MODELING FLOWS OVER NATURAL OR ENGINEERED SURFACES

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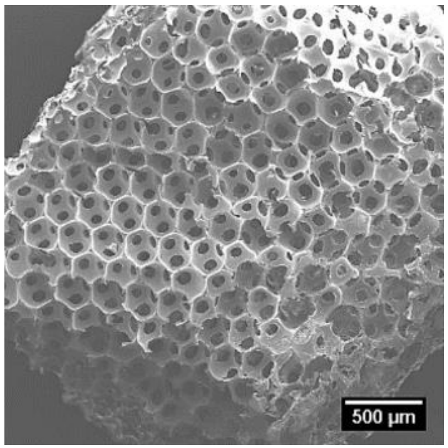
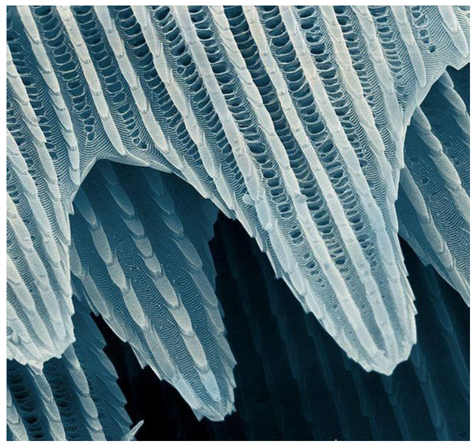
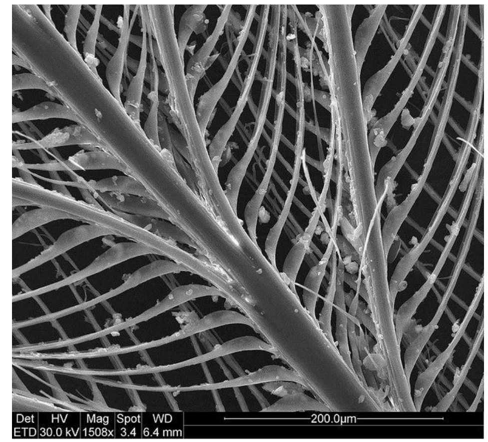
Regular, small-scale textures are the norm in nature and technology



Freeze-dried hydrogel foam

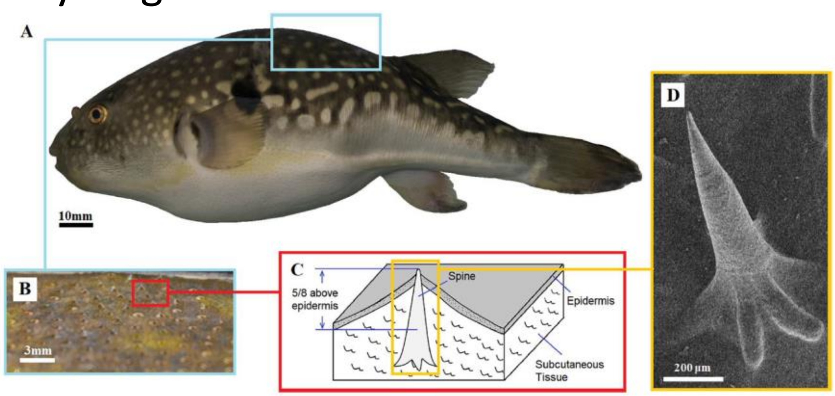


Butterfly wing scales

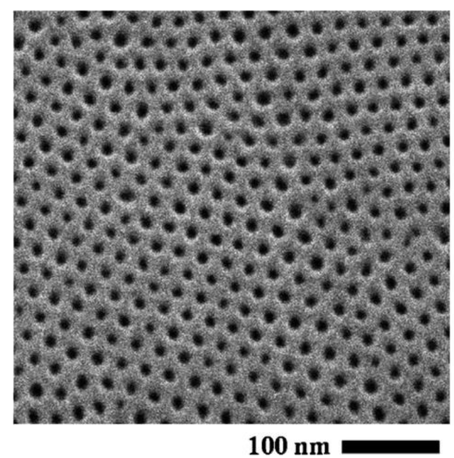


Bird feather

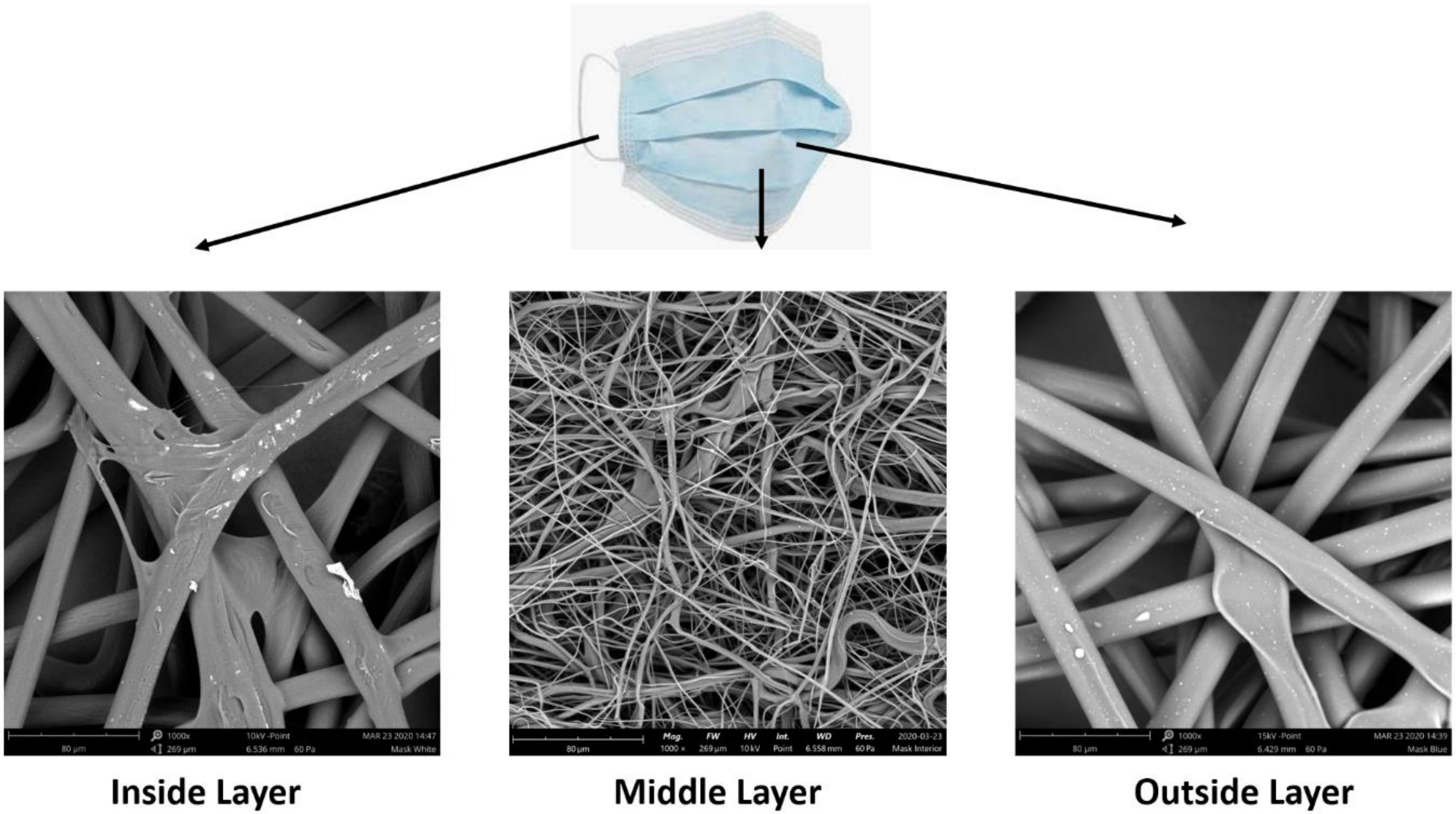
Hydraulic filter



Pufferfish spines (Tian *et al.*, ACS Omega, 2021)



Polysterene membrane for water purification



Middle layer of surgical mask is key to stopping virus particles
(D. Verma, *nanoscience.com,* 2021)

Common feature: repeated patterns, eventually with a hierarchy of scales

Question: can we model the presence of such regularly microstructured surfaces by an effective boundary condition?

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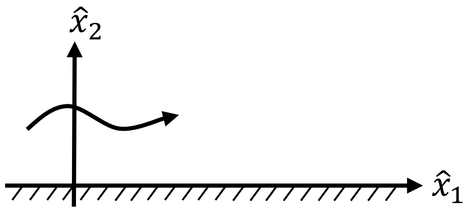
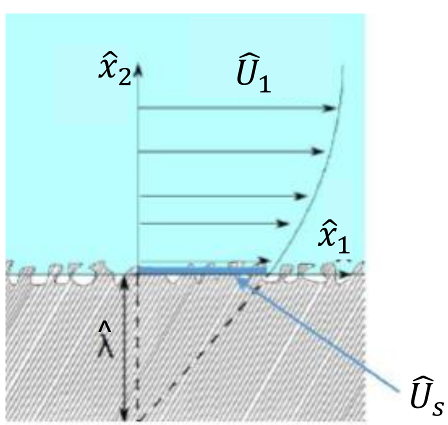
Three prototype problems:

- Rough wall (eventually superhydrophobic)
- Thin membrane
- Porous layer

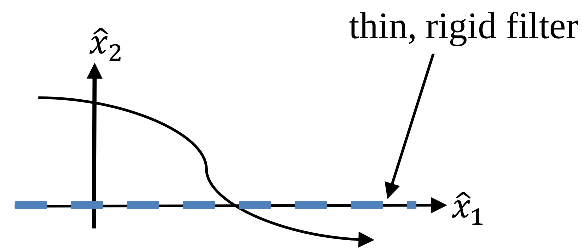


impermeable,
rough wall

Navier slip condition (1823)

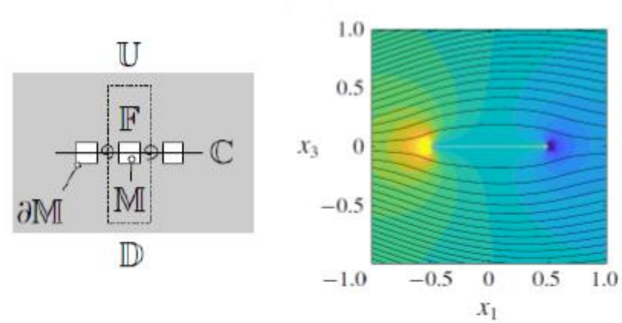


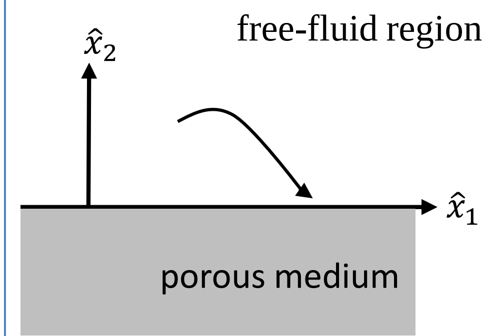
$$\hat{U}_s = \hat{\lambda} \frac{\partial \hat{U}_1}{\partial \hat{x}_2}\bigg|_{wall}$$



Darcy's law unjustified for **thin** membranes.

Zampogna & Gallaire (2020)





Beavers-Joseph-Saffman

Classical result: Beavers & Joseph, *JFM* 1967

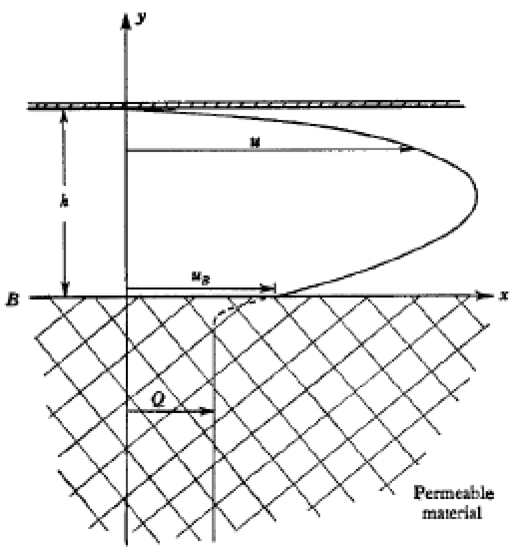


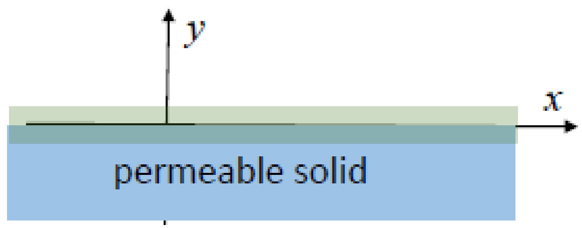
FIGURE 1. Velocity profile for the rectilinear flow in a horizontal channel formed by a permeable lower wall ($y = 0$) and an impermeable upper wall ($y = h$).

$$\left.\frac{du}{dy}\right|_{y=0_+} = \frac{\alpha}{\sqrt{k}}(u_B - Q) \qquad Q = -\frac{k}{\mu}\frac{dP}{dx}$$

with α a dimensionless function of the structural properties of the porous matrix

Block	k(in.^{2})	α	Average pore size (in.)
Foametal A	$1{\cdot}5 \times 10^{-5}$	0·78	0·016
Foametal B	$6{\cdot}1 \times 10^{-5}$	1·45	0·034
Foametal C	$12{\cdot}7 \times 10^{-5}$	4·0	0·045
Aloxite	$1{\cdot}0 \times 10^{-6}$	0·1	0·013
Aloxite	$2{\cdot}48 \times 10^{-6}$	0·1	0·027

Classical result: Saffman, *Stud. Appl. Math*. 1971



$$-\mu\nabla^2\mathbf{u} + \nabla p = 0, \qquad \text{for } y > 0$$

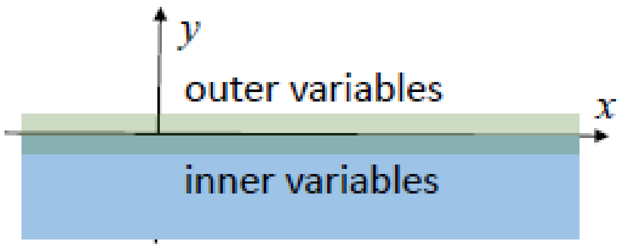
$$\mu\frac{\mathbf{U}^{(0)}}{k} + \nabla\bar{p} = 0, \qquad \text{for } y < 0$$

The velocity $\mathbf{U}$ in the *intermediate/boundary layer* must satisfy asymptotic matching conditions:

$$\lim_{y/k^{1/2}\to\infty} \mathbf{U} = \lim_{y\to 0+} \mathbf{u},$$

$$\lim_{y/k^{1/2}\to -\infty} \mathbf{U} = \lim_{y\to 0-} \mathbf{U}^{(0)},$$

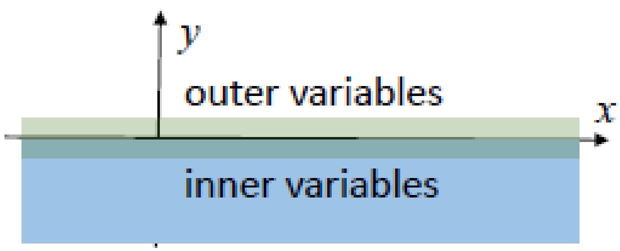
Classical result: Saffman, *Stud. Appl. Math.* 1971



Using linearity and expanding intermediate layer variables in terms of delta function derivatives, Saffman finds the asymptotic expressions of the velocity in the outer layer as $y \to 0^+$:

$$u = \frac{k^{1/2}}{\alpha}\frac{\partial u}{\partial y} + O(k) \qquad \text{on } y = 0.$$

Classical result: Saffman, *Stud. Appl. Math.* 1971



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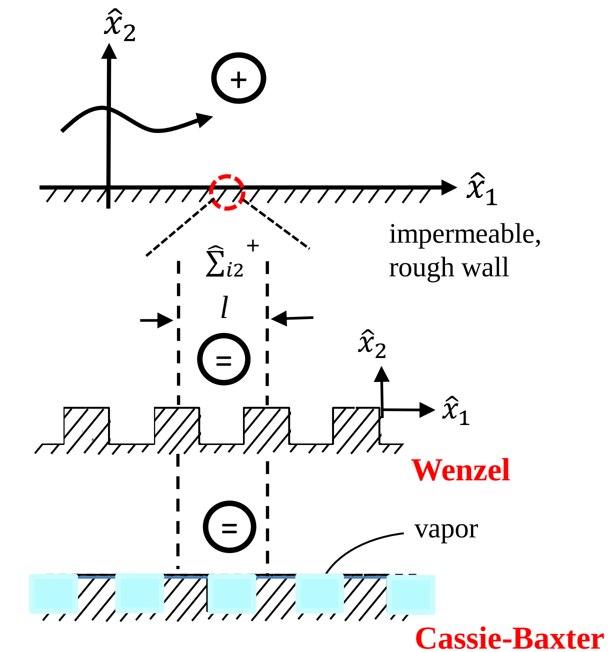
The result by Saffman permits to find the outer flow solution without iterating between inner and outer domains. Elaborating upon Saffman's results it is possible to find the order k correction:

$$\hat{u} = -\frac{B\,k}{\mu}\frac{\partial \hat{p}^-}{\partial \hat{x}} + \hat{\lambda}\frac{\partial \hat{u}}{\partial \hat{y}} \qquad \text{on } y = 0.$$

$\hat{\mathcal{K}}^{itf}$

Asymptotic homogenization approach

(Mei & Vernescu, *Homogenization Methods for Multiscale Mechanics*, 2010)



(+) Macroscopic, outer domain

(=) Near wall, inner domain

Scales:

(+) $\mathcal{U}, L, \rho\mathcal{U}^2$

(=) $\epsilon\mathcal{U}, \epsilon L = l, \mu\frac{\mathcal{U}}{L}$

$$\epsilon = l/L \ll 1$$

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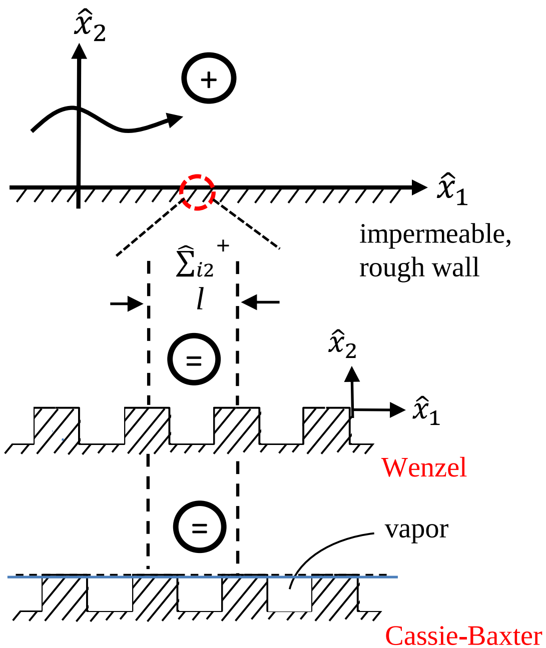

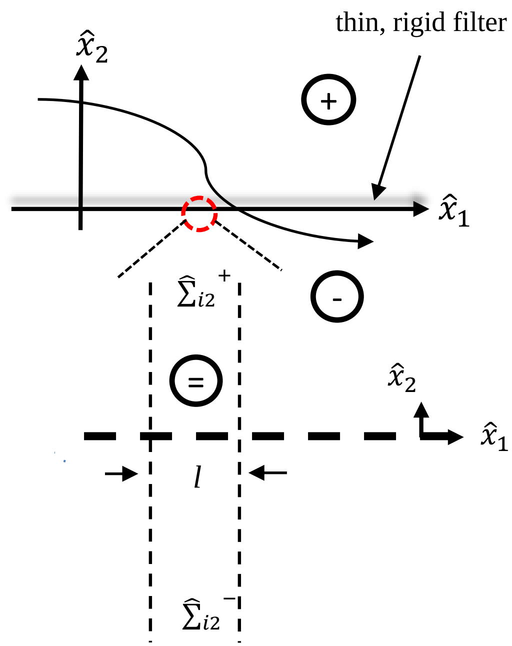

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(-) $\mathcal{U}, L, \rho\mathcal{U}^2$

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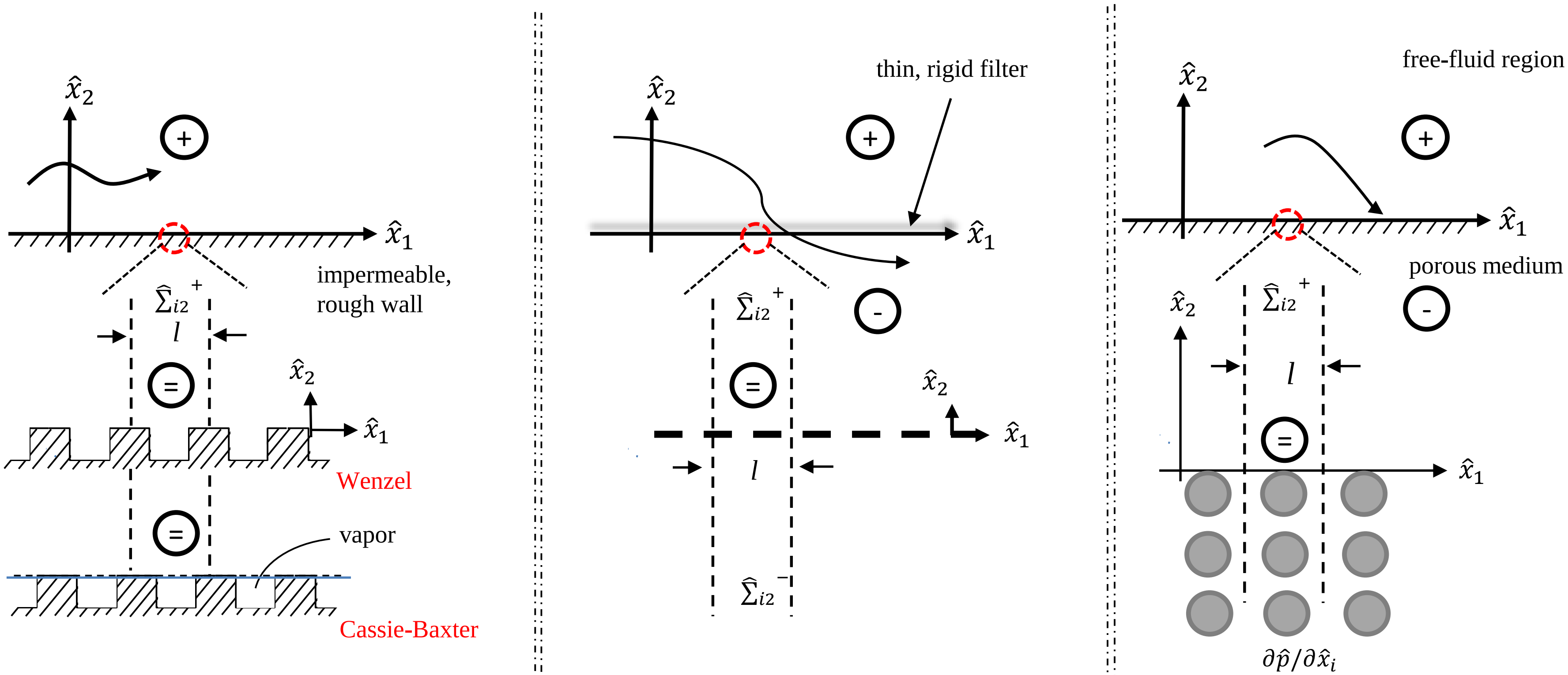
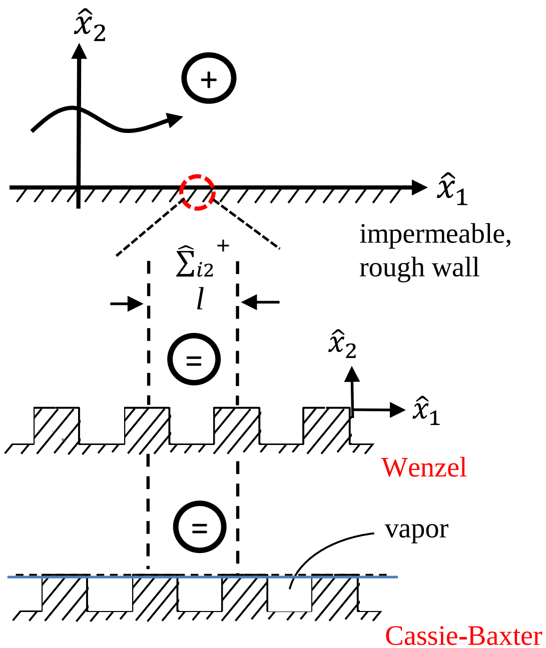


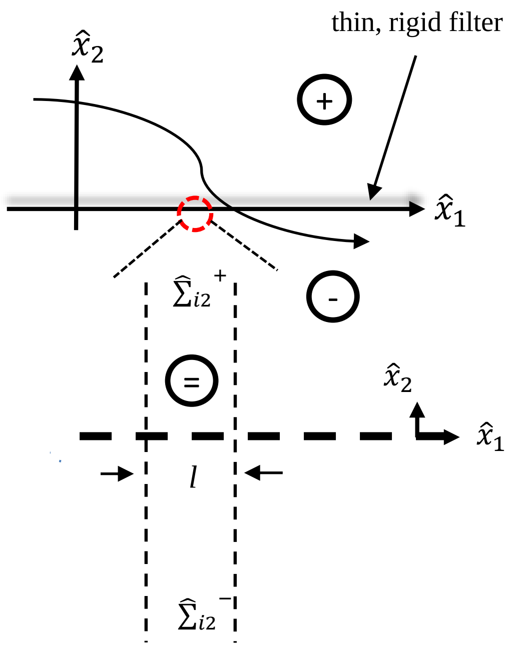
Scales:

	Rough wall	Thin, rigid filter	Porous medium
(+)	$\mathcal{U}, L, \rho\mathcal{U}^2$	$\mathcal{U}, L, \rho\mathcal{U}^2$	$\mathcal{U}, L, \rho\mathcal{U}^2$
(=)	$\epsilon\mathcal{U}, \epsilon L = l, \mu\frac{\mathcal{U}}{L}$	$\epsilon\mathcal{U}, \epsilon L = l, \mu\frac{\mathcal{U}}{L}$	$\epsilon\mathcal{U}, \epsilon L, \mu\frac{\mathcal{U}}{L}$
(-)		$\mathcal{U}, L, \rho\mathcal{U}^2$	$\epsilon^2\mathcal{U}, \epsilon L, \mu\frac{\mathcal{U}}{L}$



Scales: (+) $\mathcal{U}, L, \rho\mathcal{U}^2$

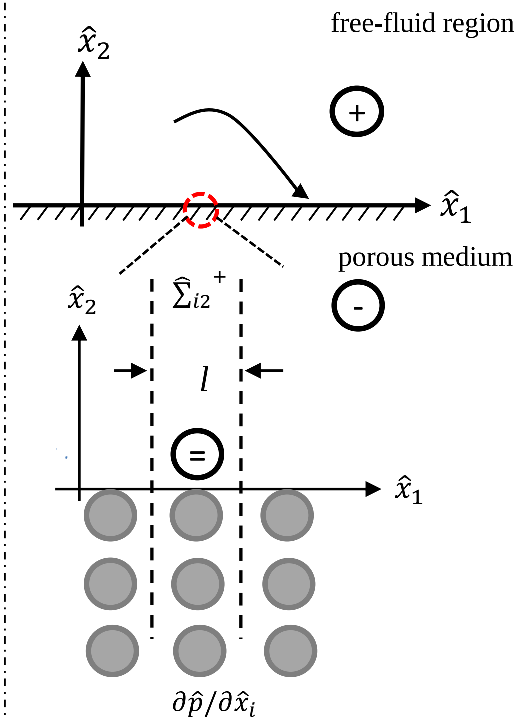
(=) $\epsilon\mathcal{U}, \epsilon L = l, \mu\frac{\mathcal{U}}{L}$



(+) $\mathcal{U}, L, \rho\mathcal{U}^2$

(=) $\epsilon\mathcal{U}, \epsilon L = l, \mu\frac{\mathcal{U}}{L}$

(-) $\mathcal{U}, L, \rho\mathcal{U}^2$



(+) $\mathcal{U}, L, \rho\mathcal{U}^2$

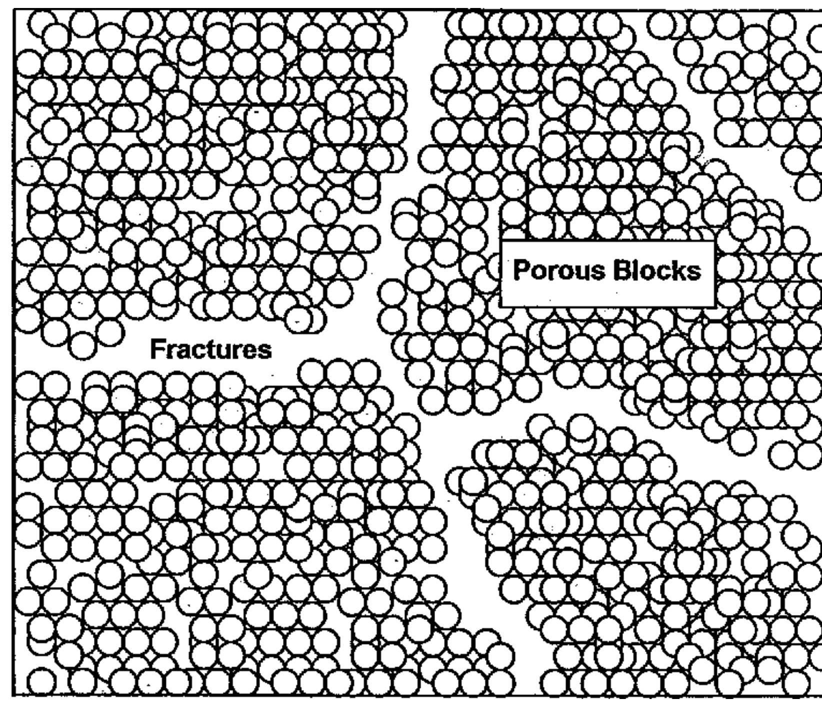
(=) $\epsilon\mathcal{U}, \epsilon L, \mu\frac{\mathcal{U}}{L}$

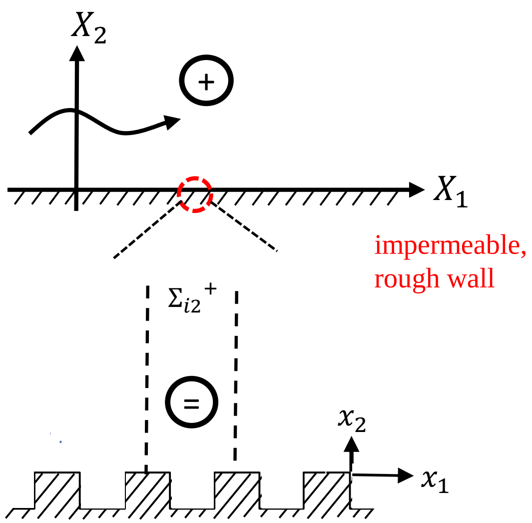
(-) $\epsilon^2\mathcal{U}, \epsilon L, \mu\frac{\mathcal{U}}{L}$

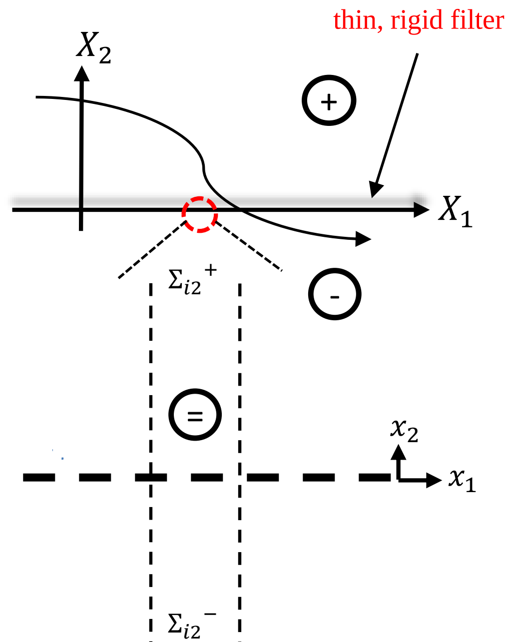
→ Dimensionless, normalized equations in each subdomain

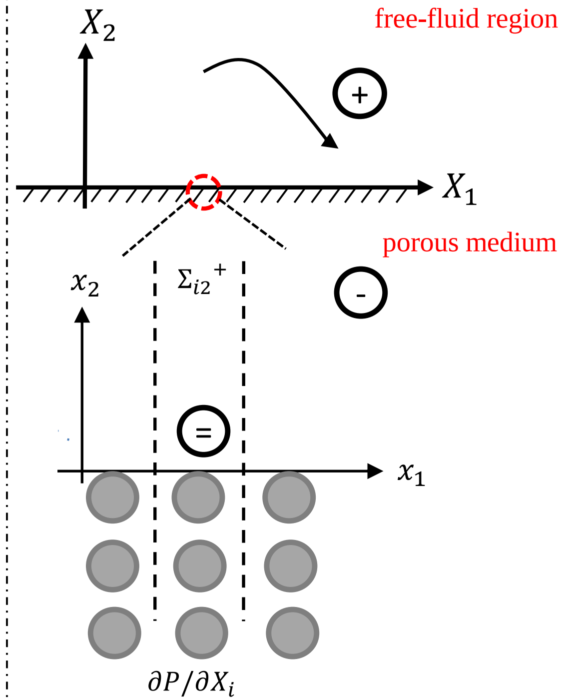
The approach followed can also be employed to study conditions near a **fracture** in a porous medium, the problem of the “interface” between **two porous media** of different porosity/microstructure or the condition at a **solid, impermeable boundary**.

All of these problems have been treated by Valdés-Parada & Lasseux (*Phys. Fluids* 2021) by the method of *volume averaging*, in the frame of the **one-domain approach.**









Dimensionless matching conditions:

$x_2 \to +\infty, \qquad X_2 \to 0^+$

$$\begin{cases} U_i{}^{+} = \epsilon U_i{}^{=} \\ \sigma_{i2}^{=} = -P^{=}\delta_{i2} + \dfrac{\partial U_i^{=}}{\partial x_2} + \dfrac{\partial U_2^{=}}{\partial x_i} = \\ \qquad = \Sigma_{i2}{}^{+} = -\,Re\, P^{+}\delta_{i2} + \dfrac{\partial U_i^{+}}{\partial X_2} + \dfrac{\partial U_2^{+}}{\partial X_i} \end{cases}$$

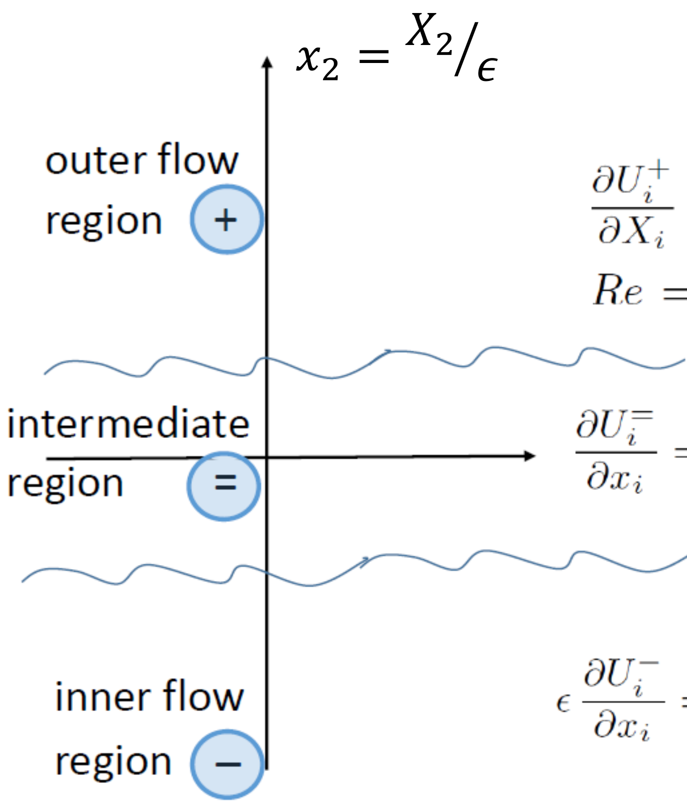
$x_2 \to +\infty, \qquad X_2 \to 0^+$

$x_2 \to -\infty, \qquad X_2 \to 0^-$

$x_2 \to +\infty, \qquad X_2 \to 0^+$

$x_2 \to -\infty \qquad$ Darcy

Equations for the free-fluid/porous case

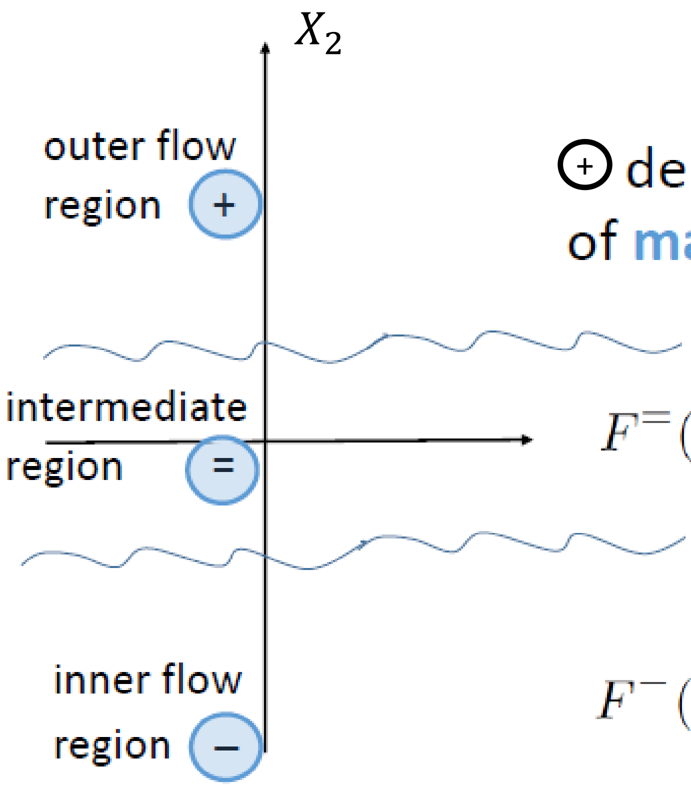


$$\frac{\partial U_i^+}{\partial X_i} = 0, \quad \frac{\partial U_i^+}{\partial t} + U_j^+ \frac{\partial U_i^+}{\partial X_j} = -\frac{\partial P^+}{\partial X_i} + \frac{1}{Re}\frac{\partial^2 U_i^+}{\partial X_j^2}$$

$$Re = \rho\,\mathcal{U}\,L/\mu$$

$$\frac{\partial U_i^=}{\partial x_i} = 0, \quad \epsilon^2 Re\left(\frac{\partial U_i^=}{\partial t} + U_j^= \frac{\partial U_i^=}{\partial x_j}\right) = -\frac{\partial P^=}{\partial x_i} + \frac{\partial^2 U_i^=}{\partial x_j^2}$$

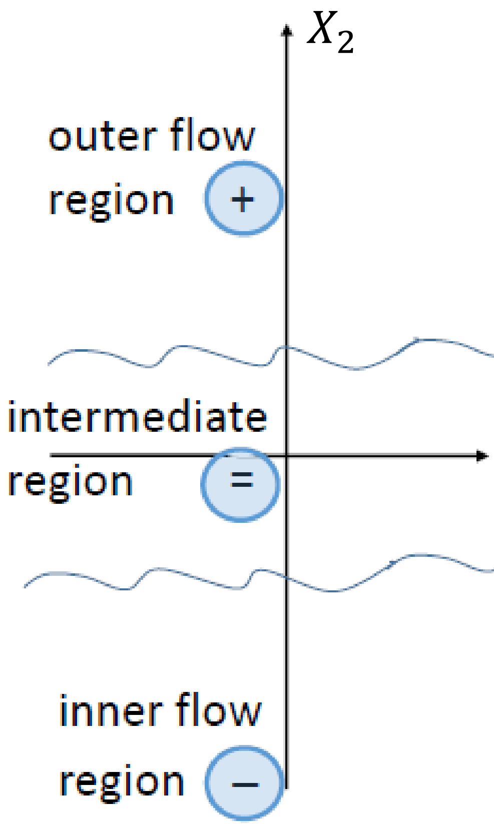
$$\epsilon\frac{\partial U_i^-}{\partial x_i} = 0, \quad \epsilon^4 Re\, U_j^- \frac{\partial U_i^-}{\partial x_j} = -\frac{\partial P^-}{\partial x_i} + \epsilon\frac{\partial^2 U_i^-}{\partial x_j^2}$$



⊕ dependent variables: only function of macroscopic independent variables

$$F^{=}(x_i, X_i, t) = F_0^{=} + \epsilon\, F_1^{=} + \epsilon^2\, F_2^{=} + \ldots$$

$$F^{-}(x_i, X_i, \cancel{t}) = F_0^{-} + \epsilon\, F_1^{-} + \epsilon^2\, F_2^{-} + \ldots$$



$$\frac{\partial}{\partial x_j} \rightarrow \frac{\partial}{\partial x_j} + \epsilon \frac{\partial}{\partial X_j}$$

$$\mathcal{O}(\epsilon^0): \quad \frac{\partial U_{i0}^{=}}{\partial x_i} = 0, \qquad -\frac{\partial P_0^{=}}{\partial x_i} + \frac{\partial^2 U_{i0}^{=}}{\partial x_j^2} = 0,$$

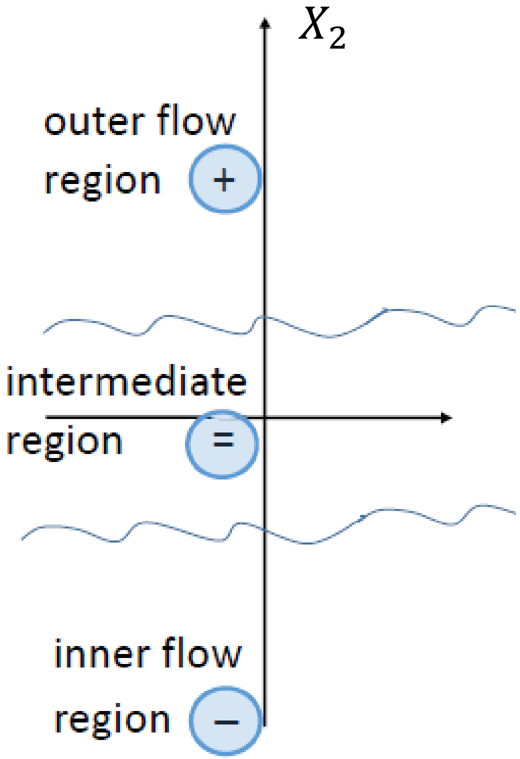
$$\mathcal{O}(\epsilon^1): \quad \frac{\partial U_{i1}^{=}}{\partial x_i} = -\frac{\partial U_{i0}^{=}}{\partial X_i}, \qquad -\frac{\partial P_1^{=}}{\partial x_i} + \frac{\partial^2 U_{i1}^{=}}{\partial x_j^2} = \frac{\partial P_0^{=}}{\partial X_i} - 2\frac{\partial^2 U_{i0}^{=}}{\partial x_j \partial X_j}.$$

$$\mathcal{O}(\epsilon^0): \quad \frac{\partial P_0^{-}}{\partial x_i} = 0,$$

$$\mathcal{O}(\epsilon^1): \quad \frac{\partial U_{i0}^{-}}{\partial x_i} = 0, \qquad -\frac{\partial P_1^{-}}{\partial x_i} + \frac{\partial^2 U_{i0}^{-}}{\partial x_j^2} = \frac{\partial P_0^{-}}{\partial X_i},$$

$$\mathcal{O}(\epsilon^2): \quad \frac{\partial U_{i1}^{-}}{\partial x_i} = -\frac{\partial U_{i0}^{-}}{\partial X_i}, \qquad -\frac{\partial P_2^{-}}{\partial x_i} + \frac{\partial^2 U_{i1}^{-}}{\partial x_j^2} = \frac{\partial P_1^{-}}{\partial X_i} - 2\frac{\partial^2 U_{i0}^{-}}{\partial x_j \partial X_j}.$$

COMPOSITE DESCRIPTION



$$u_i = u_i^{(0)} + \epsilon\, u_i^{(1)} + \mathcal{O}(\epsilon^2),$$
$$p = p^{(0)} + \epsilon\, p^{(1)} + \mathcal{O}(\epsilon^2),$$

$$u_i^{(0)} = \begin{cases} U_{i0}^{=} & y > 0, \\ \epsilon\, U_{i0}^{-} & y < 0, \end{cases}$$

$$p^{(0)} = \begin{cases} P_0^{=} & y > 0, \\ P_0^{-} + \epsilon\, P_1^{-} & y < 0, \end{cases}$$

$$u_i^{(1)} = \begin{cases} U_{i1}^{=} & y > 0, \\ \epsilon\, U_{i1}^{-} & y < 0, \end{cases}$$

$$p^{(1)} = \begin{cases} P_1^{=} & y > 0, \\ \epsilon\, P_2^{-} & y < 0. \end{cases}$$

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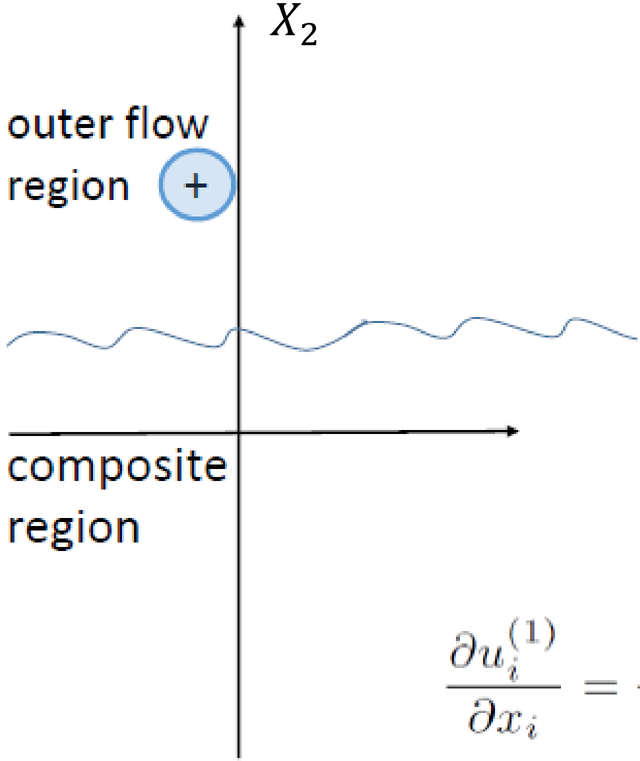

COMPOSITE DESCRIPTION

for = and - regions

(order 0 and order 1 in ϵ)

$$\frac{\partial u_i^{(0)}}{\partial x_i} = 0, \qquad -\frac{\partial p^{(0)}}{\partial x_i} + \frac{\partial^2 u_i^{(0)}}{\partial x_j^2} = 0,$$

$$\frac{\partial u_i^{(1)}}{\partial x_i} = -\frac{\partial u_i^{(0)}}{\partial X_i} \qquad -\frac{\partial p^{(1)}}{\partial x_i} + \frac{\partial^2 u_i^{(1)}}{\partial x_j^2} = \frac{\partial p^{(0)}}{\partial X_i} - 2\,\frac{\partial^2 u_i^{(0)}}{\partial x_j \partial X_j}.$$

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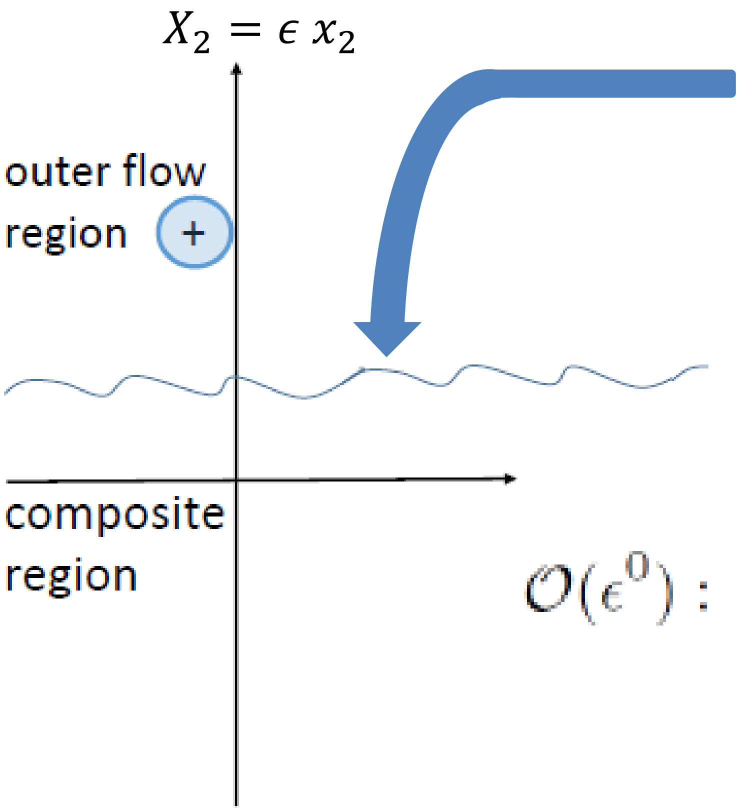


MATCHING CONDITIONS @ $x_2 \to +\infty$

$$U_i^{\;+} = \epsilon\left(u_i^{(0)} + \epsilon u_i^{(1)} + \;\ldots\right)$$

$$\mathcal{O}(\epsilon^0): \quad -p^{(0)}\delta_{i2} + \frac{\partial u_i^{(0)}}{\partial x_2} + \frac{\partial u_2^{(0)}}{\partial x_i} = \Sigma_{i2}^{+}$$

$$\mathcal{O}(\epsilon^1): \quad -p^{(1)}\delta_{i2} + \frac{\partial u_i^{(1)}}{\partial x_2} + \frac{\partial u_2^{(1)}}{\partial x_i} = -\left(\frac{\partial u_i^{(0)}}{\partial X_2} + \frac{\partial u_2^{(0)}}{\partial X_i}\right)$$

Linearity permits to express the **order 0** solution as

$$\begin{cases} u_i^{(0)} = u_{ij}^{\dagger}\, \Sigma_{j2}{}^{+} \\ p^{(0)} = p_j^{\dagger}\, \Sigma_{j2}{}^{+} + \text{K} \end{cases}$$

with $u_{ij}^{\dagger}$ and $p_j^{\dagger}$ function of only **microscopic** independent variables.

A Stokes system for the 'dagger' variables ensues, to be solved in a **periodic** (along x_1 and x_3) **elementary cell** subject to

$$-p_j^{\dagger}\delta_{i2} + \frac{\partial u_{ij}^{\dagger}}{\partial x_2} + \frac{\partial u_{2j}^{\dagger}}{\partial x_i} = \delta_{ij} \quad \text{when} \quad x_2 \to +\infty$$

plus 1-periodicity when $x_2 \to -\infty$.

The **order 1** condition at $x_2 \to \infty$ becomes

$$-p^{(1)}\delta_{i2} + \frac{\partial u_i^{(1)}}{\partial x_2} + \frac{\partial u_2^{(1)}}{\partial x_i} = -\left(u_{ij}^{\dagger} \frac{\partial \Sigma_{j2}^{+}}{\partial X_k} \delta_{k2} + u_{2j}^{\dagger} \frac{\partial \Sigma_{j2}^{+}}{\partial X_k} \delta_{ik} \right)$$

so that, on account of linearity, the couple $\left(u_i^{(1)}, p^{(1)}\right)$ has the form:

$$\begin{cases} u_i^{(1)} = u_{ijk}^{*} \dfrac{\partial \Sigma_{j2}^{+}}{\partial X_k} \\ p^{(1)} = p_{jk}^{*} \dfrac{\partial \Sigma_{j2}^{+}}{\partial X_k} + \cancel{K} \end{cases}$$

The ‘star’ variables satisfy the forced Stokes system

$$\begin{cases} \dfrac{\partial u^*_{ijk}}{\partial x_i} = -u^\dagger_{kj} \\[2ex] -\dfrac{\partial p^*_{jk}}{\partial x_i} + \dfrac{\partial^2 u^*_{ijk}}{\partial x_l^2} = -p^\dagger_j \delta_{ik} - 2\dfrac{\partial u^\dagger_{ij}}{\partial x_k} \end{cases}$$

with the condition at $x_2 \to \infty$:

$$-p^*_{jk}\delta_{i2} + \frac{\partial u^*_{ijk}}{\partial x_2} + \frac{\partial u^*_{2jk}}{\partial x_i} = -u^\dagger_{ij}\delta_{k2} - u^\dagger_{2j}\delta_{ik}$$

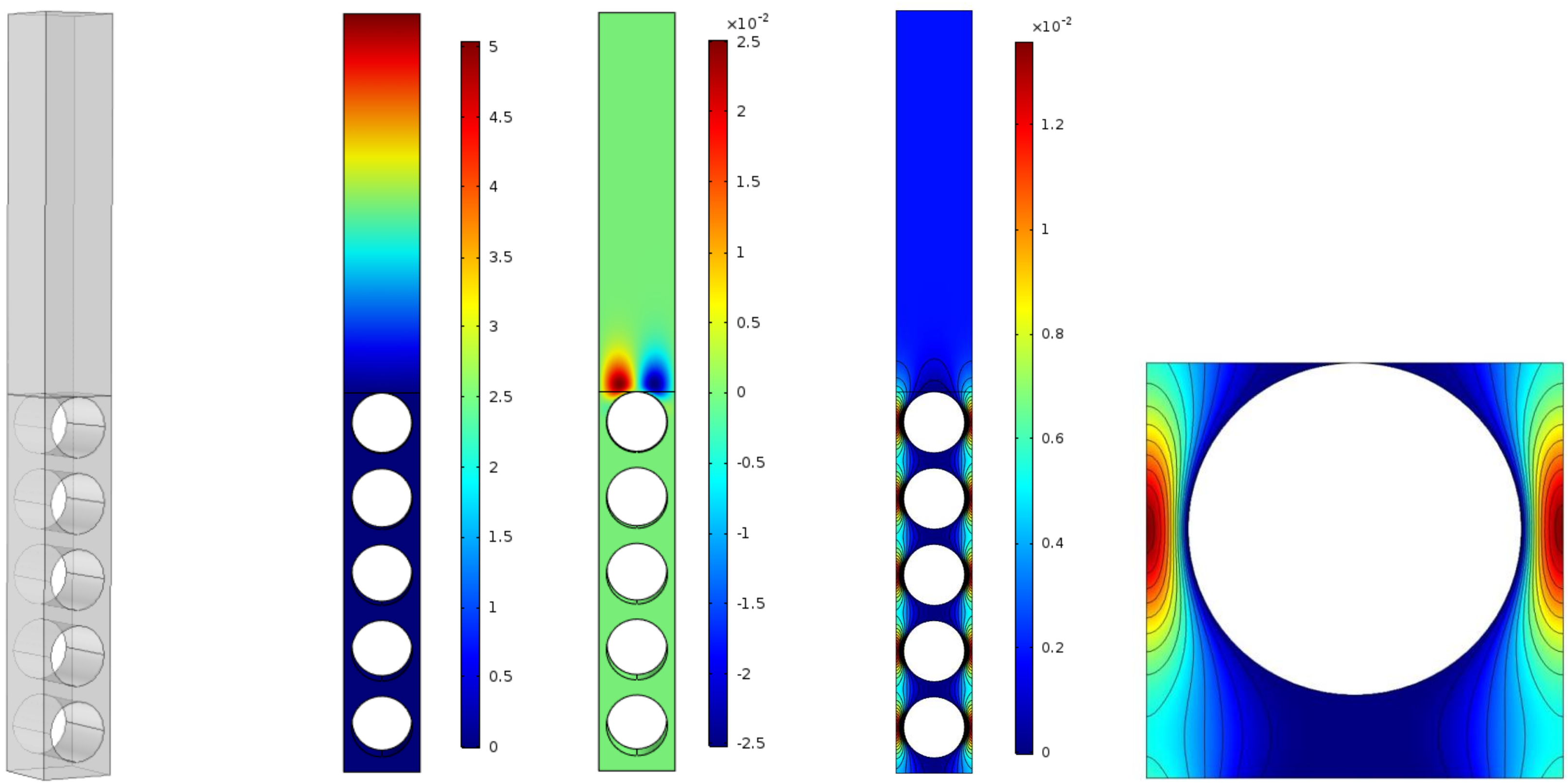
Once the 'dagger' and the 'star' systems are solved for, the macroscopic solution at $X_2 = \epsilon\, y_\infty$ is

$$U_i{}^+ = \epsilon \left(u_{ij}^\dagger\, \Sigma_{j2}{}^+ + \epsilon\, u_{ijk}^* \frac{\partial \Sigma_{j2}{}^+}{\partial X_k} \right) + \mathcal{O}(\epsilon^3)$$

The variable $u_{ij}^\dagger$ evaluated at $x_2 = y_\infty$ is a **Navier slip tensor**; the variable u_{ijk}^* is a rank-3 **permeability tensor**, and it includes an *interface permeability* effect.

Solutions can be pursued also at order ϵ^3 and higher (Bottaro & Naqvi, *Meccanica* 2020)

Porous medium: cylinders aligned along x_3, porosity $\theta = 0.5$



$u_{11}^{\dagger}$ $u_{21}^{\dagger}$ u_{222}^{*}

Sufficiently far from the axis' origin in $x_2 = 0$, results are independent of x_1 and x_3, and the non-trivial solutions of interest are:

$$u_{11}^{\dagger} = \lambda_x + x_2 \qquad u_{33}^{\dagger} = \lambda_z + x_2$$

$$u_{121}^{*} = -u_{211}^{*} = K_{xy}^{itf} + \lambda_x x_2 + \frac{1}{2} x_2^2$$

$$u_{323}^{*} = -u_{233}^{*} = K_{zy}^{itf} + \lambda_z x_2 + \frac{1}{2} x_2^2$$

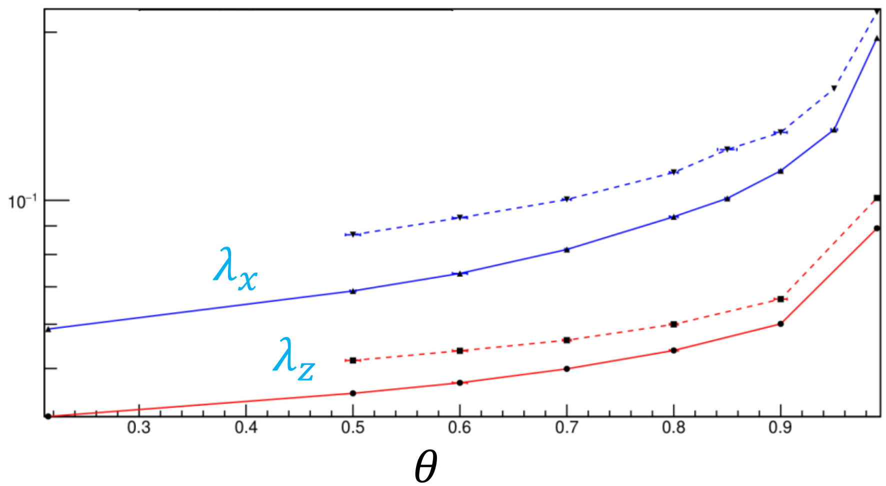
$$u_{222}^{*} = K_{yy}$$

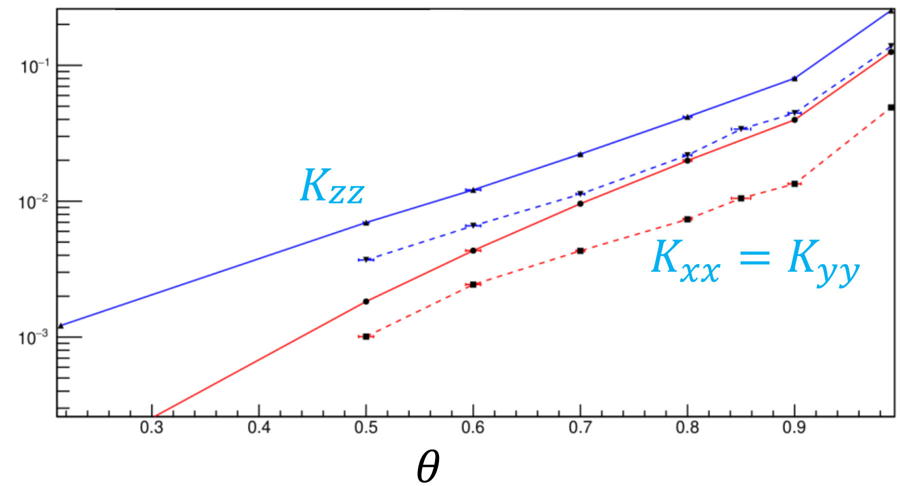
Transferring the interface condition from $X_2 = \epsilon y_\infty$ to $X_2 = 0$:

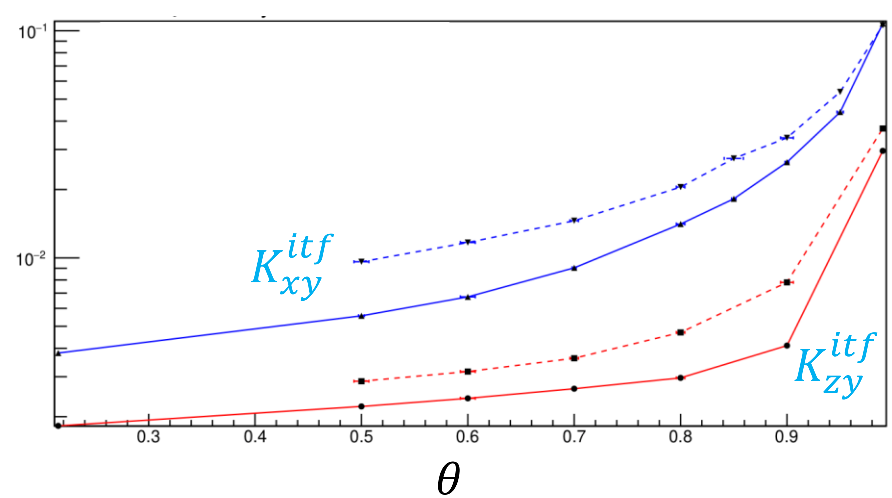
$$U_1^+\Big|_{X_2=0} = \epsilon\lambda_x \, \Sigma_{12}^+\Big|_{X_2=0} + \epsilon^2 K_{xy}^{itf} \left.\frac{\partial \Sigma_{22}^+}{\partial X_1}\right|_{X_2=0} + \mathcal{O}(\epsilon^3)$$

$$U_2^+\Big|_{X_2=0} = -\epsilon^2 K_{xy}^{itf} \left.\frac{\partial \Sigma_{12}^+}{\partial X_1}\right|_{X_2=0} - \epsilon^2 K_{zy}^{itf} \left.\frac{\partial \Sigma_{32}^+}{\partial X_3}\right|_{X_2=0} + \epsilon^2 K_{yy} \left.\frac{\partial \Sigma_{22}^+}{\partial X_2}\right|_{X_2=0} + \mathcal{O}(\epsilon^3)$$

$$U_3^+\Big|_{X_2=0} = \epsilon\lambda_z \, \Sigma_{32}^+\Big|_{X_2=0} + \epsilon^2 K_{zy}^{itf} \left.\frac{\partial \Sigma_{22}^+}{\partial X_3}\right|_{X_2=0} + \mathcal{O}(\epsilon^3)$$







Two different virtual origins, for longitudinal and tranverse flow

In **dimensional form** the interface conditions reduce to

$$\hat{u}|_{0^+} \approx \hat{\lambda}_x \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)\Big|_{0^+} + \frac{\hat{\mathcal{K}}_{xy}^{itf}}{\mu} \frac{\partial}{\partial \hat{x}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+}$$

$$\hat{v}|_{0^+} \approx \frac{\hat{\mathcal{K}}_{yy}}{\mu} \frac{\partial}{\partial \hat{y}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+} - \hat{\mathcal{K}}_{xy}^{itf} \frac{\partial}{\partial \hat{x}} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)\Big|_{0^+} - \hat{\mathcal{K}}_{zy}^{itf} \frac{\partial}{\partial \hat{z}} \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right)\Big|_{0^+}$$

$$\hat{w}|_{0^+} \approx \hat{\lambda}_z \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right)\Big|_{0^+} + \frac{\hat{\mathcal{K}}_{zy}^{itf}}{\mu} \frac{\partial}{\partial \hat{z}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+}$$

NO EMPIRICAL COEFFICIENTS!

In **dimensional form** the interface conditions reduce to

$$\hat{u}|_{0^+} \approx \hat{\lambda}_x \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)\Big|_{0^+} + \frac{\hat{\mathcal{K}}^{itf}_{xy}}{\mu} \frac{\partial}{\partial \hat{x}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+}$$

$$\hat{v}|_{0^+} \approx \frac{\hat{\mathcal{K}}_{yy}}{\mu} \frac{\partial}{\partial \hat{y}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+} - \hat{\mathcal{K}}^{itf}_{xy} \frac{\partial}{\partial \hat{x}} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)\Big|_{0^+} - \hat{\mathcal{K}}^{itf}_{zy} \frac{\partial}{\partial \hat{z}} \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right)\Big|_{0^+}$$

$$\hat{w}|_{0^+} \approx \hat{\lambda}_z \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right)\Big|_{0^+} + \frac{\hat{\mathcal{K}}^{itf}_{zy}}{\mu} \frac{\partial}{\partial \hat{z}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+}$$

These can be further modified using the balance of normal forces at the interface:

$$\hat{p}\Big|_{0^-} \approx \hat{p}\Big|_{0^+} - 2\mu \frac{\partial \hat{v}}{\partial \hat{y}}\Big|_{0^+}$$

… yielding an extended set of **Saffman's conditions**:

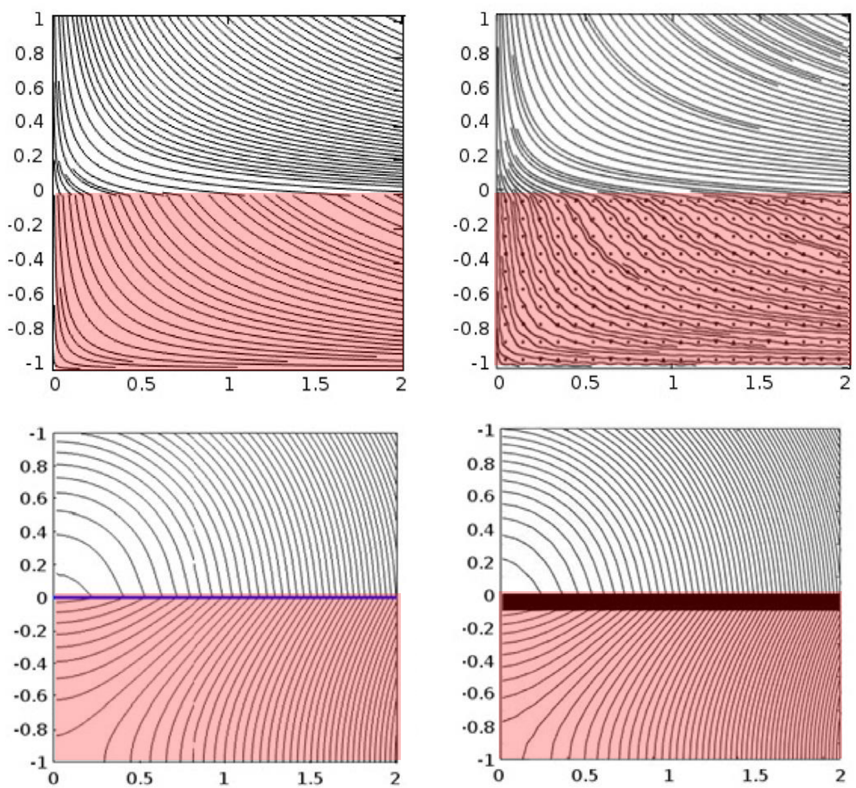
$$\hat{u}|_{0+} \approx \hat{\lambda}_x \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) \Big|_{0+} - \frac{\hat{\mathcal{K}}^{itf}_{xy}}{\mu} \frac{\partial \hat{p}}{\partial \hat{x}} \Big|_{0-}$$

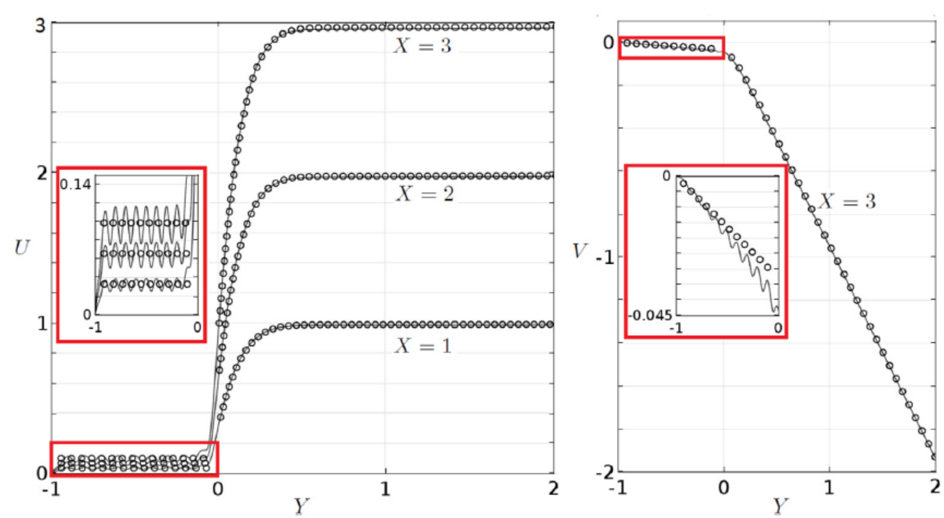
$$\hat{v}|_{0+} \approx -\frac{\hat{\mathcal{K}}_{yy}}{\mu} \frac{\partial \hat{p}}{\partial \hat{y}} \Big|_{0-} - \hat{\mathcal{K}}^{itf}_{xy} \frac{\partial}{\partial \hat{x}} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) \Big|_{0+} - \hat{\mathcal{K}}^{itf}_{zy} \frac{\partial}{\partial \hat{z}} \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right) \Big|_{0+}$$

$$\hat{w}|_{0+} \approx \hat{\lambda}_z \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right) \Big|_{0+} - \frac{\hat{\mathcal{K}}^{itf}_{zy}}{\mu} \frac{\partial \hat{p}}{\partial \hat{z}} \Big|_{0-}$$

(which require, however, coupling with the solution for the pressure within the porous medium ...)

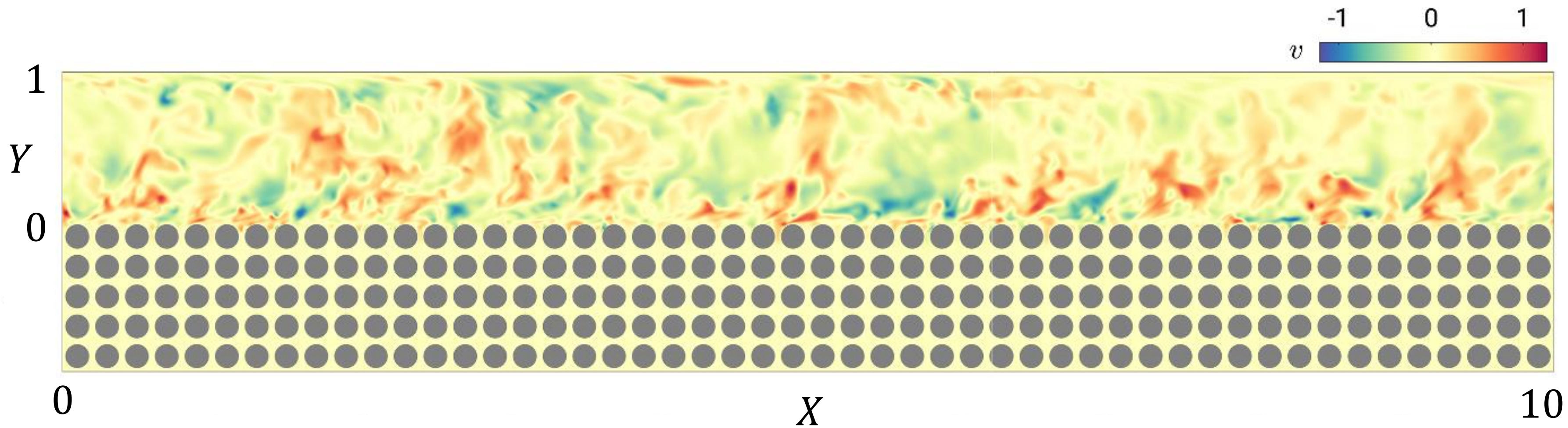
These interface conditions have been extensively tested (Naqvi & Bottaro, *Int. J. Multiphase Flow*, 2021) including cases with significant infiltration within the porous medium.





$\epsilon = 0.1, \qquad \theta = 0.99$

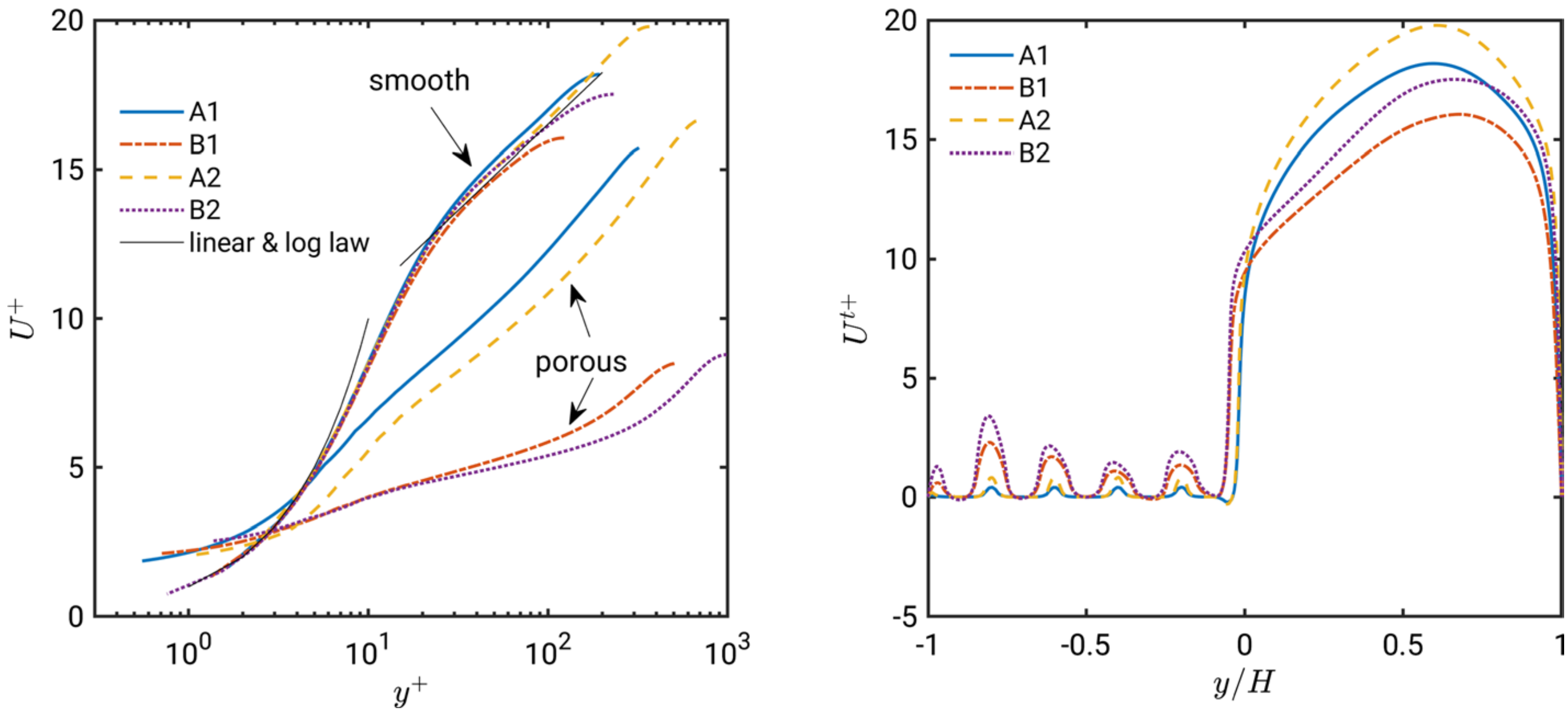
More laborious test case: turbulent flow in a channel, with one permeable wall



$\theta = 0.5$

Chu *et al.*, *Transp. Porous Media*, 2021
Wang *et al.*, *JFM*, 2021

(Chu *et al.*, 2021)



$Re_\tau \approx 200$ based on boundary layer thickness and friction velocity

$\epsilon = 0.4 \Longrightarrow l = 0.4\,L \Longrightarrow l^+ = \frac{l\,u_\tau}{\nu} = 0.4\,Re_\tau \approx 80$ **(probably too large for modeling via an effective boundary condition!)**

Chavarin *et al*. (*JFM* 2021) have shown that anisotropic permeable substrates can hamper the near-wall turbulent cycle, leading to drag reduction, in a manner similar to that of riblets, producing an offset between the virtual origin felt by the mean flow and that by the turbulent fluctuations.

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Porous medium: **longitudinal cylinders** (with driving pressure gradient along the same direction).

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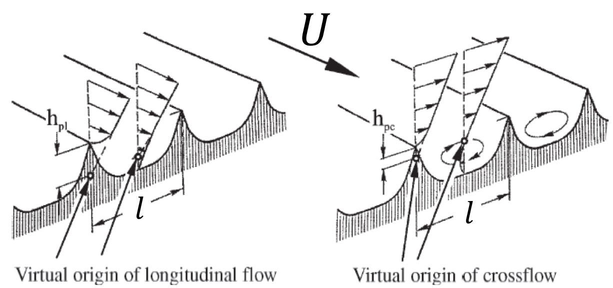
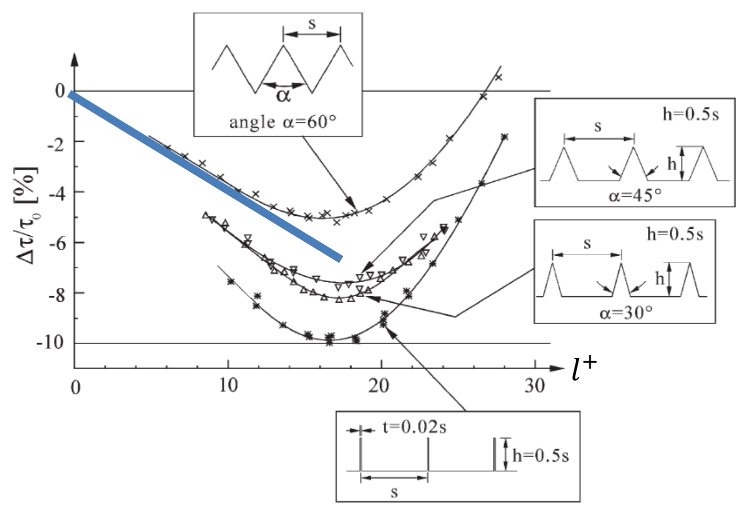


Porous medium: **longitudinal cylinders**

This mimics **riblets**, with the added effect of transpiration through the pores. For riblets, to leading order, the skin friction coefficient is reduced proportionally to $\lambda_z - \lambda_x$

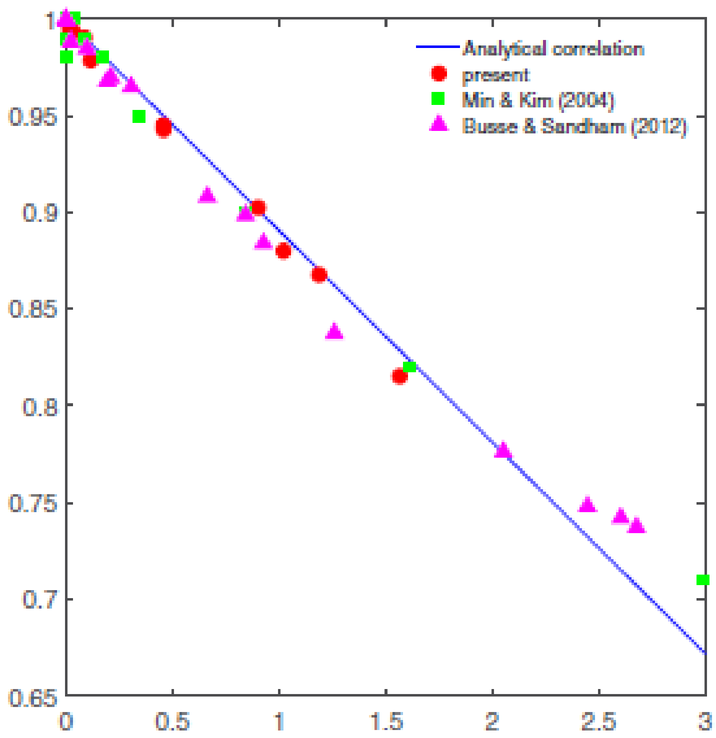


(Bechert & Hage, *WIT Trans*., 2006
Luchini *et al*., *JFM* 1991)

$$\frac{C_f}{C_{f_0}} \approx 1 - \frac{\Delta\lambda^+}{(2C_{f_0})^{-0.5} + (2k)^{-1}}$$

Luchini, 1996

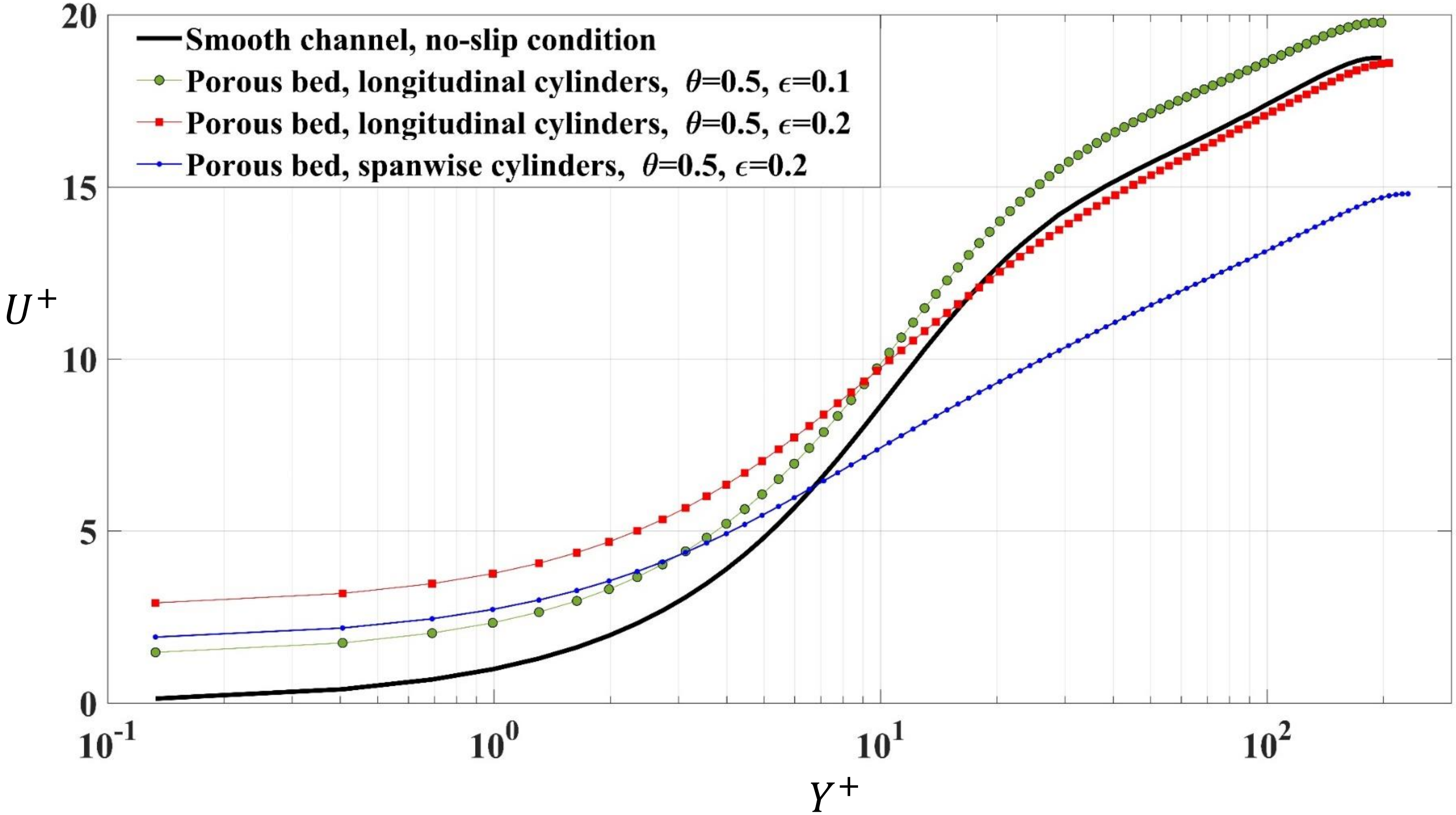
$\frac{C_f}{C_{f_0}}$



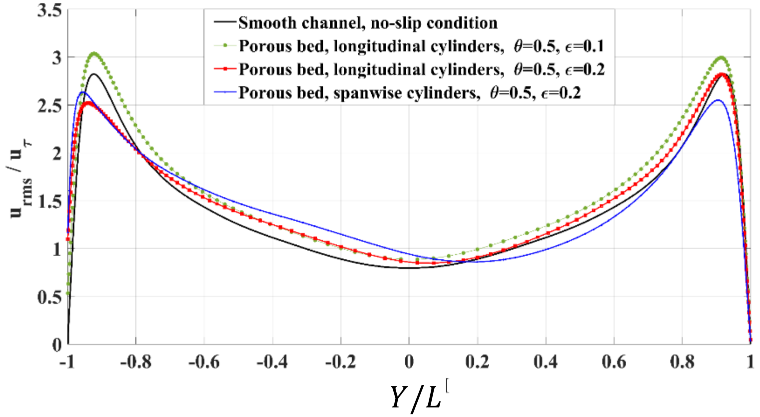
$\Delta\lambda^+$

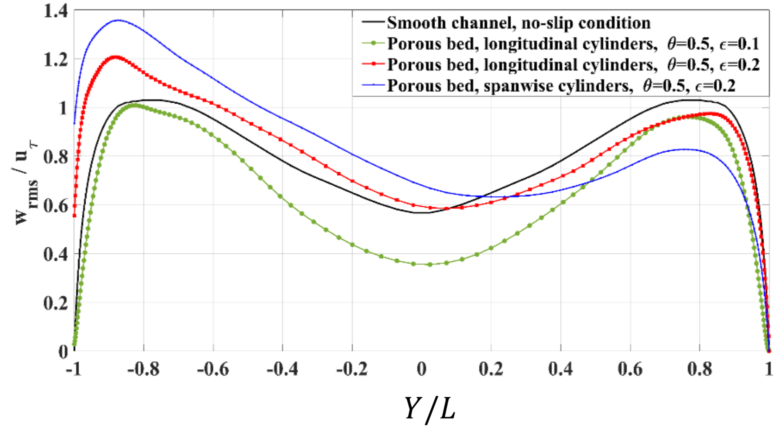
Alinovi & Bottaro, *PRF*, 2018

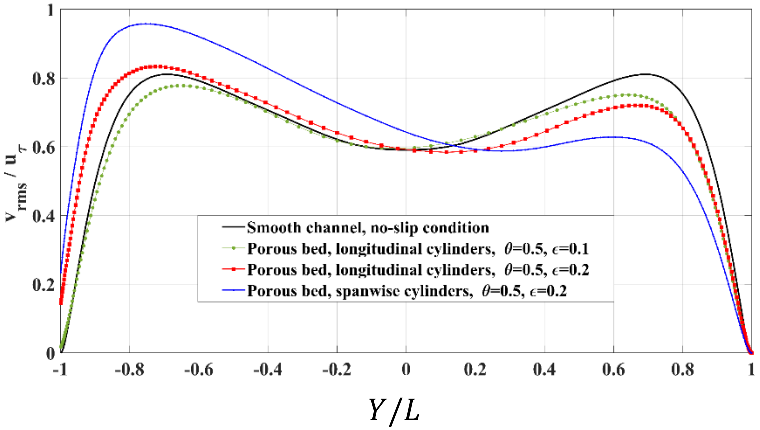
DNS results for different arrangements of solid cylindrical inclusions within the porous layer

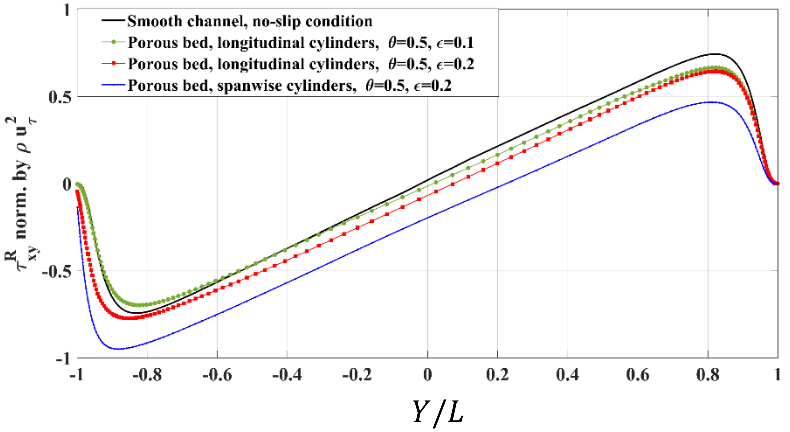


MODELING FLOWS OVER NATURAL OR ENGINEERED SURFACES



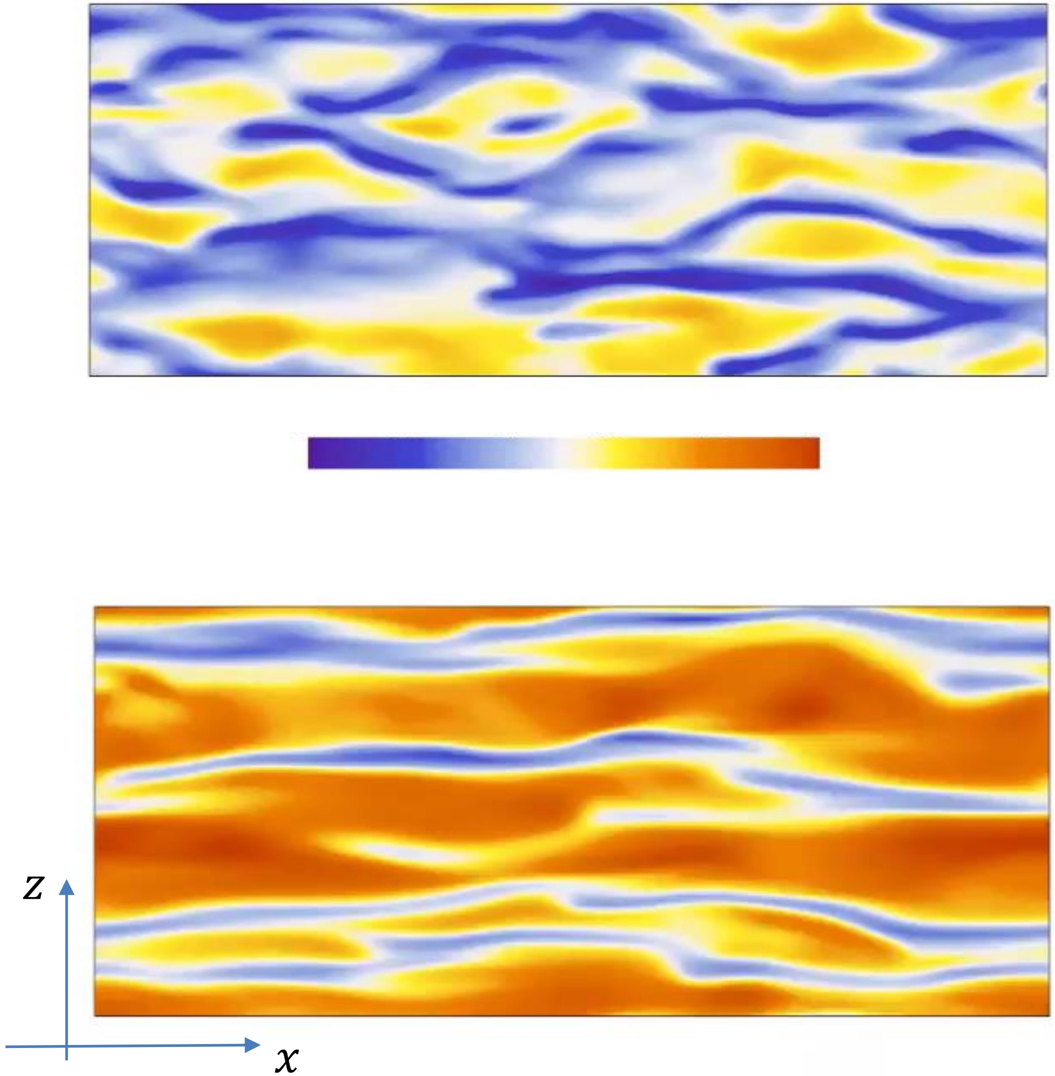






Isolines of U at $y^+ = 20$

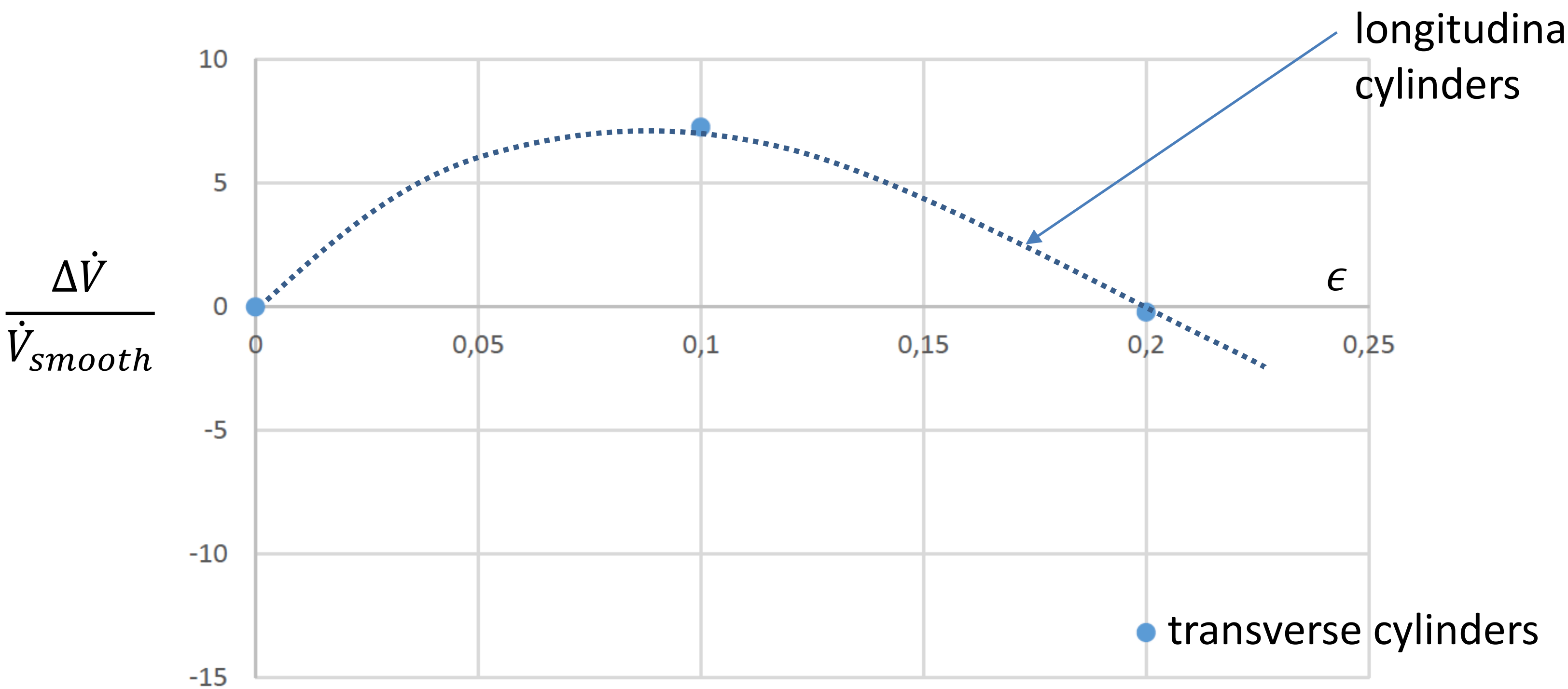
Channel of dimensions $2\pi \times 2 \times \pi$ ($\Delta x^+ = 9.5$, $\Delta y^+_{wall} = 0.28$, $\Delta z^+ = 6.3$)



z-aligned cylinders $\epsilon = 0.2$

x-aligned cylinders $\epsilon = 0.1$

For the **same driving pressure gradient** ($\theta = 0.5$) we have



$\epsilon_{max} \approx 0.1$ ⇒ $l_{max} \approx 0.1\, L$ ⇒ $l^+_{max} = \frac{l_{max}\, u_\tau}{\nu} = 0.1\, Re_\tau \approx 20$

Optimal anisotropic porous layers can be designed (quickly) by employing effective conditions.

Drawback: should maintain $l^+ = \mathcal{O}(10)$

If the wall is impermeable the conditions at a **rough wall** are recovered:

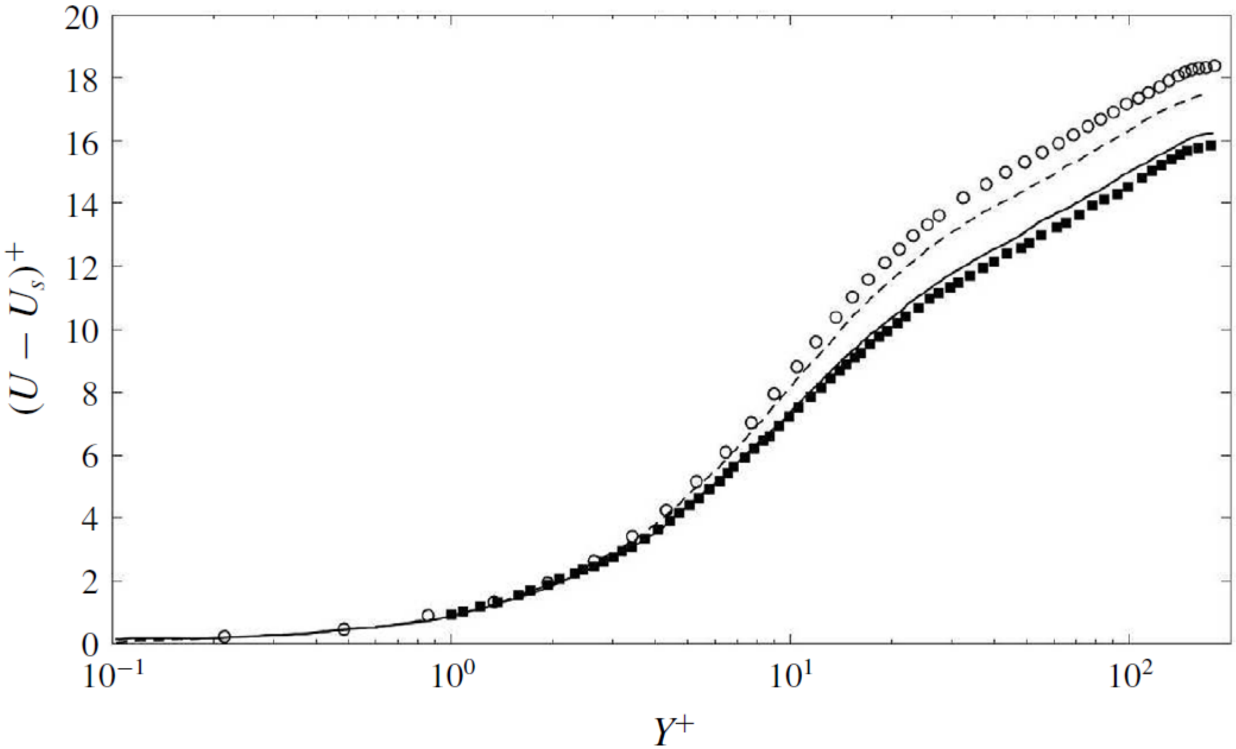
$$\hat{u}|_{0^+} \approx \hat{\lambda}_x \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)\Big|_{0^+} + \frac{\hat{\mathcal{K}}^{itf}_{xy}}{\mu} \frac{\partial}{\partial \hat{x}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+}$$

$$\hat{v}|_{0^+} \approx \frac{\hat{\mathcal{K}}_{yy}}{\mu} \frac{\partial}{\partial \hat{y}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+} - \hat{\mathcal{K}}^{itf}_{xy} \frac{\partial}{\partial \hat{x}} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)\Big|_{0^+} - \hat{\mathcal{K}}^{itf}_{zy} \frac{\partial}{\partial \hat{z}} \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right)\Big|_{0^+}$$

$$\hat{w}|_{0^+} \approx \hat{\lambda}_z \left(\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} \right)\Big|_{0^+} + \frac{\hat{\mathcal{K}}^{itf}_{zy}}{\mu} \frac{\partial}{\partial \hat{z}} \left(-\hat{p} + 2\mu \frac{\partial \hat{v}}{\partial \hat{y}} \right)\Big|_{0^+}$$

(Bottaro, *JFM* 2019; Lacis *et al.*, *JFM* 2020; Bottaro & Naqvi, *Meccanica*, 2020)

Flow over natural or engineered surfaces



Bottaro, *JFM* 2019
Lacis *et al.*, *JFM* 2020

CONCLUSIONS

'Microscopic', repeated features can be treated via **multiple scale homogenization**, yielding effective conditions at the *interface* which permit to avoid the numerical resolution of very small scale details. This would allow the rapid modeling of geometrical micro-features, to identify, e.g., the most efficient drag-reducing textures, or the most suitable structure of a porous membrane, etc.

M.C. Escher, *Angels and Demons*, 1960