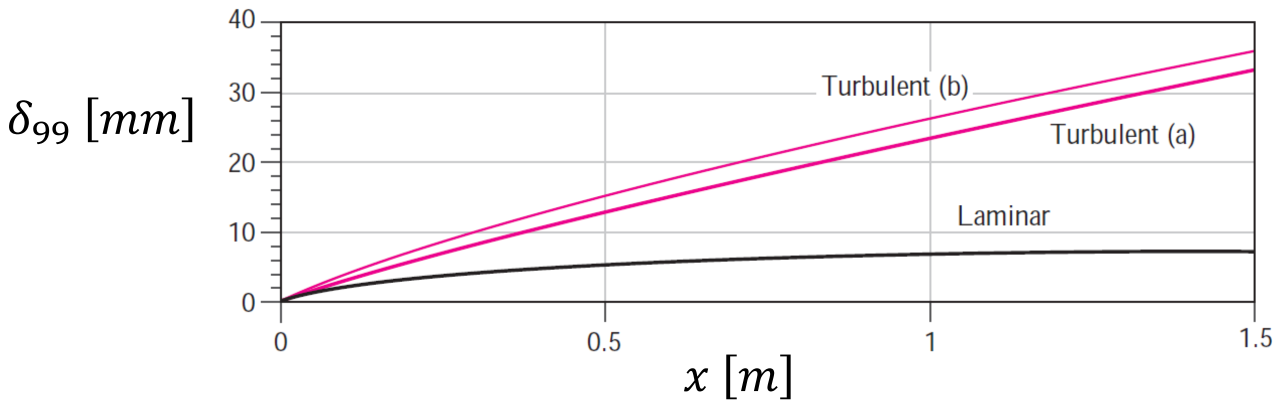



with $\bar{u}$ and $\overline{U}$ time-averaged (or ensemble-averaged) velocities.

However, such empirical laws are **inaccurate** near the wall

# Turbulent flat plate boundary layer

| Property | Laminar | (a) Turbulent | (b) Turbulent |
|---|---|---|---|
| Boundary layer thickness | $\frac{\delta_{99}}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$ | $\frac{\delta_{99}}{x} \cong \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$ | $\frac{\delta_{99}}{x} \cong \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$ |
| Displacement thickness | $\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$ | $\frac{\delta^*}{x} \cong \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$ | $\frac{\delta^*}{x} \cong \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$ |
| Momentum thickness | $\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$ | $\frac{\theta}{x} \cong \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$ | $\frac{\theta}{x} \cong \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$ |
| Local skin friction coefficient | $c_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$ | $c_f \cong \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$ | $c_f \cong \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$ |




Flow of air at $10\ [m/s]$ over a smooth flat plate of length $L = 1.5\ [m]$

# Turbulent flat plate boundary layer

A better semi-empirical approximation of the near-wall boundary layer flow starts from the definition of the *friction velocity*:

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

**Why $\boldsymbol{u_\tau}$?** The streamwise *Reynolds-averaged* momentum equation in the near-wall region of a fully developed turbulent boundary layer can be approximated as:

$$\mu \frac{\partial^2 \bar{u}}{\partial y^2} - \rho \frac{\partial}{\partial y}\left(\overline{u'v'}\right) = 0$$

and integrating once over $y$:

$$\tau_w = \mu \frac{\partial \bar{u}}{\partial y} - \rho\, \overline{u'v'}$$

# Turbulent flat plate boundary layer

**Normalize** by an as-yet unknown velocity scale $u^*$ and viscous length scale $\nu/u^*$, so that the equation reduces to:

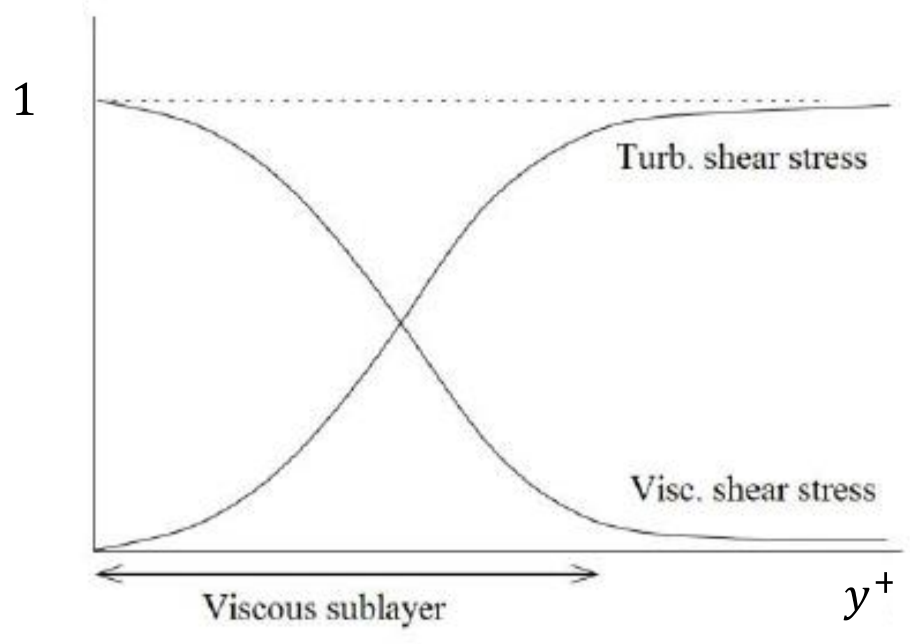
$$\frac{\tau_w}{\rho u^{*2}} = \frac{\partial(\bar{u}/u^*)}{\partial(yu^*/\nu)} - \frac{\overline{u'v'}}{u^{*2}}$$

Now, let us choose $u^* = u_\tau$ for the left-hand-side of the equation to become equal to one. The terms on the right-hand-side must also be of order 1. We thus have:

$$1 = \frac{\partial u^+}{\partial y^+} - \frac{\overline{u'v'}}{u_\tau^2}$$

# Turbulent flat plate boundary layer

$$1 = \frac{\partial u^+}{\partial y^+} - \frac{\overline{u'v'}}{u_\tau^2}$$



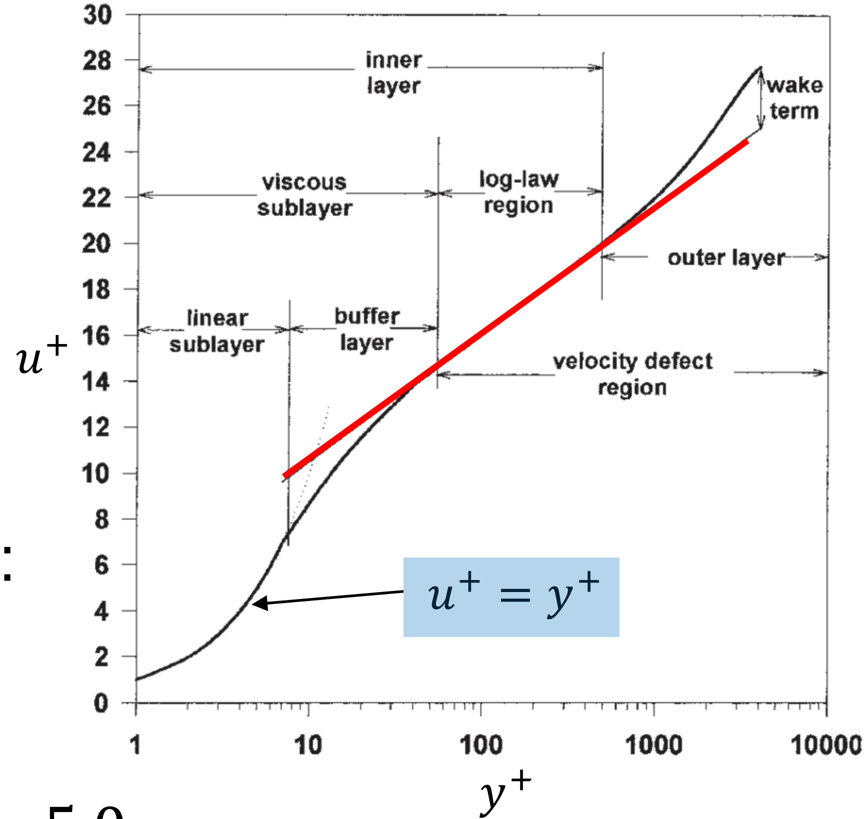
# Turbulent flat plate boundary layer

Near the wall the proper velocity scale (for both mean flow and fluctuations) is $u_\tau$ and the proper scale of length is $\nu/u_\tau$. These are the so-called + **(or viscous) variables.**

In the log-law (*inertial*) region the mean velocity has the form:

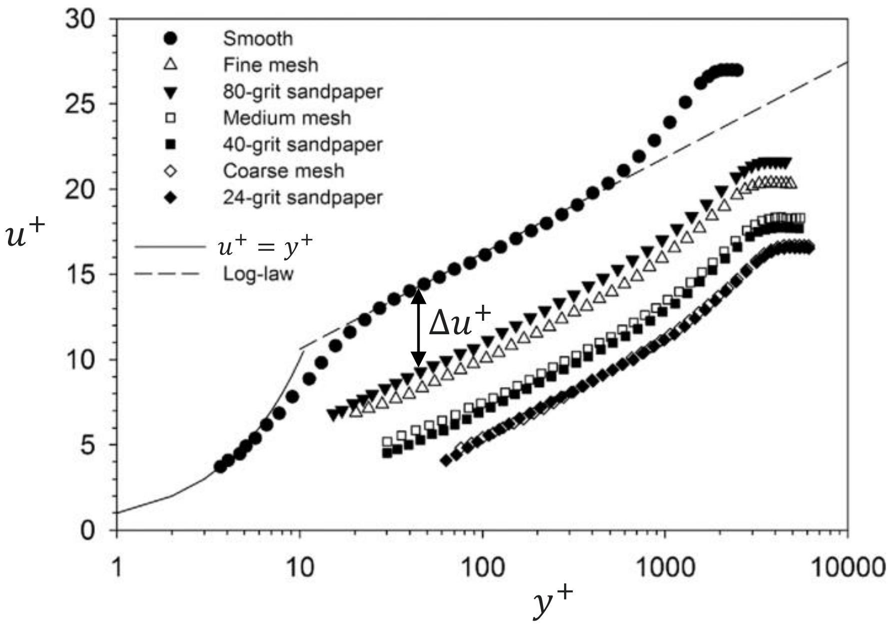
$$u^+ = \frac{1}{\kappa}\ln y^+ + A$$

$\kappa \approx 0.41$ *Karman constant*, $A \approx 5.0$



# Turbulent flat plate boundary layer

Wall roughness shifts the logarithmic law, without changing the logarithmic behavior



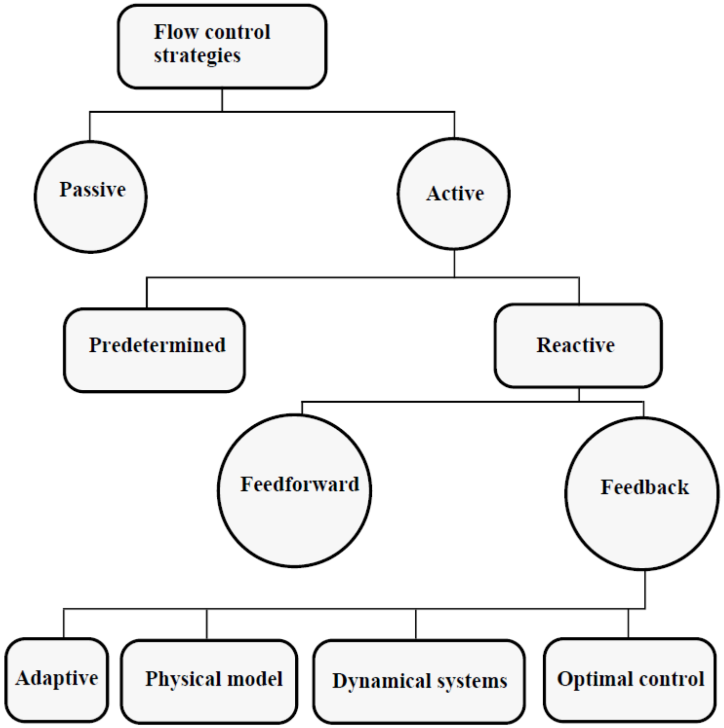

$$u^+ = \frac{1}{\kappa} \ln y^+ + A - \Delta u^+$$

$\Delta u^+$ : Clauser roughness function

The log-law is also affected by pressure gradient, curvature, etc. It should not be used “blindly” in a CFD code!

# Boundary layer control

How can a boundary layer be *controlled*?
What objective(s) are we trying to pursue?
How do we want to achieve it/them?




(M. Gad-el-Hak 1996)

# Boundary layer control

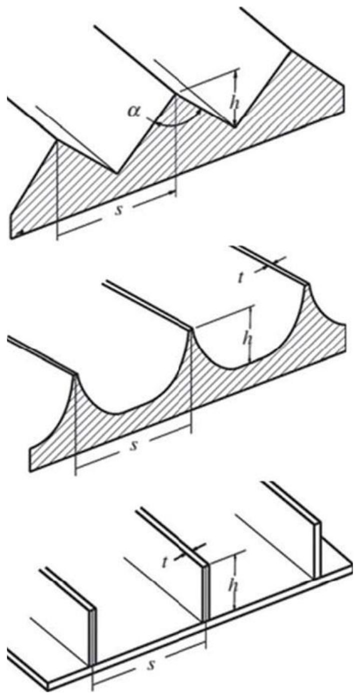
**Passive**

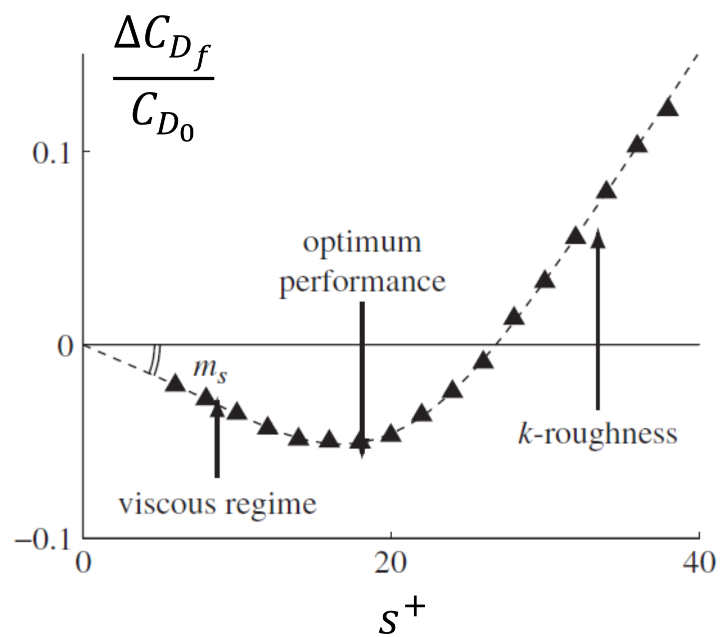
Ex. **Riblets** for skin-friction drag reduction in a turbulent boundary layer (ineffective in laminar flow!)

sawtooth

scalloped

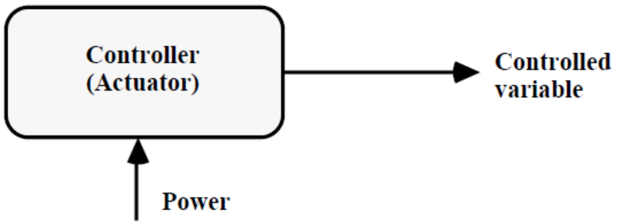
blade riblets



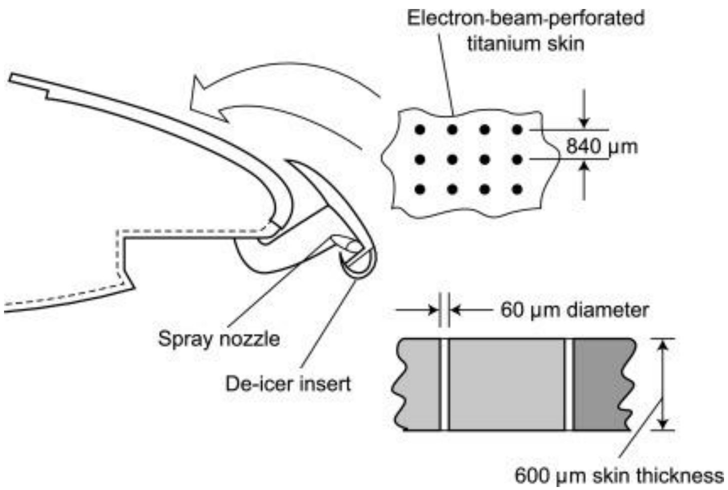


# Boundary layer control

**Active, predetermined**

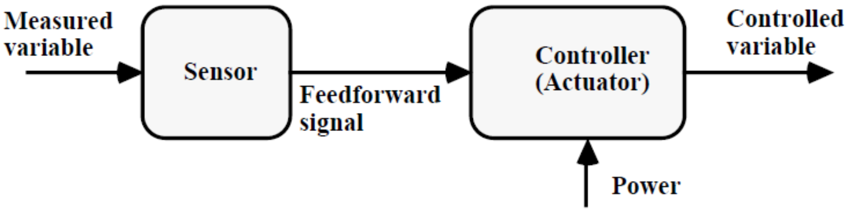



Ex. **Suction-type laminar flow control** for transition delay

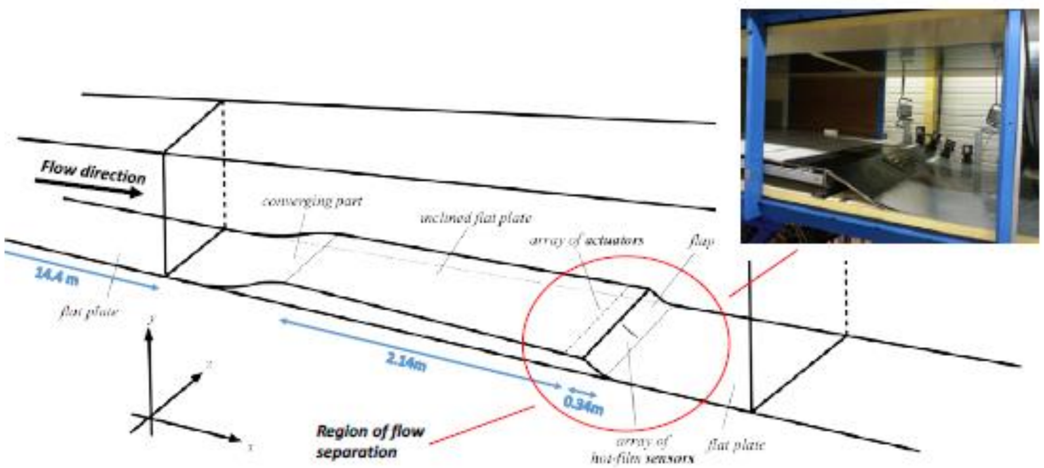



Leading-edge arrangement for 1983–1987 flight tests conducted on a JetStar aircraft at NASA's Dryden Flight Research Center. Important features: (1) suction on upper surface only; (2) suction through electron-beam-perforated skin; (3) leading-edge shield extended for insect protection; (4) de-icer insert on shield for ice protection; (5) supplementary spray nozzles for protection from insects and ice

## Reactive, feedforward




Ex. Open loop **control of flow separation**

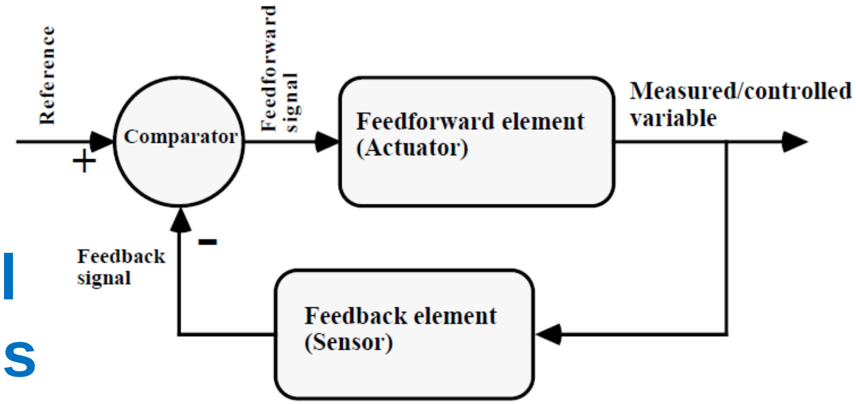



Laboratoire de Mécanique de Lille

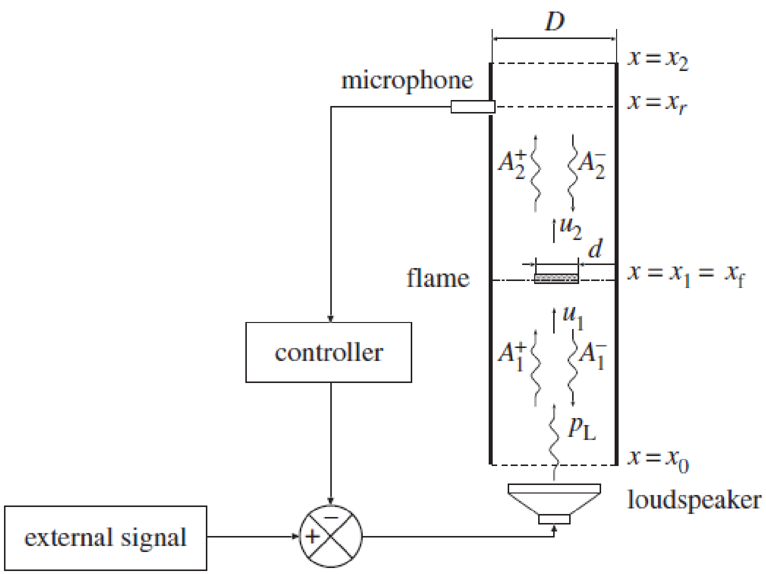
Important features: (1) an array of 22 round jet air blowers, parallel to the flap's edge, is used as actuators to reduce/eliminate the separation region; (2) hot film sensors along the flap are used to measure the gain in skin friction

# Another example of flow control problem

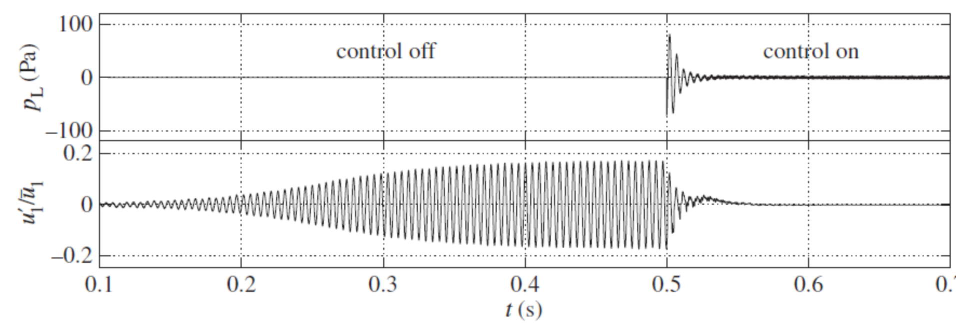
## Reactive, feedback




Ex. Closed loop, robust **control of combustion instabilities**




Rijke tube




Evolution of $p_L$ and $u_1'/\bar{u}_1$ with time. The controller is switched on when $t = 0.5$ s. $x_f = 0.2$ m.

(Li & Morgans, 2016)

# Exercises

Ex. 1 Do all the steps which lead to the Falkner-Skan ODE equation (slide 18)

Ex. 2 For the model boundary layer flow

$$\begin{cases} \dfrac{u}{U_\infty} = 2\eta - 2\eta^3 + \eta^4 & \text{for} \quad 0 \le \eta = \dfrac{y}{\delta_{99}} \le 1 \\ \dfrac{u}{U_\infty} = 1 & \text{for} \quad \eta > 1 \end{cases}$$

compute $\delta^*$, $\vartheta$, $c_f$ and $C_{D_f}$, and compare to the Blasius solution

Ex. 3 Resolve the problem of the "asymptotic" suction boundary layer: a flat plate is placed parallel to a constant stream at $U_\infty$ and suction is applied through the wall so that a constant (negative) velocity $V_w$ is present at $y = 0$

Ex. 4 Revise problems 4.8 to 4.11 of the book by Anderson (6th ed.) from p. 382

Ex. 5 Using the results on slide 27, evaluate the roughness amplitude $\varepsilon$ of a flat plate of length $L = 1\,m$, with a turbulent boundary layer of speed $U_\infty = 150\,m/s$, $C_{D_f} = 0.0048$. The fluid is air, $\nu = 1.5 \times 10^{-5}\,m^2/s$