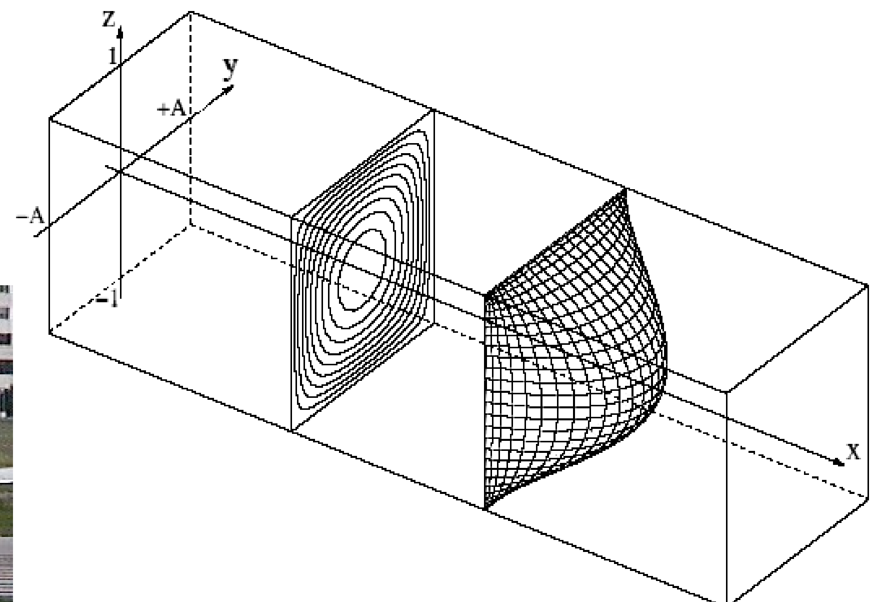
Secondary vortices in turbulent square-duct flow

A. Bottaro, H. Soueid & B. Galletti
DIAM, Università di Genova & DIASP, Politecnico di Torino



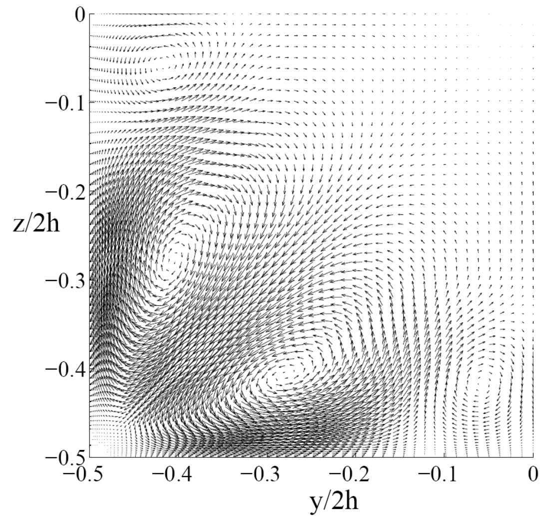
Goal: hydrodynamic stability-based approach to make progress towards understanding the formation of secondary flows in turbulent ducts. Proper prediction of secondary vortices crucial in many applications.

Problem being investigated since Nikuradse (*Ph.D. Thesis*, Göttingen, 1926)

Experiments:	Brundett & Baines (1964), Gessner (1973)
Reynolds-averaged simulations:	Launder & Ying (1973), Demuren & Rodi (1984) Mompean (1998)
DNS/LES:	Madabhushi & Vanka (1991), Gavrilakis (1992), Huser & Biringen (1993)

- Secondary flows near corners induced by anisotropic turbulent fluctuations
- Second-order closure underpredicts secondary vortices, possibly because of inadequate modeling of secondary shear stress components
- "*A theory of the flow structures that give rise to the observed mean flow is not yet available*"

Qualitative picture of the mean secondary flow:



TURBULENT FLOW: AVAILABLE DNS

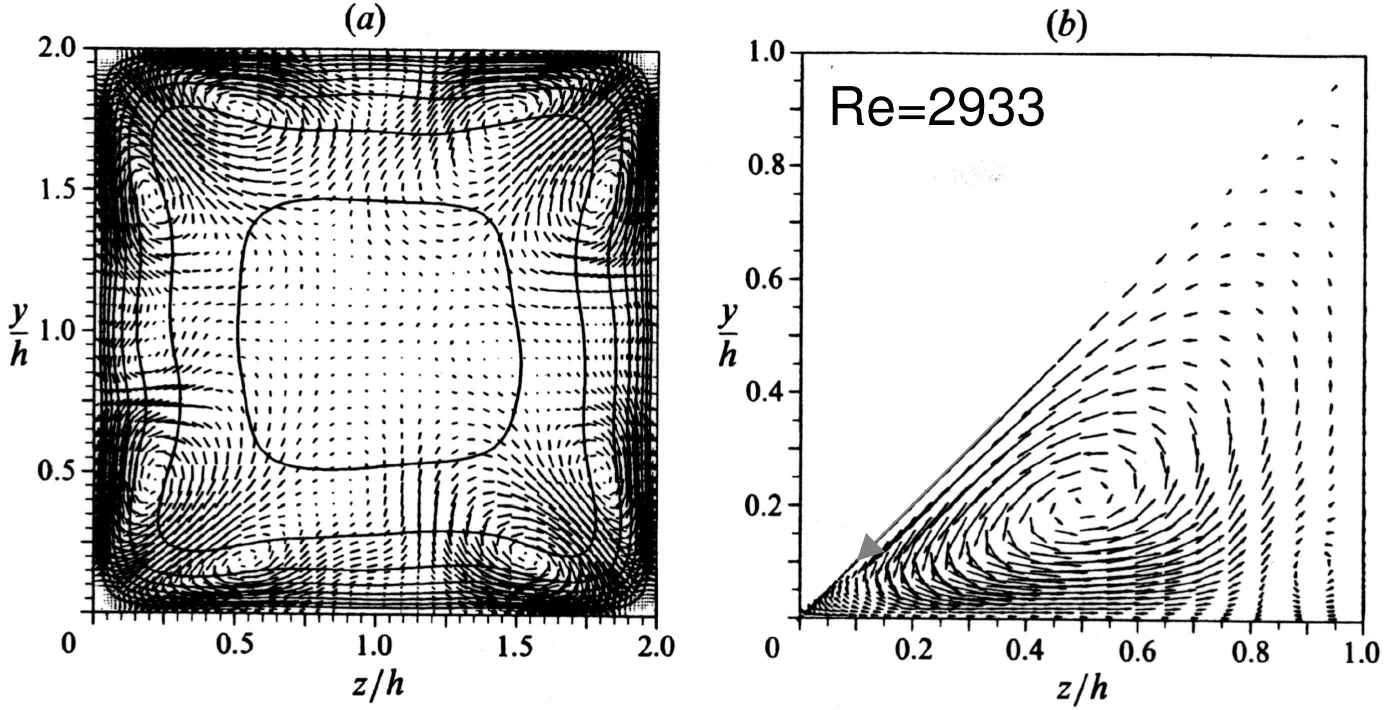
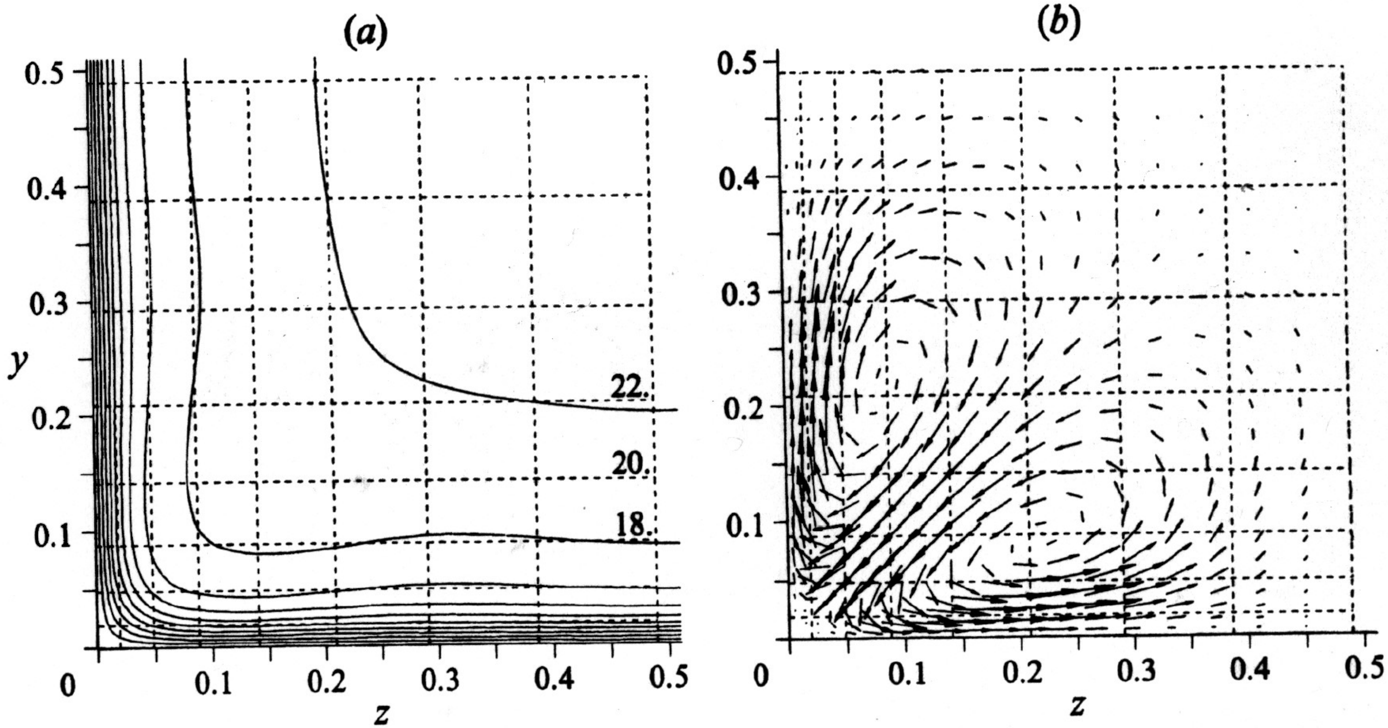


FIGURE 6. (*a*) Mean secondary velocity vectors and mean streamwise flow contours. The contour increment is $4u_\tau$ with the lowest value contour being nearest to the duct walls representing $4u_\tau$ units. (*b*) Vector field in (*a*) averaged over all octants. Only half the vectors in each direction are shown.

S. Gavrilakis, *JFM* 1992

TURBULENT FLOW: AVAILABLE DNS



Re = 6880

FIGURE 4. Ensemble-averaged mean velocities, Run A: (a) $\bar{u}$-contours, increment = 2; (b) $\bar{v}$, $\bar{w}$ velocity vectors.

A. Huser & S. Biringen, *JFM* 1993 " … the secondary flow is produced by the secondary Reynolds stresses."

TURBULENT FLOW: AVAILABLE LES

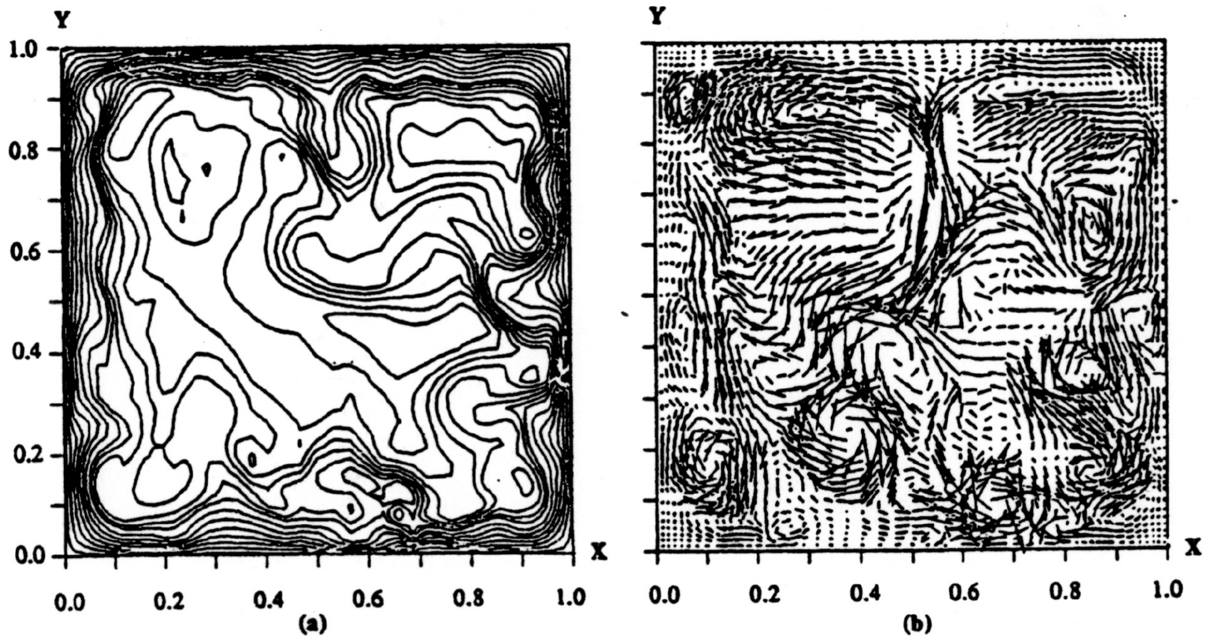


FIG. 3. (a) Instantaneous streamwise velocity contours, and (b) instantaneous secondary velocity vectors in the $Z = 0$ plane

Re = 3873

R.K. Madabhushi & S.P. Vanka, *PoF* 1991

TURBULENT FLOW: AVAILABLE RANS

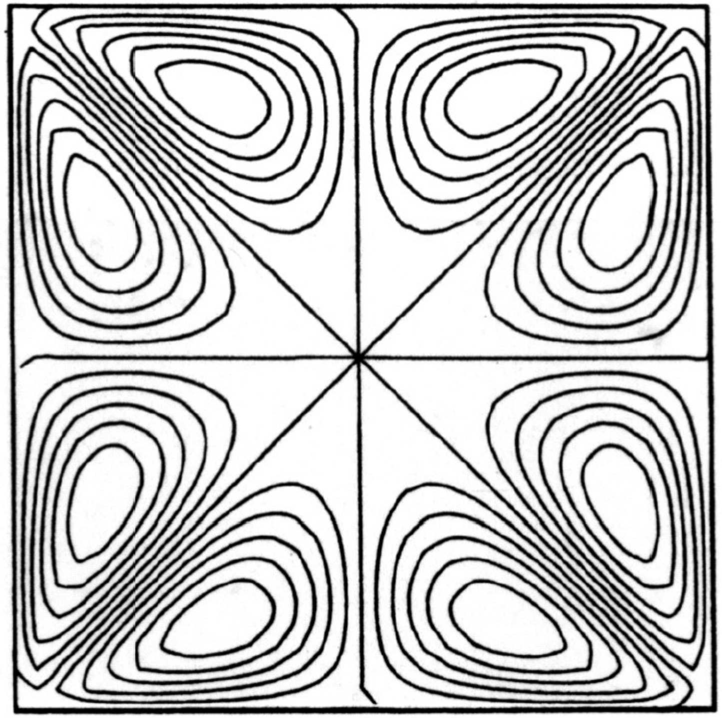
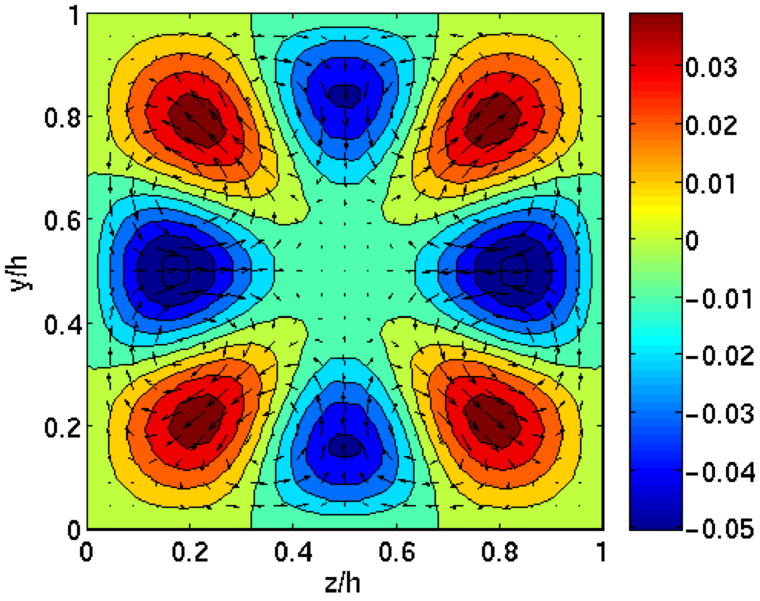
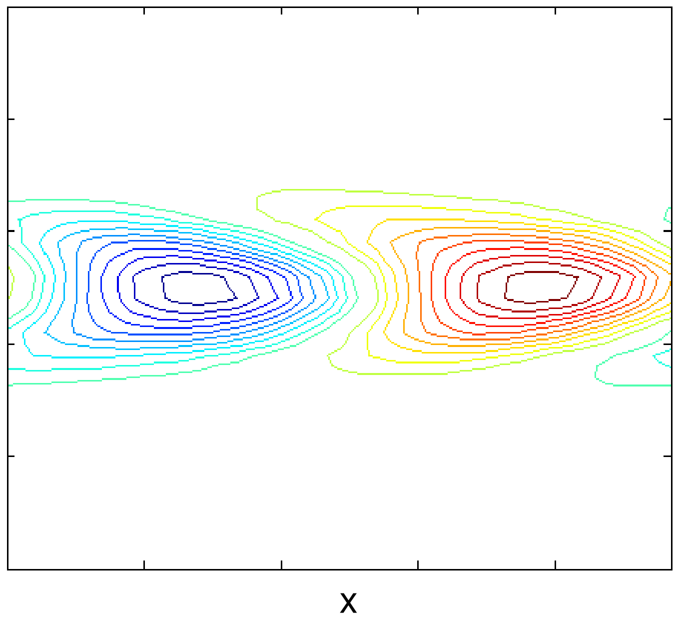


Figure 6. Fully developed secondary flow streamlines in a rectangular duct obtained using the nonlinear K–l model.

C.G. Speziale, *JFM* 1987 "… linear models of turbulence can give rise to highly inaccurate predictions …"

“TURBULENT” FLOW: *ECS*





Isolines of the transverse velocity in a mid-plane

OPTIMAL COHERENT STATES

Triple decomposition of the variables:

$$(u, v, w, p) \quad = \quad [U(y,z), 0, 0, P(x)] + [\tilde{u}(x,y,z), \tilde{v}(x,y,z), \tilde{w}(x,y,z), \tilde{p}(x,y,z)] + [u', v', w', p'](x,y,z,t)$$

with the coherent part $\tilde{u}(x,y,z), \tilde{v}(x,y,z), \tilde{w}(x,y,z), \tilde{p}(x,y,z)$ "small"

F

$$\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0,$$

$$U\tilde{u}_x + \tilde{v}U_y + \tilde{w}U_z = -\frac{1}{\rho}\frac{dP}{dx} - \frac{1}{\rho}\tilde{p}_x + \nu(U_{yy} + U_{zz} + \tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz}) - (\overline{u'u'})_x - (\overline{u'v'})_y - (\overline{u'w'})_z$$

$$U\tilde{v}_x = -\frac{1}{\rho}\tilde{p}_y + \nu(\tilde{v}_{xx} + \tilde{v}_{yy} + \tilde{v}_{zz}) - (\overline{v'u'})_x - (\overline{v'v'})_y - (\overline{v'w'})_z$$

$$U\tilde{w}_x = -\frac{1}{\rho}\tilde{p}_z + \nu(\tilde{w}_{xx} + \tilde{w}_{yy} + \tilde{w}_{zz}) - (\overline{w'u'})_x - (\overline{w'v'})_y - (\overline{w'w'})_z$$

Boussinesq hypothesis:

$$-\overline{u'_i u'_j} = -\frac{P_T}{\rho}\delta_{ij} + (\overline{\nu_t} + \tilde{\nu}_t)\left[\frac{\partial(U_i + \tilde{u}_i)}{\partial x_j} + \frac{\partial(U_j + \tilde{u}_j)}{\partial x_i}\right]$$

$\overline{\nu_t}$ "large"

$\tilde{\nu}_t$ "small"

Mean 1D motion	$\frac{P+P_T}{\rho U_0^2}$	$\frac{U}{U_0}$	$\frac{x}{L}$	$\frac{y,z}{h}$

Leading order streamwise momentum equation:

$$0 = -\frac{1}{\rho}\frac{dP^*}{dx} + \nu(U_{yy} + U_{zz}) + (\overline{\nu}_t U_y)_y + (\overline{\nu}_t U_z)_z$$

Further scales to be employed for the terms left at higher order are:

Coherent motion	$\frac{x}{h/\varepsilon}$	$\frac{y,z}{h}$	$\frac{\tilde{u}}{\tilde{U}}$	$\frac{\tilde{v},\tilde{w}}{\varepsilon\tilde{U}}$	$\frac{\tilde{p}}{\varepsilon^2\rho\tilde{U}U_0}$

ε not yet specified, $\tilde{U}$ unspecified velocity scale for the coherent motion

$\tilde{U} << U_0$, $\tilde{U} << u_\tau$ ← scale of turbulent velocity fluctuations

Equations for the coherent motion:

$$\underbrace{\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0}_{\mathcal{O}(\epsilon \tilde{U}/h)},$$

$$\underbrace{U\tilde{u}_x + \tilde{v}U_y + \tilde{w}U_z}_{\mathcal{O}(\epsilon \tilde{U} U_0/h)} = \underbrace{-\tilde{p}_x/\rho}_{\mathcal{O}(\epsilon^3 \tilde{U} U_0/h)} + \underbrace{\nu \tilde{u}_{xx}}_{\mathcal{O}(\epsilon^2 \nu \tilde{U}/h^2)} + \underbrace{\nu(\tilde{u}_{yy} + \tilde{u}_{zz})}_{\mathcal{O}(\nu \tilde{U}/h^2)} + \underbrace{(\overline{\nu}\tilde{u}_x)_x}_{\mathcal{O}(\epsilon^2 \tilde{U} u_\tau/h)} + \underbrace{(\overline{\nu}_t \tilde{u}_y + \tilde{\nu}_t U_y)_y + (\overline{\nu}_t \tilde{u}_z + \tilde{\nu}_t U_z)_z}_{\mathcal{O}(\tilde{U} u_\tau/h)},$$

$$\underbrace{U\tilde{v}_x}_{\mathcal{O}(\epsilon^2 \tilde{U} U_0/h)} = \underbrace{-\tilde{p}_y/\rho}_{\mathcal{O}(\epsilon^2 \tilde{U} U_0/h)} + \underbrace{\nu \tilde{v}_{xx}}_{\mathcal{O}(\epsilon^3 \nu \tilde{U}/h^2)} + \underbrace{\nu(\tilde{v}_{yy} + \tilde{v}_{zz})}_{\mathcal{O}(\epsilon \nu \tilde{U}/h^2)} + \underbrace{(\overline{\nu}_t \tilde{u}_y + \tilde{\nu}_t U_y)_x + (2\overline{\nu}_t \tilde{v}_y)_y + [\overline{\nu}_t(\tilde{v}_z + \tilde{w}_y)]_z}_{\mathcal{O}(\epsilon \tilde{U} u_\tau/h)},$$

$$\underbrace{U\tilde{w}_x}_{\mathcal{O}(\epsilon^2 \tilde{U} U_0/h)} = \underbrace{-\tilde{p}_z/\rho}_{\mathcal{O}(\epsilon^2 \tilde{U} U_0/h)} + \underbrace{\nu \tilde{w}_{xx}}_{\mathcal{O}(\epsilon^3 \nu \tilde{U}/h^2)} + \underbrace{\nu(\tilde{w}_{yy} + \tilde{w}_{zz})}_{\mathcal{O}(\epsilon \nu \tilde{U}/h^2)} + \underbrace{(\overline{\nu}_t \tilde{u}_z + \tilde{\nu}_t U_z)_x + [\overline{\nu}_t(\tilde{v}_z + \tilde{w}_y)]_y + (2\overline{\nu}_t \tilde{w}_z)_z}_{\mathcal{O}(\epsilon \tilde{U} u_\tau/h)}.$$

By imposing that the Reynolds stresses are of the same order of the convective terms (G. L. Mellor, *Int. J. Eng. Sci.* 1972) the small parameter ε, that expresses The ratio of cross-stream to streamwise length scales, is found to be:

$$\boxed{\epsilon = \frac{u_\tau}{U_0}}$$

Neglecting formally small terms:

$$\begin{cases} \tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0, \\ U\tilde{u}_x + \tilde{v}U_y + \tilde{w}U_z = \nu(\tilde{u}_{yy} + \tilde{u}_{zz}) + (\overline{\nu}_t\tilde{u}_y + \tilde{\nu}_t U_y)_y + (\overline{\nu}_t\tilde{u}_z + \tilde{\nu}_t U_z)_z, \\ U\tilde{v}_x = -\tilde{p}_y/\rho + \nu(\tilde{v}_{yy} + \tilde{v}_{zz}) + (\overline{\nu}_t\tilde{u}_y + \tilde{\nu}_t U_y)_x + (2\overline{\nu}_t\tilde{v}_y)_y + [\overline{\nu}_t(\tilde{v}_z + \tilde{w}_y)]_z, \\ U\tilde{w}_x = -\tilde{p}_z/\rho + \nu(\tilde{w}_{yy} + \tilde{w}_{zz}) + (\overline{\nu}_t\tilde{u}_z + \tilde{\nu}_t U_z)_x + [\overline{\nu}_t(\tilde{v}_z + \tilde{w}_y)]_y + (2\overline{\nu}_t\tilde{w}_z)_z. \end{cases}$$

and the equations are closed by finding a suitable representation of the turbulent viscosity, e.g.:

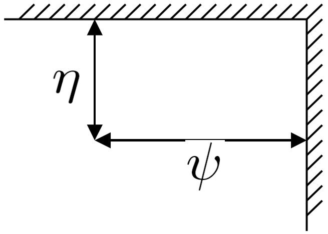
$$\nu_t = \overline{\nu}_t + \tilde{\nu}_t = c_2\,(U + \tilde{u})\,l_m$$

Mixing length:

$$l_m = 2\frac{\eta\psi}{\eta+\psi}$$

$$\begin{cases} c_2 = 22 \\ l_m \end{cases}$$

l_m harmonic mean between the distances from two orthogonal walls



Numerics

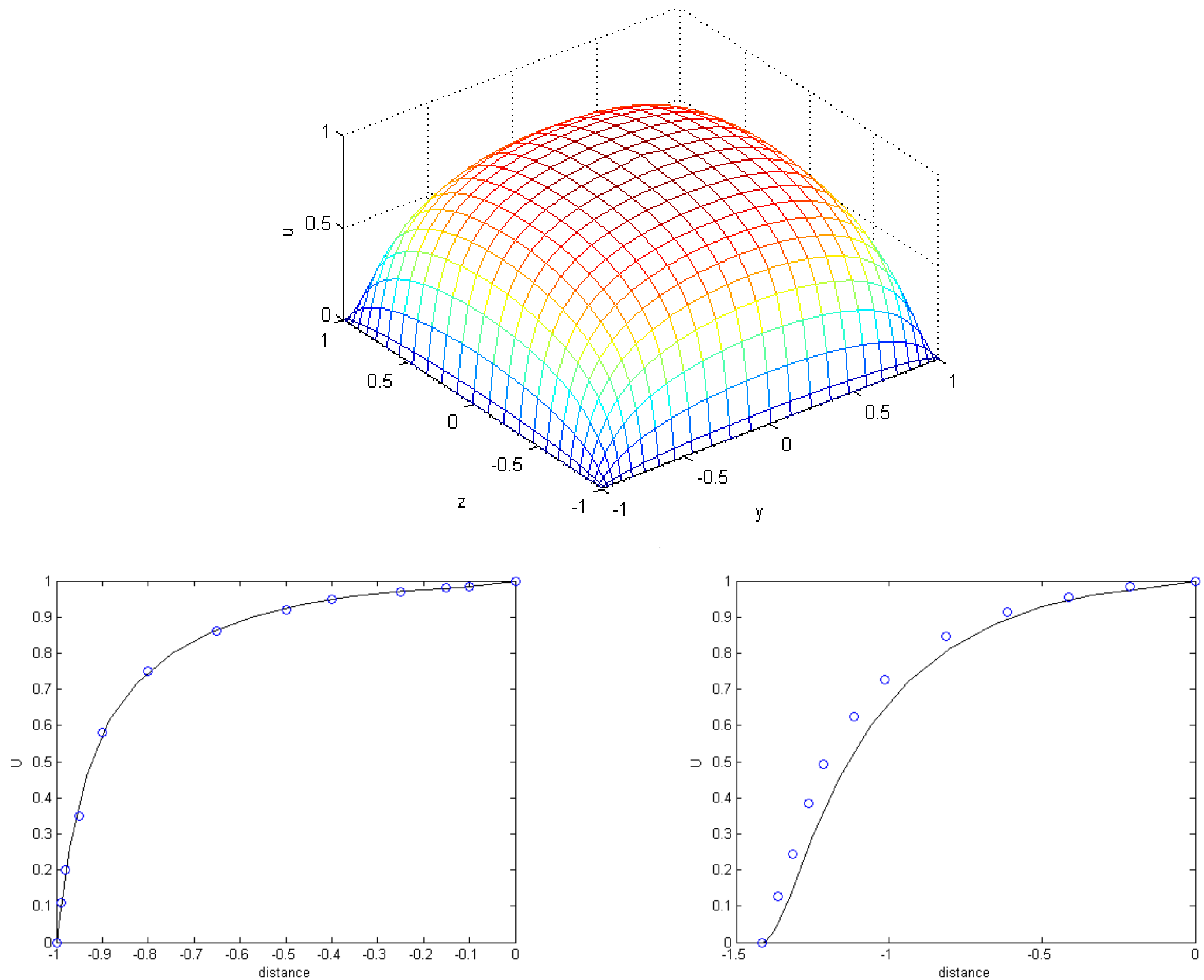
Collocation technique on Gauss-Lobatto grid points (y_i, z_j)

$$y_i = cos\pi(i-1)/(N-1) \text{ with } i = 1, ..., N$$

$$z_j = cos\pi(j-1)/(N-1) \text{ with } j = 1, ..., N$$

$$U(y,z) = \sum_{i=1}^{N}\sum_{j=1}^{N} U_{ij}\phi_i(y)\phi_j(z),$$

ϕ_i and ϕ_j Lagrangian interpolating polynomials



Computed mean 1D flow and comparisons with the DNS data of Gavrilakis (1992)

Numerics (the coherent motion)

The direct equation reads: $\boxed{\mathbf{Q}\mathbf{q}_x = \mathbf{R}\mathbf{q}}$ with $\mathbf{q} = [\mathbf{p}, \mathbf{u}, \mathbf{v}, \mathbf{w}]^T$

and upon *x*-discretization a recursive system is found:

$$\mathbf{q}_0 = [\mathbf{0}, \mathbf{0}, \mathbf{v}(0), \mathbf{w}(0)]^T, \quad \mathbf{q}_1 = \mathbf{G}_1 \mathbf{q}_0,$$

$$\mathbf{q}_{n+1} = \mathbf{G}_2(4\mathbf{q}_n - \mathbf{q}_{n-1}), \quad n = 1, ..., N_L - 1$$

$\mathbf{q}_n = \mathbf{q}(n\Delta x)$, $\mathbf{G}_1 = (\mathbf{Q} - \mathbf{R}\Delta x)^{-1}\mathbf{Q}$, $\mathbf{G}_2 = (3\mathbf{Q} - 2\mathbf{R}\Delta x)^{-1}\mathbf{Q}$, and $L = N_L \Delta x$

solved with *Singular Value Decomposition.*

Constraint: $$\frac{1}{2}\int_{-1}^{-1}\int_{-1}^{-1}[\tilde{v}(0,y,z)^2 + \tilde{w}(0,y,z)^2]dy\,dz = E_0 = 1$$

Questions:

What initial condition?
Is there some extremum principle?

In "classical" stability theory it is customary to focus on the transient growth of disturbances, and to search for the initial condition that maximizes a disturbance norm (such as a kinetic energy-like norm), that reads in the present case:

$$E(x) = \frac{1}{2} \int_{-1}^{-1} \int_{-1}^{-1} \tilde{u}(x,y,z)^2 + \boldsymbol{\varepsilon}^2 [\tilde{v}(x,y,z)^2 + \tilde{w}(x,y,z)^2] dy\, dz$$

In turbulent flows there are suggestions (Malkus 1956, Busse 1970, Plasting & Kerswell 2005) that statistical extreme states are reached, related to the degree of disorganization (entropy) of the motion.

It might thus be a sensible thing to maximize the rate of viscous dissipation …

$$2\overline{s'_{ij} s'_{ij}}$$

with $s'_{ij} = (\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i})/2$ the fluctuating rate of strain

In conditions of mechanical-energy equilibrium we can consider production instead of dissipation:

$$\mathcal{P}(x) = -\int_{-1}^{-1}\int_{-1}^{-1} \overline{u_i' u_j'} S_{ij} dy\, dz$$

Which at order ε reads:

$$\mathcal{P}_\epsilon = \int_{-1}^{-1}\int_{-1}^{-1} \tilde{\nu}_t \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} + \tilde{\nu}_t (\frac{\partial U_j}{\partial x_i})^2 + \overline{\nu_t}\frac{\partial U_j}{\partial x_i}\frac{\partial \tilde{u}_i}{\partial x_j} + \overline{\nu_t}\frac{\partial U_j}{\partial x_i}\frac{\partial \tilde{u}_j}{\partial x_i} dy\, dz.$$

However, an appropriate (quadratic) functional might be:

$$\boxed{\mathcal{I}(x) = \int_{-1}^{1}\int_{-1}^{1} (\tilde{u}_y^2 + \tilde{u}_z^2)\, dy\, dz}$$

provided we can consider the turbulent viscosity a property of small-scale incoherent processes and can thus rule it out …

The optimization and the discrete adjoint system

The cost function is written in a generic way as

$$\mathcal{J} = \alpha_1 \mathcal{I}(L) + \frac{\alpha_2}{L} \int_0^L \mathcal{I}(x)\, dx$$

to either target the functional at the final position ($\alpha_2 = 0$) or as an integral over x ($\alpha_1 = 0$).

In discrete form:

$$\mathcal{J}_n = \frac{1}{2}\alpha_1 \mathbf{q}_{N_L}^T \mathbf{A} \mathbf{q}_{N_L} + \frac{\alpha_2}{2L} \sum_{n=1}^{N_L} \mathbf{q}_n^T \mathbf{A} \mathbf{q}_n \Delta x$$

with the initial constraint: $\frac{1}{2}\mathbf{q}_0^T \mathbf{B} \mathbf{q}_0 = E_0$

The Lagrangian functional

The constrained optimization is transformed to an unconstrained one by introducing:

$$\mathcal{L}_n = \frac{1}{2}\alpha_1 \mathbf{q}_{N_L}^T \mathbf{A} \mathbf{q}_{N_L} + \mathbf{r}_0^T(\mathbf{q}_1 - \mathbf{G}_1\mathbf{q}_0) + \sum_{n=1}^{N_L-1} \{\mathbf{r}_n^T[\mathbf{q}_{n+1} - \mathbf{G}_2(4\mathbf{q}_n - \mathbf{q}_{n-1})] + \frac{\alpha_2}{2L}\mathbf{q}_n^T\mathbf{A}\mathbf{q}_n\Delta x\} +$$

$$+\frac{\alpha_2}{2L}\mathbf{q}_{N_L}^T\mathbf{A}\mathbf{q}_{N_L}\Delta x + \lambda_0(\frac{1}{2}\mathbf{q}_0^T\mathbf{B}\mathbf{q}_0 - E_0),$$

so that an optimum is obtained when stationarity is enforced with respect to all independent variables, leading to the following discrete adjoint system to be integrated backward in space:

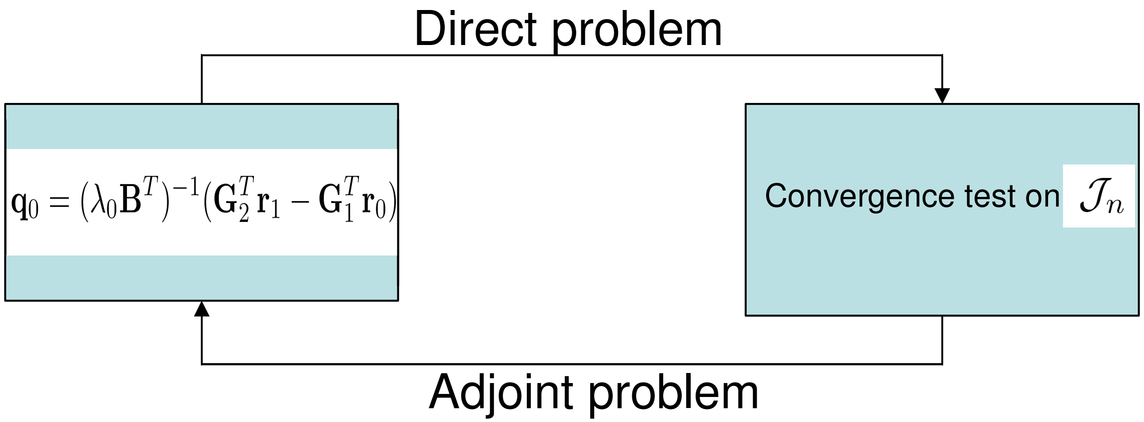
$$\mathbf{r}_{N_L} = [\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}]^T, \quad \mathbf{r}_{N_L-1} = -(\alpha_1 + \Delta x\tfrac{\alpha_2}{L})\mathbf{A}^T\mathbf{q}_N,$$

$$\mathbf{r}_{n-1} = \mathbf{G}_2^T(4\mathbf{r}_n - \mathbf{r}_{n+1}) - \Delta x\tfrac{\alpha_2}{L}\mathbf{A}^T\mathbf{q}_n, \quad n = N_L-1, ..., 1$$

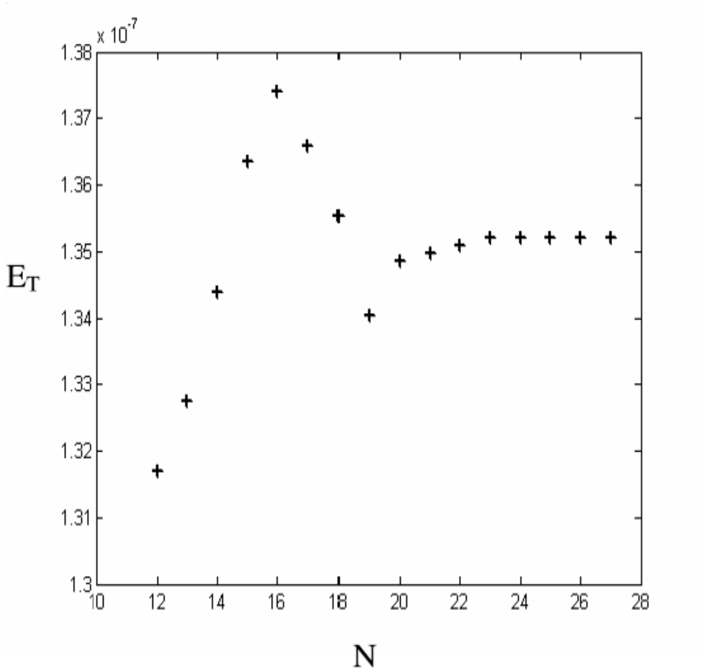
together with the optimality condition:

$$\mathbf{q}_0 = (\lambda_0 \mathbf{B}^T)^{-1}(\mathbf{G}_2^T \mathbf{r}_1 - \mathbf{G}_1^T \mathbf{r}_0)$$

that permits to iteratively update the inflow solution of the direct problem.



Accuracy study for the direct problem



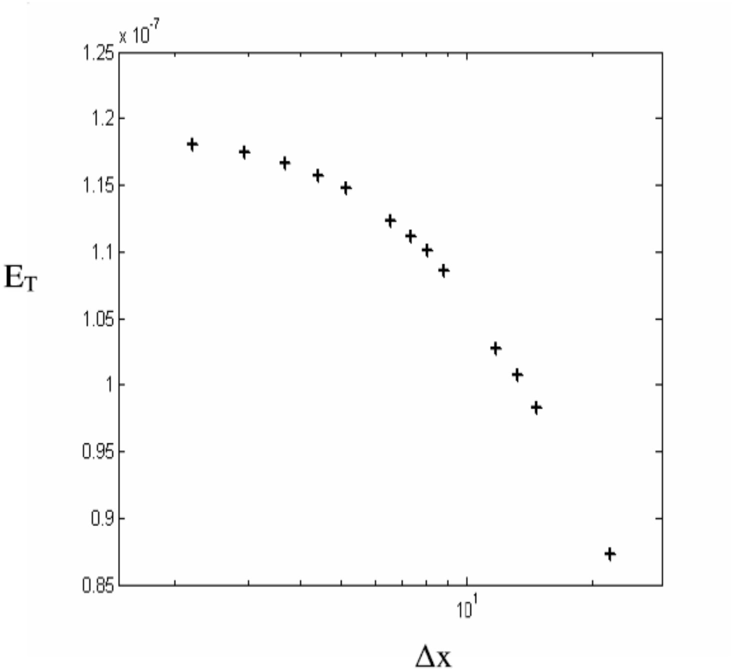
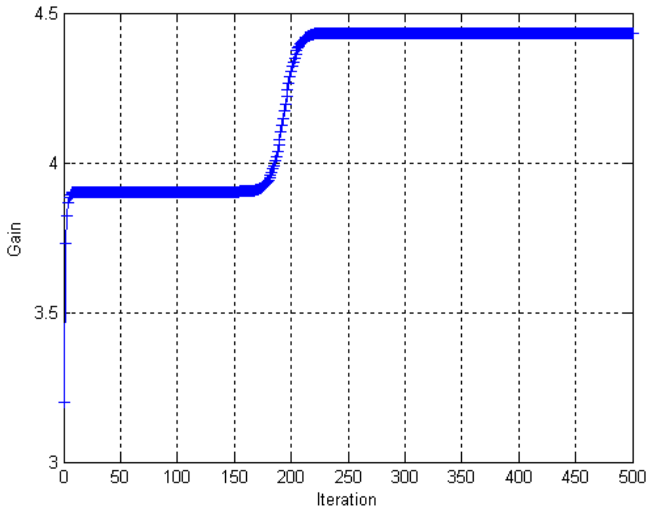
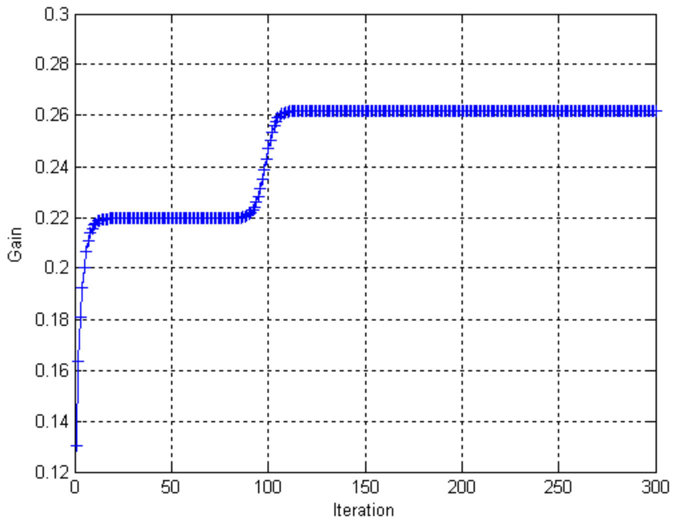


Figure 3. Grid resolution study. To obtain the figure on the left the streamwise step has been fixed at $\Delta x = 2.2$ with $L = 220$; for the figure on the right all calculations have been performed with $N = 23$ and $L = 396$.

The direct-adjoint iterative procedure can be very long, pointing to the weak selectivity of the final “optimal” states (Luchini 2000).

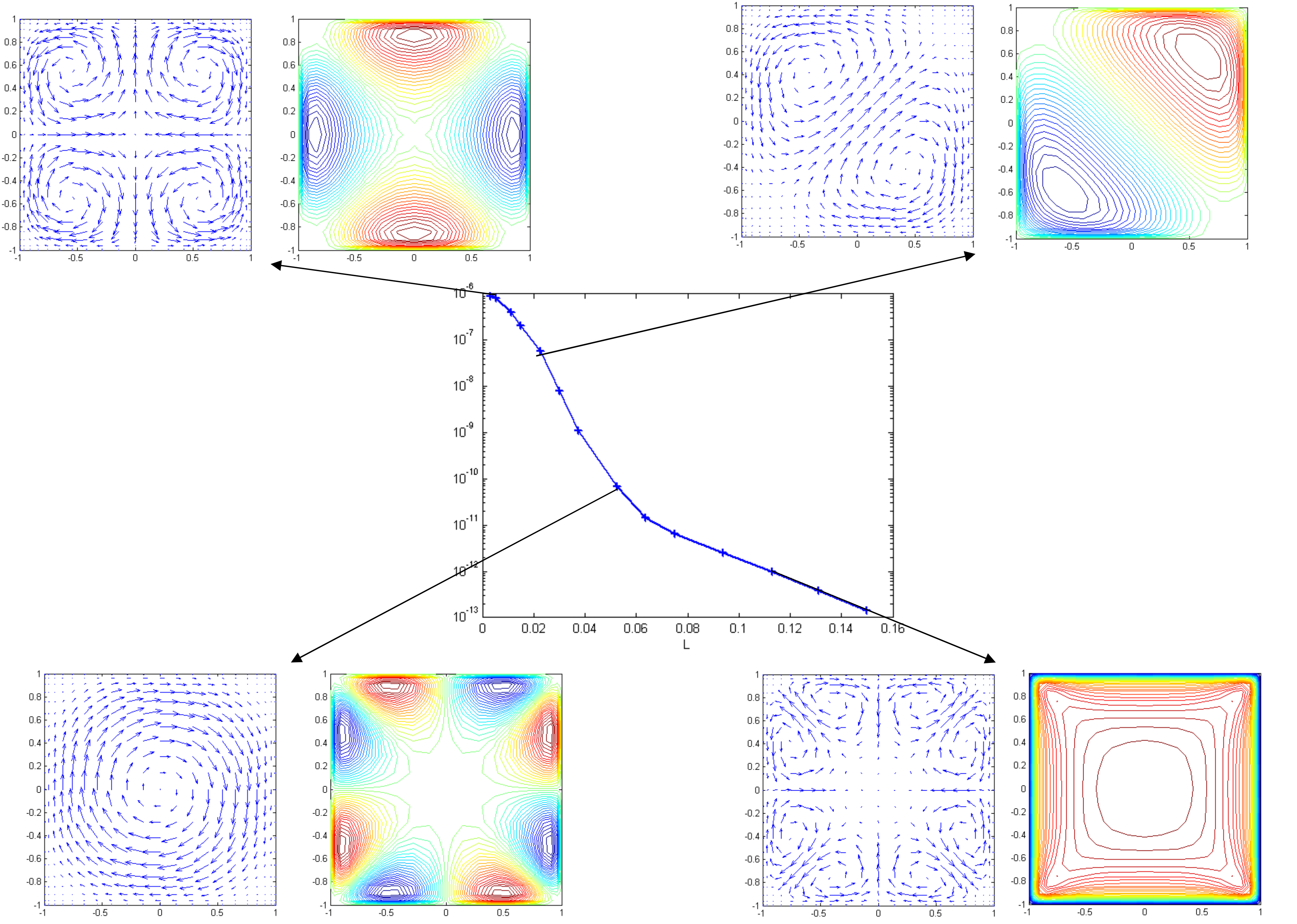




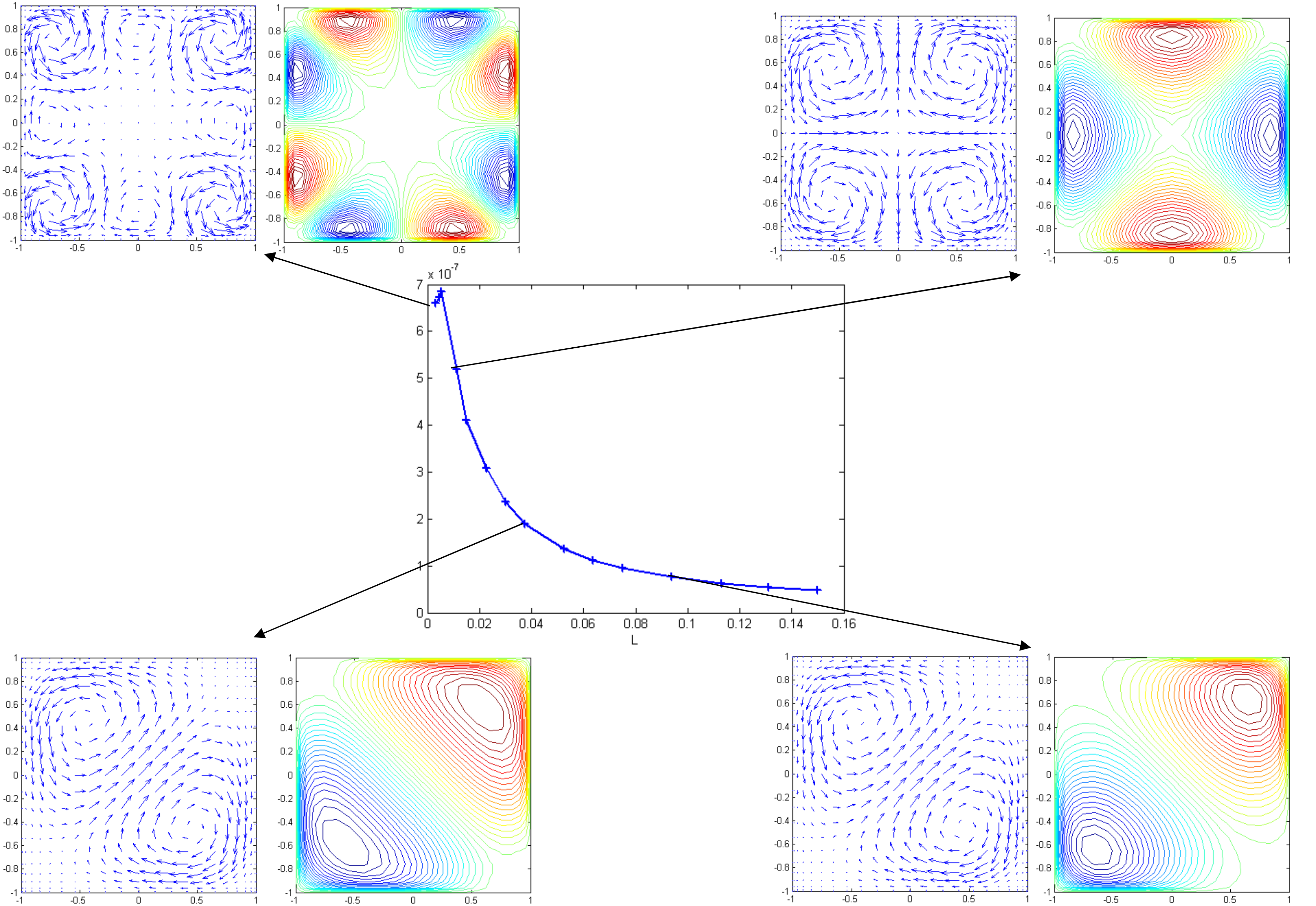
Examples of the presence of *plateaux* in the course of iterations

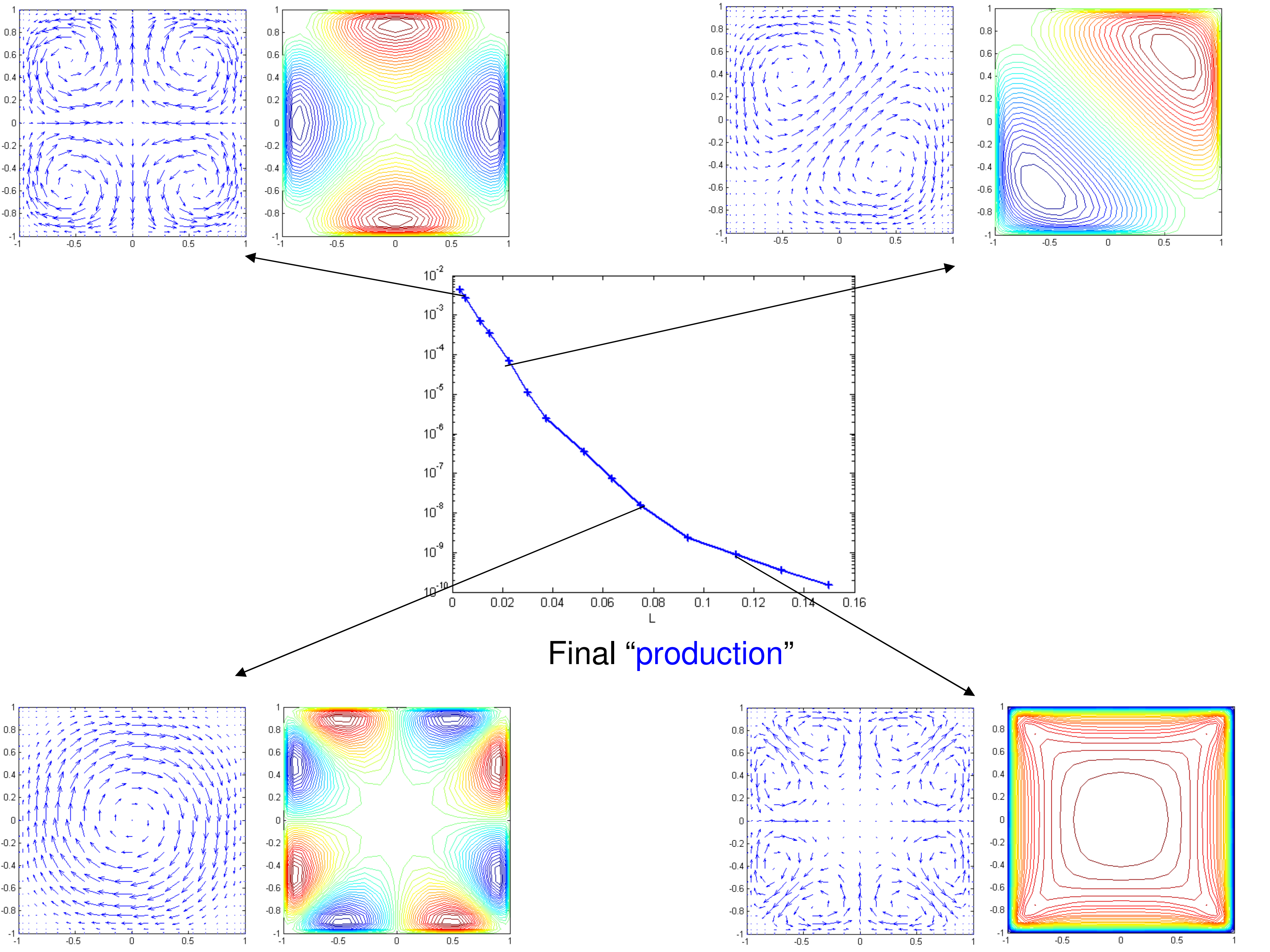
500 iterations are needed to reach convergence in short ducts (left), whereas 300 are sufficient in long ducts (right)

Functional: kinetic energy-like norm at the final position



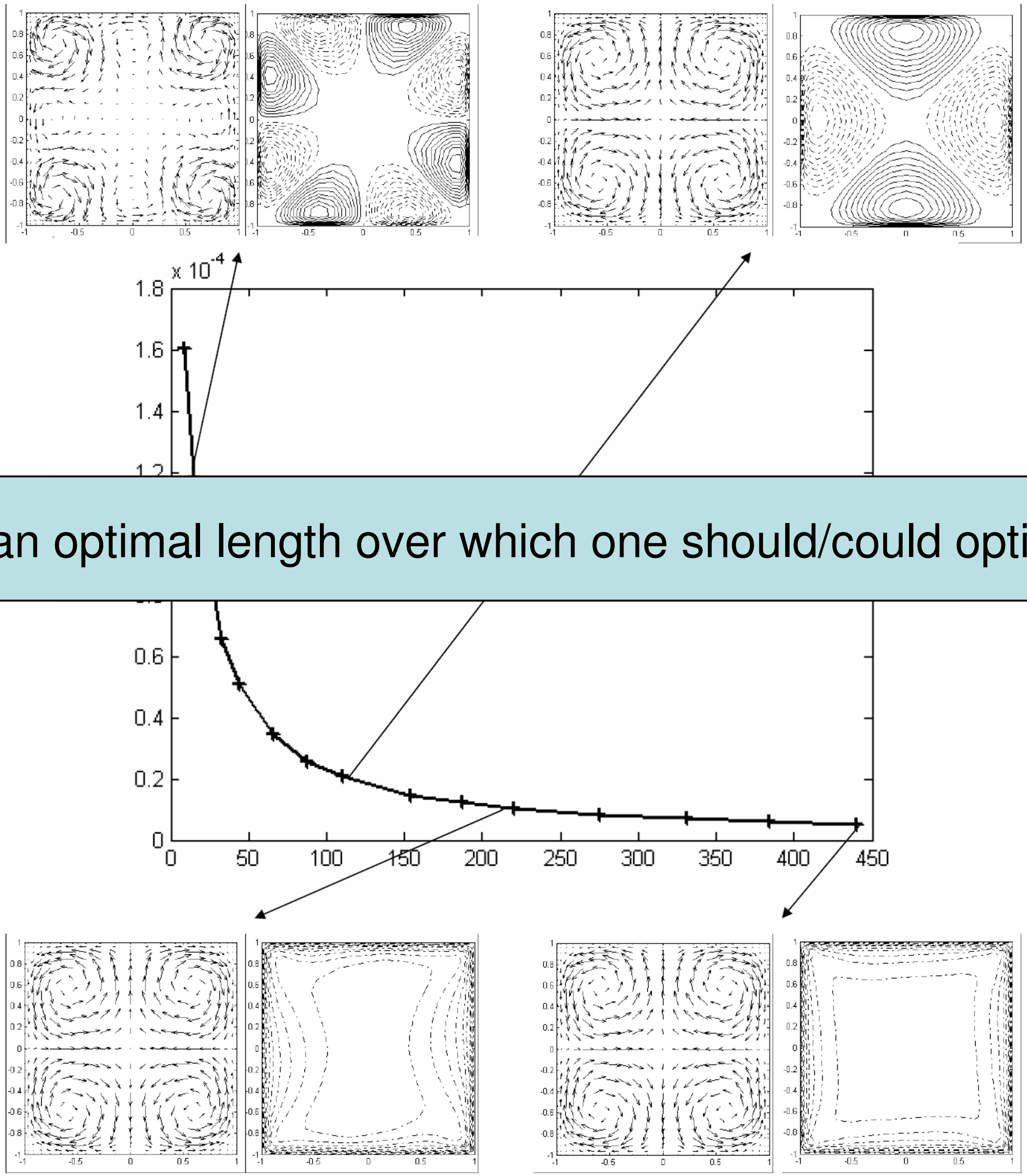
Functional: integral over *x* of the kinetic energy-like norm





Final “production”

Integral “production”



Is there an optimal length over which one should/could optimize?

Discussion

Streamwise vorticity equation (linearized in the coherent field):

$$U\tilde{\omega}_x = \nu(\tilde{\omega}_{yy} + \tilde{\omega}_{zz}) + \underbrace{U_z\tilde{v}_x - U_y\tilde{w}_x}_{P_1} + \underbrace{(\overline{v'v'} - \overline{w'w'})_{yz}}_{P_2}$$

$$+ \underbrace{(\partial_{zz} - \partial_{yy})\overline{v'w'}}_{P_3} + \underbrace{\partial_x[(\overline{u'v'})_z - (\overline{u'w'})_y]}_{P_4}$$

Nonlinear, fully developed flow: $\tilde{v}\tilde{\omega}_y + \tilde{w}\tilde{\omega}_z = \nu(\tilde{\omega}_{yy} + \tilde{\omega}_{zz}) + P_2 + P_3$

A classical Boussinesq closure with a constant eddy viscosity for P_2 and P_3 would decouple this equation from the streamwise momentum equation, and yield $\tilde{v} = \tilde{w} = 0,$ which is the reason why traditionally a nonlinear relation for the Reynolds stress tensor has been considered necessary.

However, here P_1 (*mean shear skewing*) and P_4 ($\tilde{\omega}$ - $\tilde{u}$ coupling) are also present, and can trigger secondary coherent vortices through spatially transient effects.

Outlook and conclusions

- Turbulent square duct: triple decomposition of the flow variables, very simple closure, coherent state of small amplitude
- Technique borrowed from optimal control, of common use in stability theory: direct-adjoint optimization to search for cross-stream (coherent) vortices capable of maximizing certain functionals
- One-dimensional turbulent-like mean flow $U(y,z)$ can sustain the growth of secondary motion of a cross-stream scale comparable to that found in experiments
- Nonlinearities (not considered here) are responsible for *maintaining* the coherent states
- Variety of structures present and weak selectivity of the "optimal" states; the functional is the largest in the limit of very small duct lengths, *suggestive* pictures are found for ducts sufficiently long
- No need for anisotropic turbulent viscosity when a streamwise inhomogeneous approach is taken (*i.e.* when vortices are considered to arise out of transients)

- No clear indications on the existence of extremum principles in turbulent flow

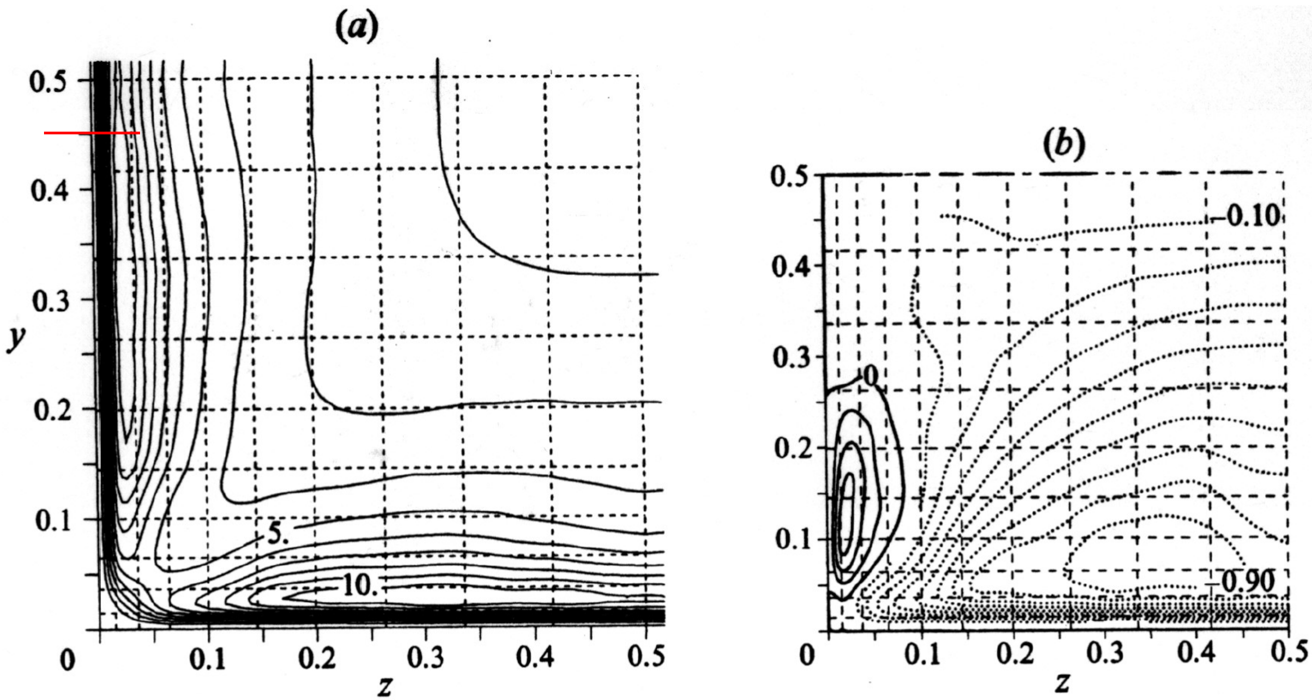


FIGURE 10. Primary Reynolds stress contours in the lower left quadrant of the transverse plane from Run B; every fifth grid-line is dashed: (a) $\overline{u'^2}$-contours, increment = 1.0; (b) $\overline{u'v'}$-contours, increment = 0.1.