A LINEAR MODEL OF TRANSITION TO TURBULENCE

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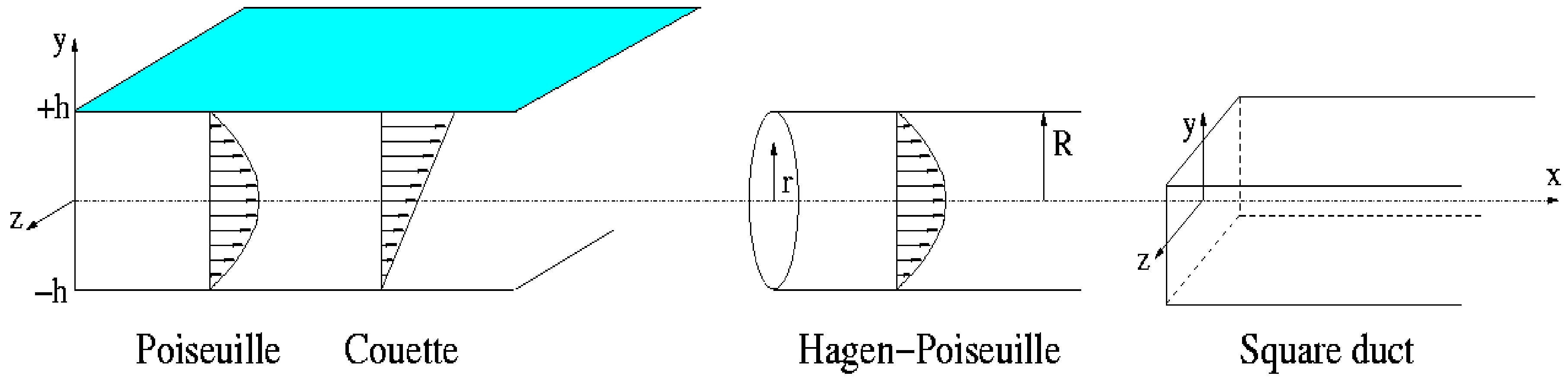
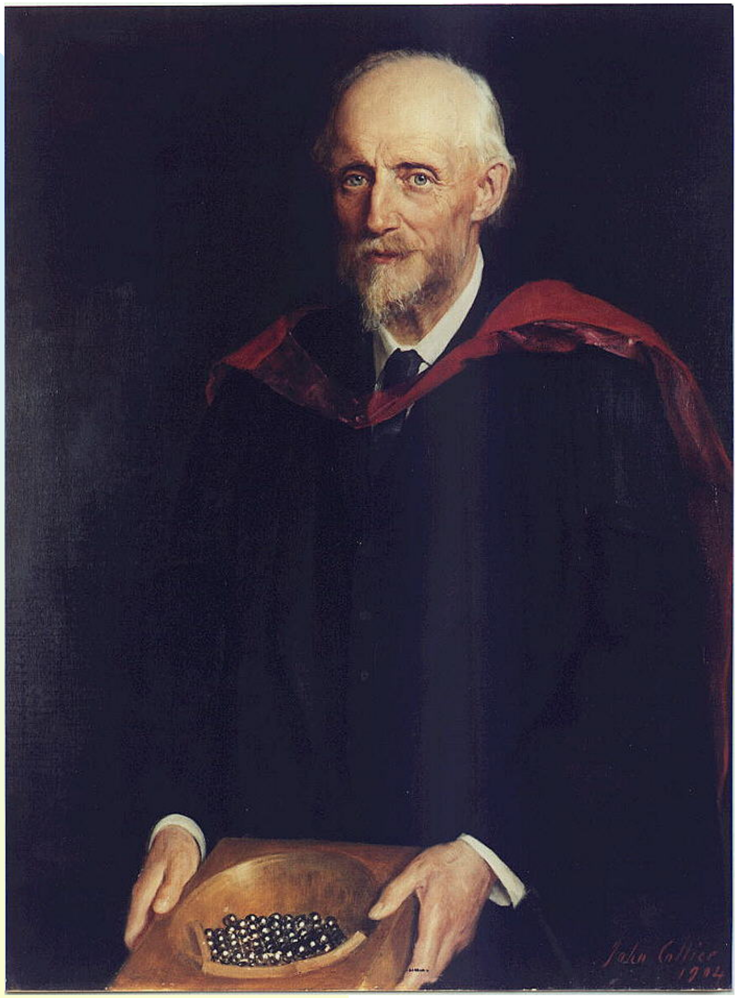
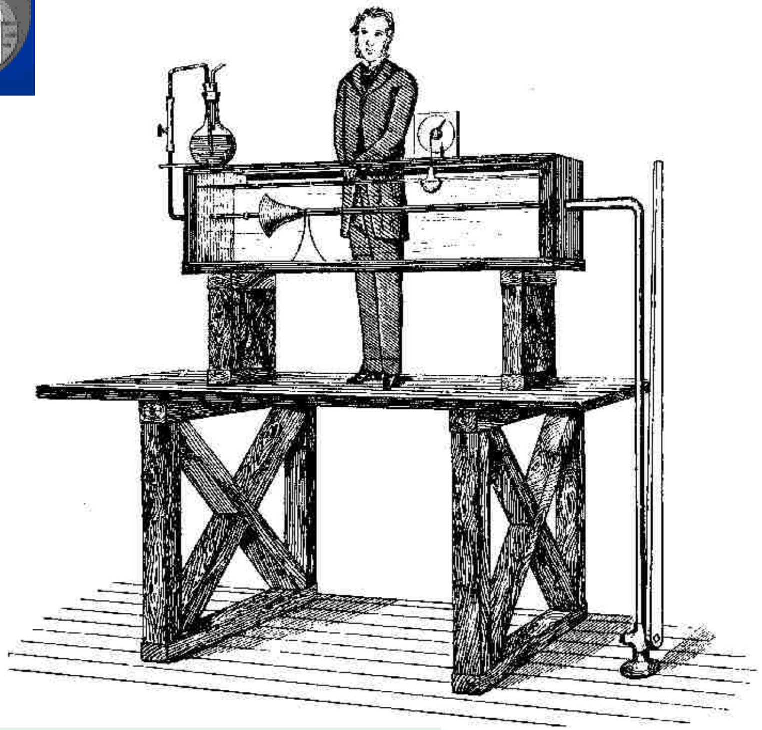


Relevant references: Galletti & Bottaro, *JFM* 2004; Bottaro et al. *AIAA J*. 2006; Biau et al. *JFM* 2008; Wedin et al. PoF 2008; Biau & Bottaro, *Phil. Trans*. 2008, *Special Issue on the 125th Anniversary …*

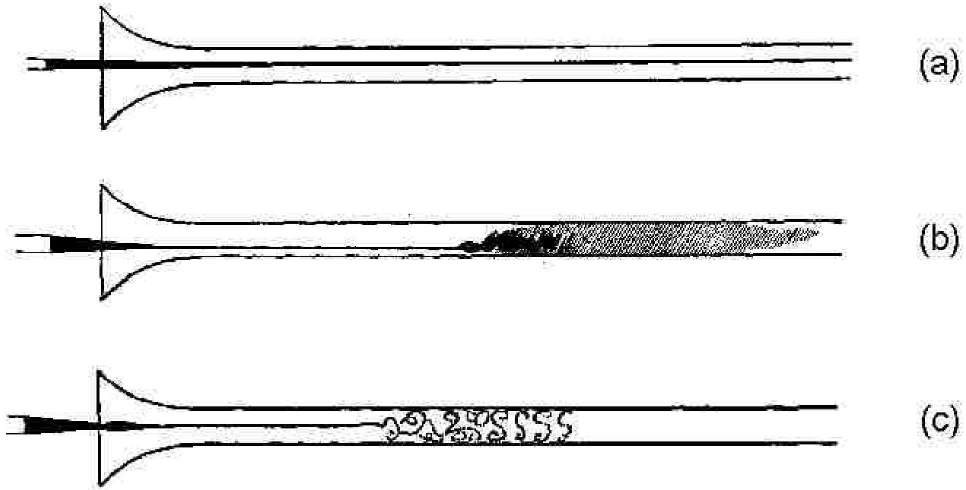
Osborne Reynolds, 1842-1912



"An experimental investigation of the circumstances which determine whether motion of water shall be direct or sinuous and of the law of resistance in parallel channels", *Royal Society, Phil. Trans*. 1883



`the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water ... On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies.'



$$Re_{trans} \approx 13000$$

Hydrodynamic stability theory followed a few years later:

W. M'F.Orr, 'The stability or instability of the steady motions of a perfect liquid and of a viscous liquid', *Proc. Roy. Irish Academy*, 1907

A. Sommerfeld, 'Ein Beitrag zur hydrodynamischen Erklaerung der turbulenten Fluessigkeitsbewegungen', *Proc. 4th International Congress of Mathematicians*, Rome, 1908

Hints on the solution of the stability equations for the flow in a pipe arrived only much later (C.L. Pekeris, 1948), just to show that

$$\mathrm{Re}_{crit} \to \infty \qquad (!!)$$

STILL TODAY, TRANSITION IN SHEAR FLOWS IS STILL NOT FULLY UNDERSTOOD. For the simplest parallel flows there is poor agreement between predictions from the classical linear stability theory (Re_{crit}) and experimentals results (Re_{trans})

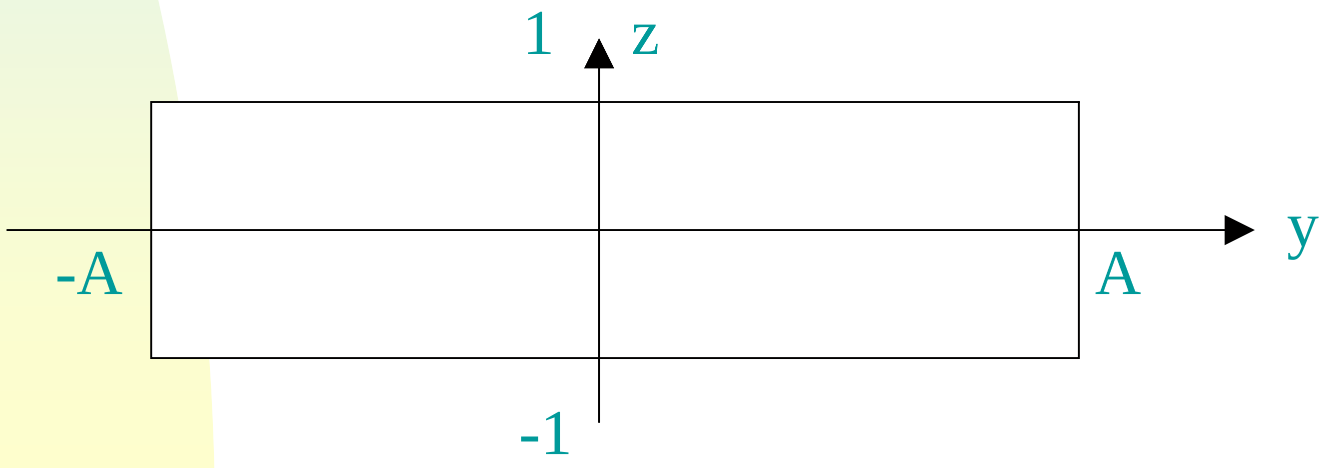
	Poiseuille	Couette	Hagen-Poiseuille	Square duct
Re_{crit}	5772	∞	∞	∞
Re_{trans}	~2000	~400	~2000	~2000

LAMINAR FLOW IN A DUCT: THE CLASSICAL LINEAR STABILITY THEORY

T. Tatsumi & T. Yoshimura, *JFM* 1990

Base flow: analytical solution in terms of a series of trigonometric and hyperbolic functions (Saint-Venant 1855)

Temporal stability analysis



LAMINAR FLOW IN A DUCT: THE CLASSICAL LINEAR STABILITY THEORY

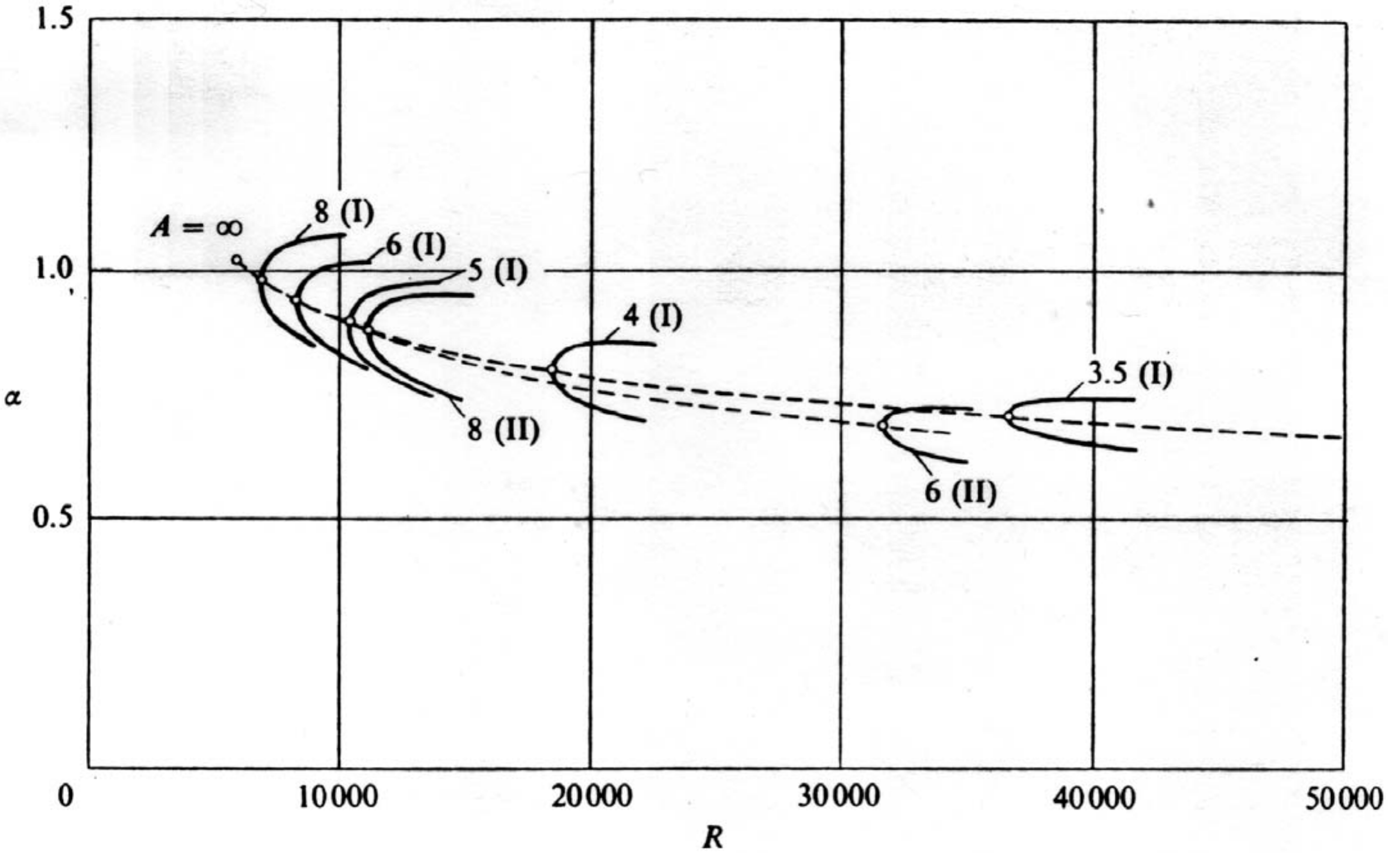


FIGURE 4. Neutral curves near the critical points for various aspect ratios: $A = 8, 6, 5, 4$ and 3.5 for mode I; and $A = 8$ and 6 for mode II.

LAMINAR FLOW IN A DUCT: THE EXPERIMENTS

T.W. Kao & C. Park, *JFM*, 1970

Unsteady artificial excitation (mechanically driven vibrating ribbon) to trigger growing instability waves, A = 8

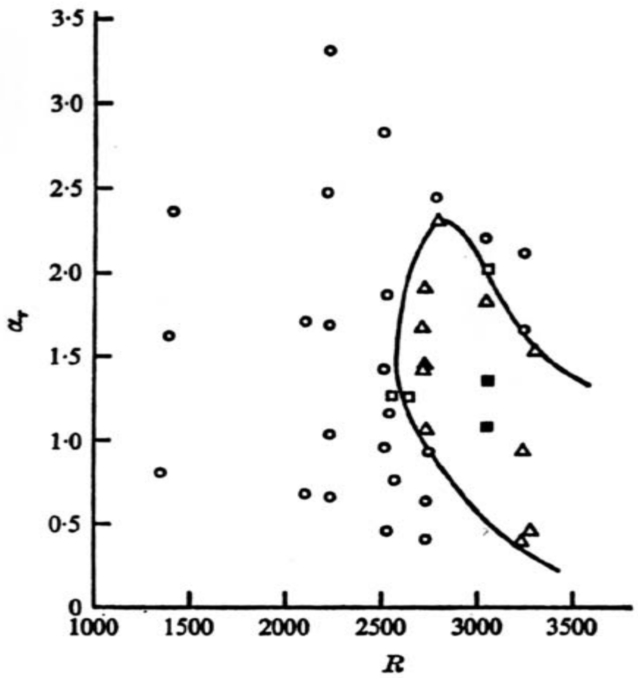


FIGURE 17. α_r, R plane showing regions of damping and growing and neutral stability boundary. ○, damped; △, growing; □, neutral; ■, growing but less certain.

LAMINAR FLOW IN A DUCT: THE EXPERIMENTS

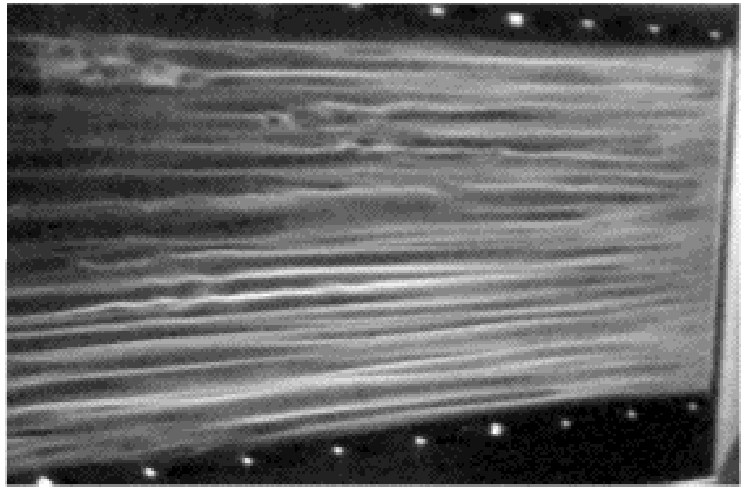
Conclusions reported:

1. For A = 8 a critical Reynolds number of 1200, α = 0.42, is obtained. “The result of the critical Reynolds number using artificial excitation is in good agreement with the naturally occurring one”
2. “The critical Reynold for rectangular channels is larger at larger aspect ratio and approaches the plane Poiseuille value when the aspect ratio becomes very large”
3. “The disturbances are three-dimensional in the experiments”
4. The unstable region includes the zero-frequency, zero streamwise wavenumber range.

Little mention of the receptivity environment.

LAMINAR FLOW : TRANSIENT GROWTH

- **THE MECHANISM**: a stationary algebraic instability exists in the inviscid system ("lift-up" effect). In the viscous case the growth of the disturbance energy is hampered by diffusion $\Rightarrow$ transient growth



P.H. Alfredsson and M. Matsubara (1996); streaky structures in a boundary layer. Free-stream speed: 2 [m/s], free-stream turbulence level: 6%

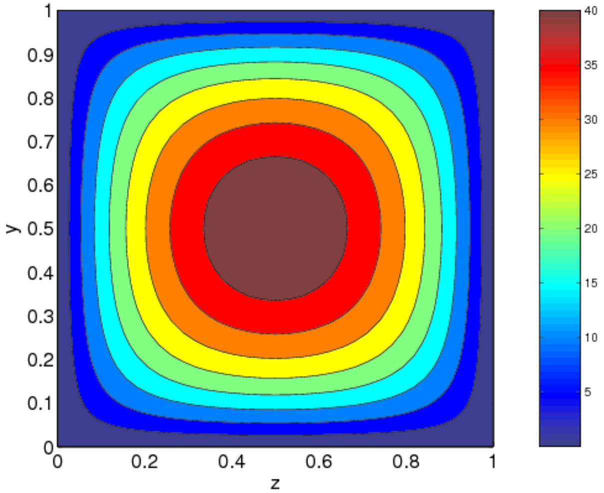
HYDRODYNAMIC STABILITY THEORY

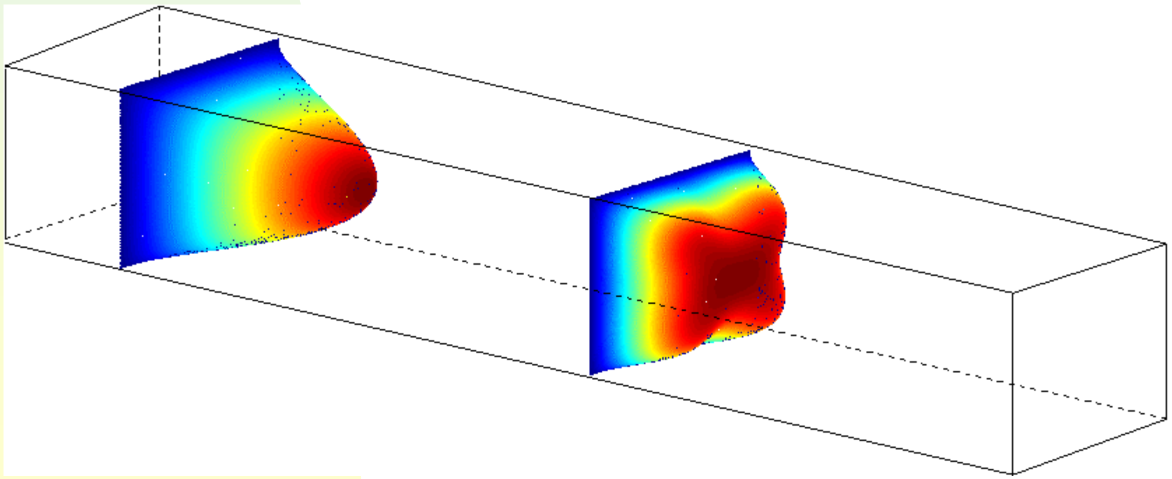
- Is transient growth the solution?

  Butler & Farrell (1993)
  Trefethen, Trefethen Reddy & Driscoll (1993)
  Schmid & Henningson (2001)

Transient growth is related to <u>unstructured</u> operator's perturbations, *i.e.* to the pseudospectrum of the linear stability operator: $[L(U, \omega; \alpha, \beta, Re) + \Delta]\, v = 0$

The case studied





dP/dx = <u>const</u>.

<u>characteristic length</u> = *h*
[channel height]

<u>characteristic speed</u> = u_τ
[$u_\tau^2 = -(h/4\rho)\, dp/dx$]

$Re_\tau = 150$

(*MARGINAL* VALUE)

Governing equations:

$$\left\{\begin{array}{l} u_x + v_y + w_z = 0, \\ u_t + uu_x + vu_y + wu_z = -p_x + \frac{1}{Re}\Delta u + 4, \\ v_t + uv_x + vv_y + wv_z = -p_y + \frac{1}{Re}\Delta v, \\ w_t + uw_x + vw_y + ww_z = -p_z + \frac{1}{Re}\Delta w, \end{array}\right.$$

Linearized disturbance equations:

$$\left\{\begin{array}{rcl} i\alpha u + v_y + w_z & = & 0, \\ u_t + i\alpha U u + vU_y + wU_z & = & -i\alpha p + \frac{1}{Re_\tau}(-\alpha^2 u + u_{yy} + u_{zz}), \\ v_t + i\alpha U v & = & -p_y + \frac{1}{Re_\tau}(-\alpha^2 v + v_{yy} + v_{zz}), \\ w_t + i\alpha U w & = & -p_z + \frac{1}{Re_\tau}(-\alpha^2 w + w_{yy} + w_{zz}). \end{array}\right.$$

with $u = v = w = 0$ on the walls

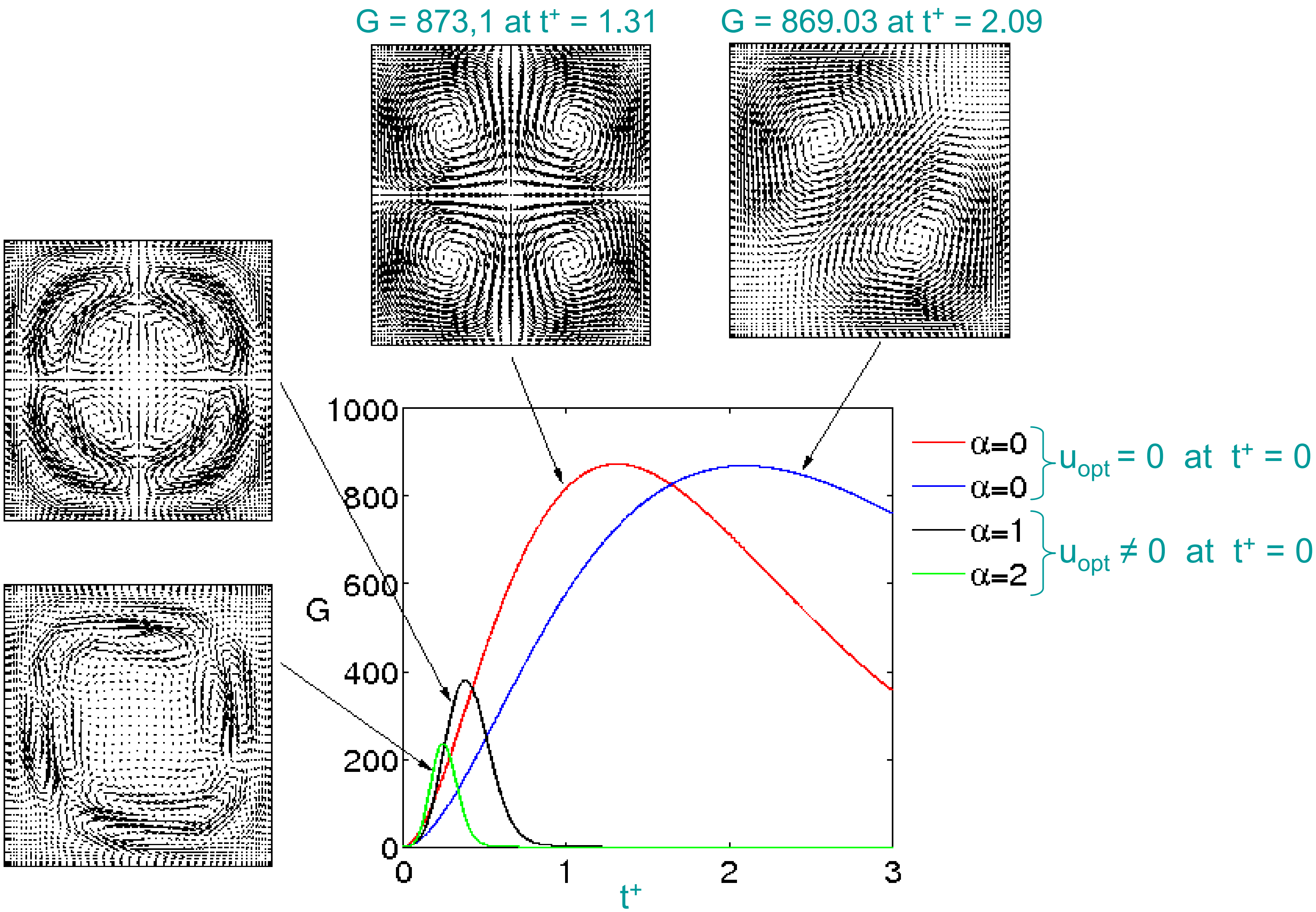
OPTIMAL PERTURBATIONS: Traditional functional optimization with adjoints

$$G(T) = \frac{E(T)}{E(0)}, \quad with \quad E = \frac{1}{2}\int_y \int_z (\bar{u}u + \bar{v}v + \bar{w}w) \quad dy\, dz$$

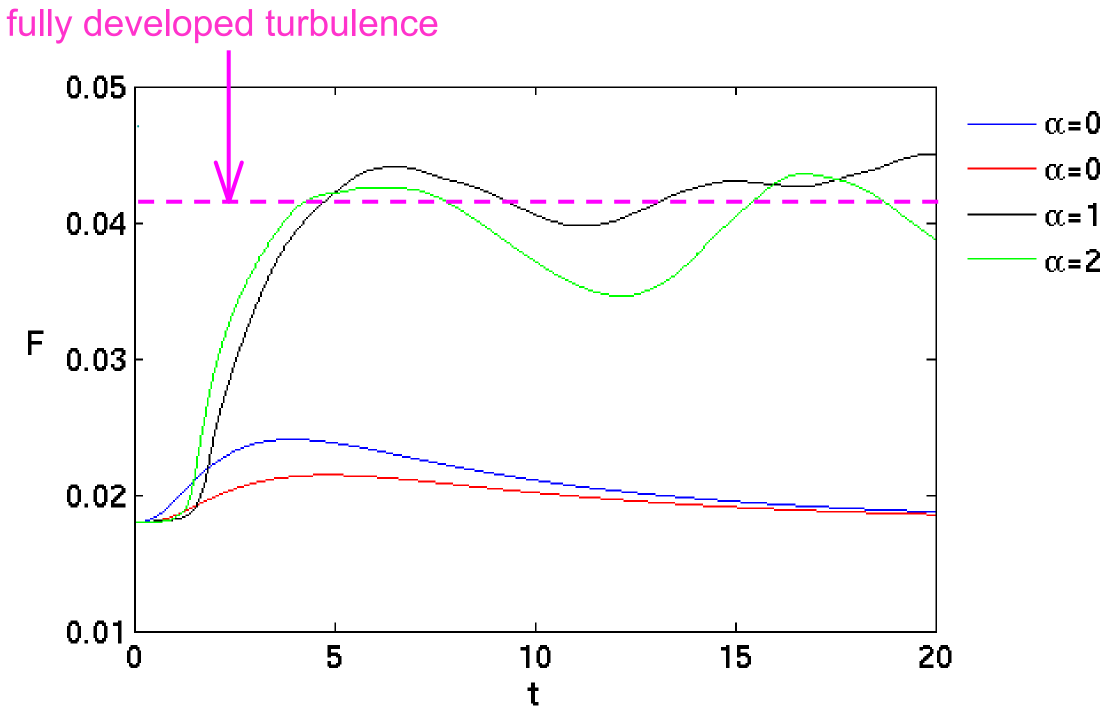
adjoint problem:

$$\left\{ \begin{array}{rcl} i\alpha u^\dagger + v_y^\dagger + w_z^\dagger &=& 0, \\ -u_t^\dagger - i\alpha U u^\dagger &=& -i\alpha p^\dagger + \frac{1}{Re_\tau}(-\alpha^2 u^\dagger + u_{yy}^\dagger + u_{zz}^\dagger), \\ -v_t^\dagger - i\alpha U v^\dagger + u^\dagger U_y &=& -p_y^\dagger + \frac{1}{Re_\tau}(-\alpha^2 v^\dagger + v_{yy}^\dagger + v_{zz}^\dagger), \\ -w_t^\dagger - i\alpha U w^\dagger + u^\dagger U_z &=& -p_z^\dagger + \frac{1}{Re_\tau}(-\alpha^2 w^\dagger + w_{yy}^\dagger + w_{zz}^\dagger). \end{array} \right.$$

$$\begin{array}{ccc} \mathbf{q}(t=0) & \xrightarrow{q_t = Lq} & \mathbf{q}(t=T) \\ \uparrow & & \downarrow \\ \mathbf{a}(t=T) & \xleftarrow{-a_t = L^\dagger a} & \mathbf{a}(t=0) \end{array}$$


G = 873,1 at t+ = 1.31
G = 869.03 at t+ = 2.09
1000
800
600
400
200
0
G
0
1
2
3
t+
α=0
α=0
α=1
α=2
uopt = 0 at t+ = 0
uopt ≠ 0 at t+ = 0

Nonlinear evolution of the optimal disturbances (+ random noise)
(streamwise periodic duct of length 4π)

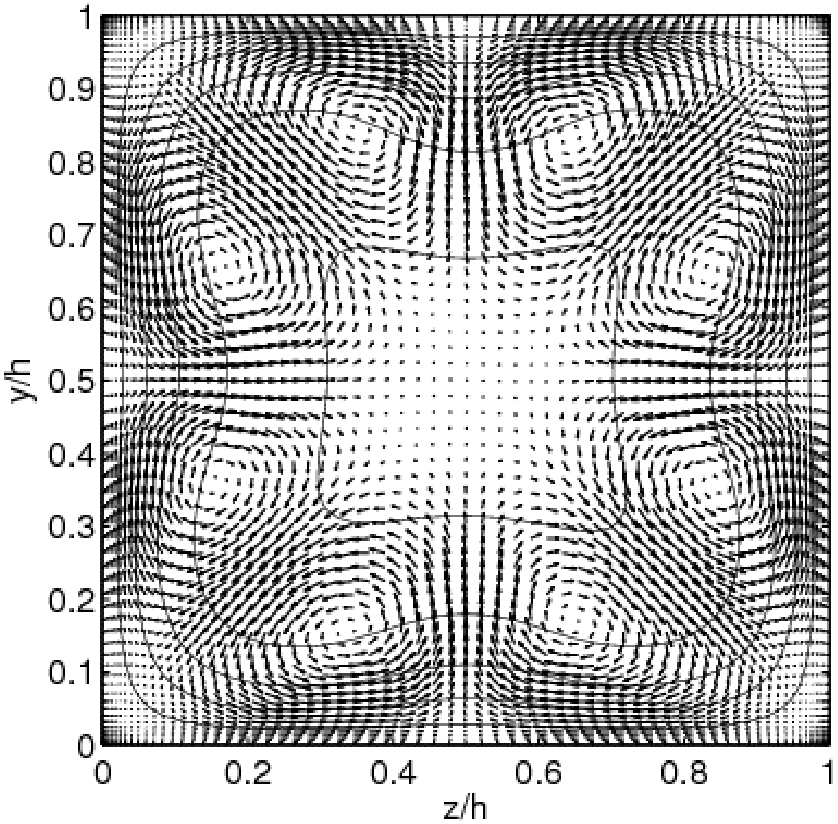


F = skin friction factor (F = 0.0415 from DNS at Re_τ = 150 (Re_b = 2084))

At t = 0:

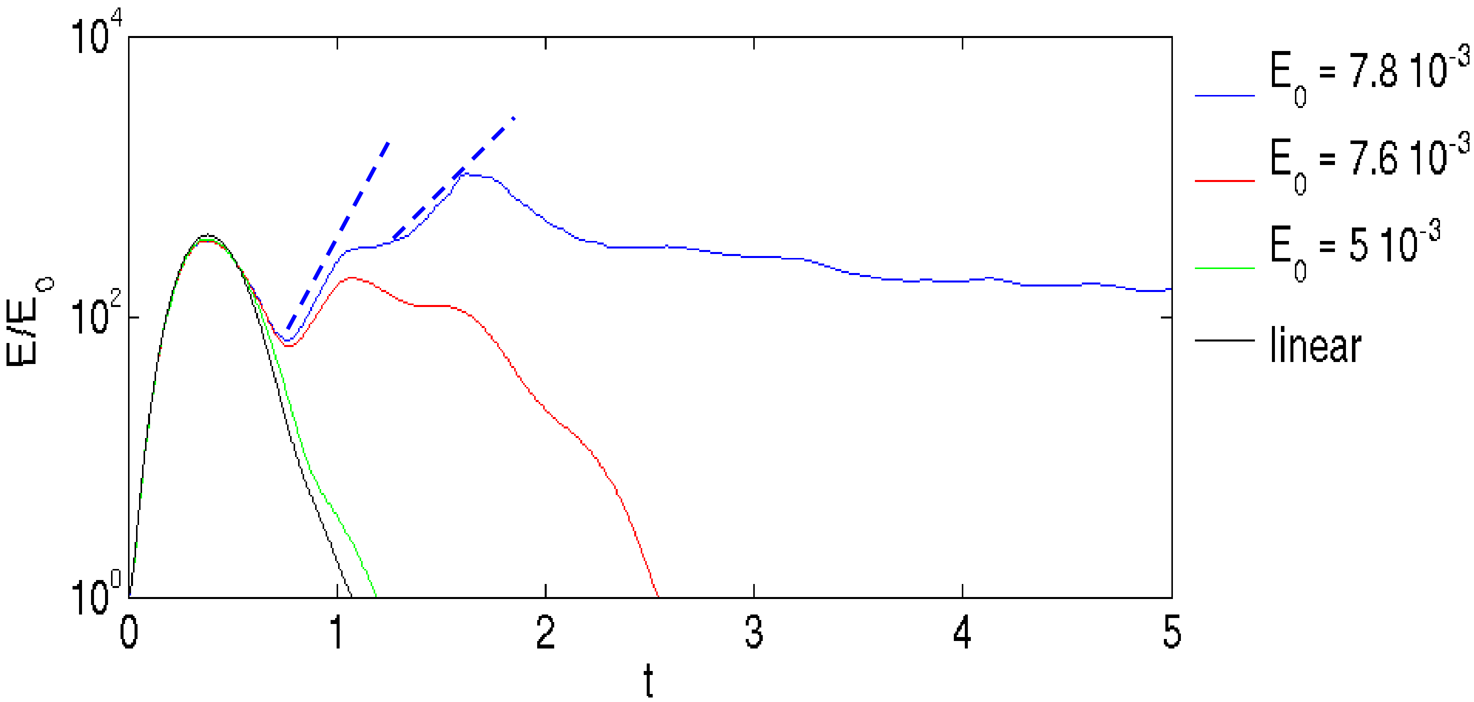
α = 0	**$E_0 = 10^{-1}$** !!
α = 1	$E_0 = 7.8 \times 10^{-3}$
α = 2	$E_0 = 4.4 \times 10^{-3}$

Streamwise/time averaged turbulent flow



Excellent agreement with Gavrilakis, ***JFM*** 1992.

The α = 1 mode



The threshold value is $E_0 = 7.8 \times 10^{-3}$. In fact the optimal perturbation for $\alpha = 1$ is not so important. What matters is the **distorted field** which emerges at $t \sim 0.8$. Such a field is subject to a strong amplification.

Partial conclusions 1

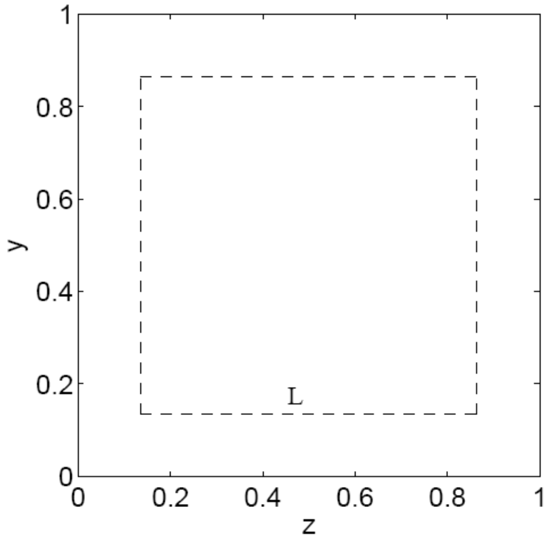
- Global optimal disturbances are interesting concepts, with probably little connection to transition.

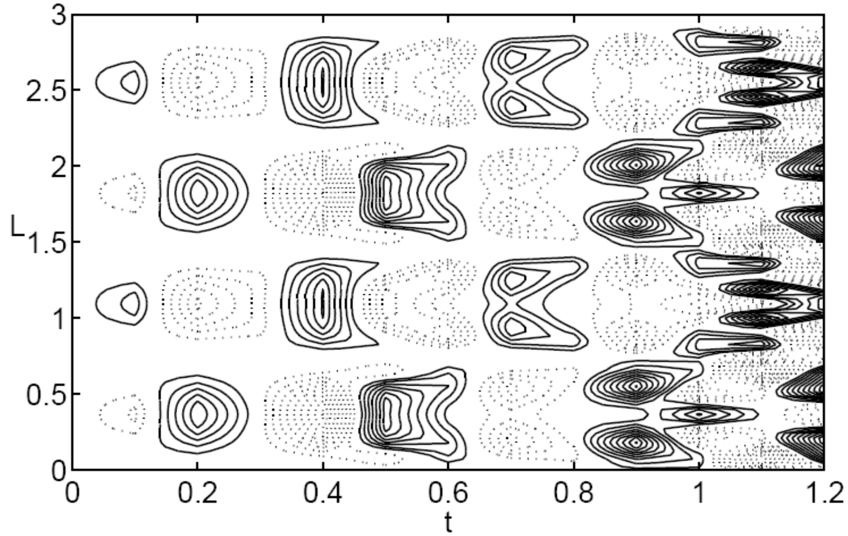
- The key to transition is to set up a distorted base flow with certain features

- The distorted base flow can be set up efficiently by sub-optimal disturbances in the form of streamwise travelling waves. In fact, any initial condition in the form of a travelling wave of sufficiently large amplitude is capable to do it!

- Is there any scope for studying optimal perturbations?

What happens past t ~ 0.8?

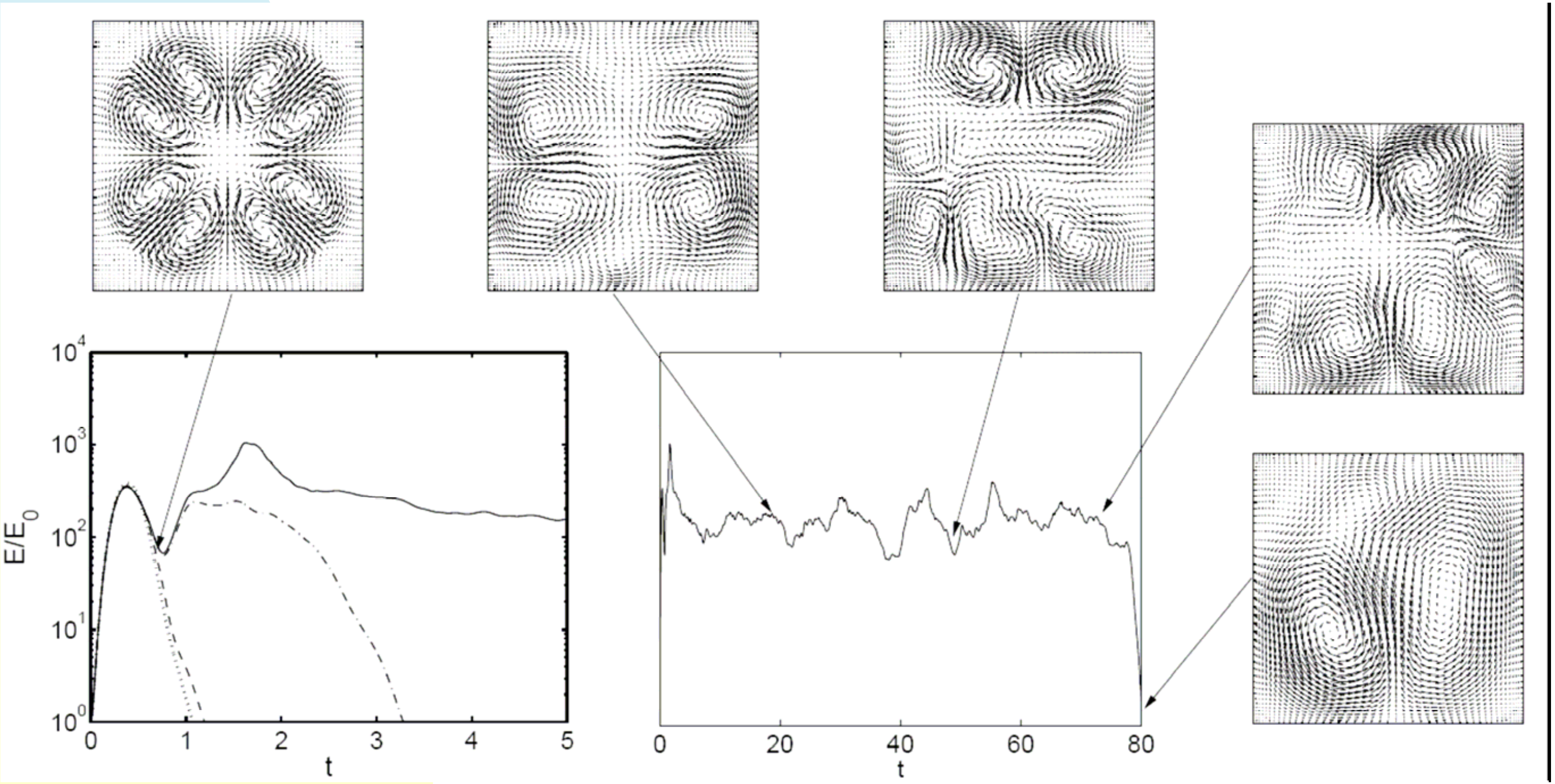




Streamwise velocity showing a non-staggered array of Λ vortices (similar to H-type transition)

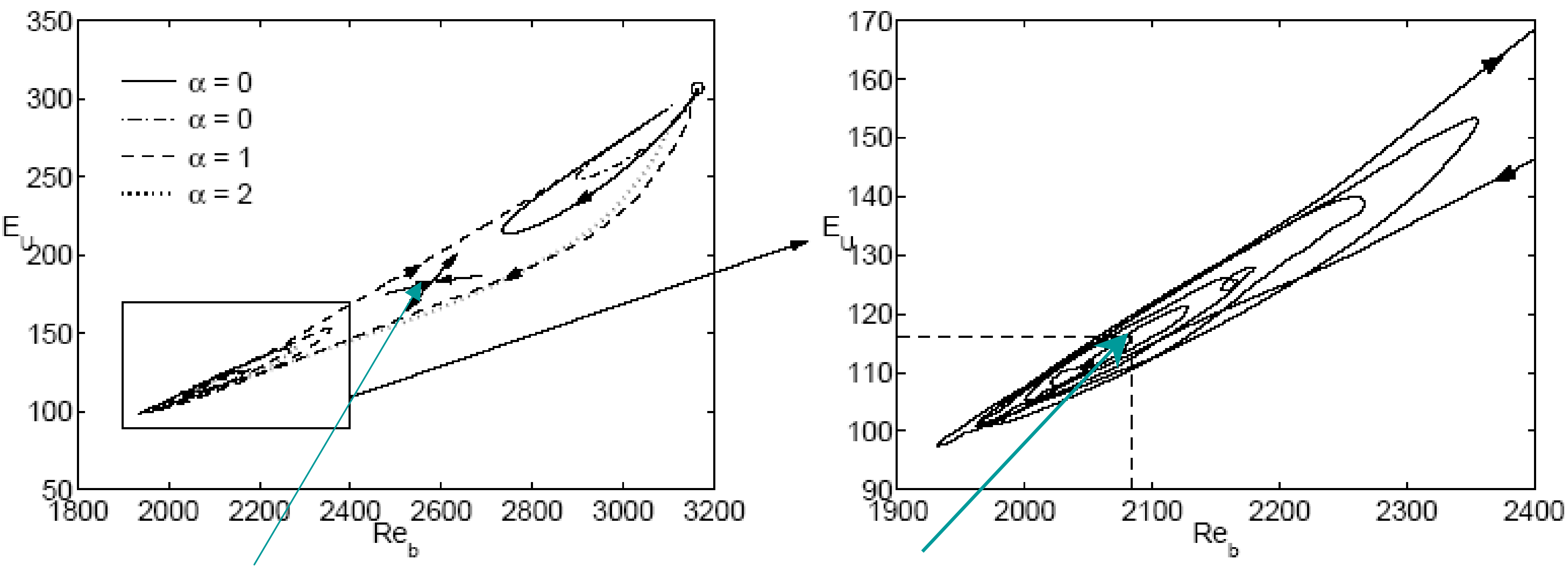
What happens past t ~ 0.8?

The secondary flow does not oscillate around the 8-vortex state, but around a **4-vortex state**, with active walls which alternate ...
… eventually relaminarisation ensues (and the lifetime of the turbulence depends non-monotonically on the streamwise dimension of the computational box)



THE PHASE SPACE PICTURE

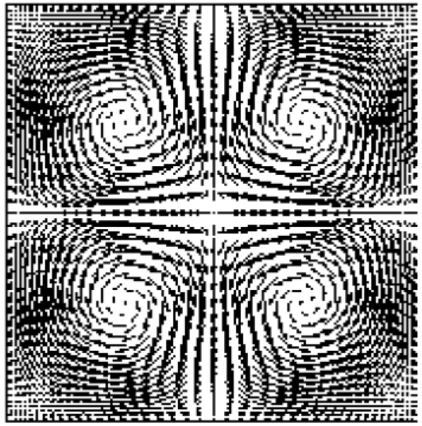
E_U = mean flow energy
Re_b = Reynolds number based on bulk velocity

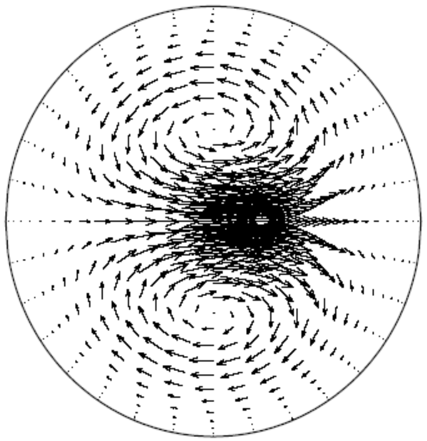


lower branch solution (qualitative)

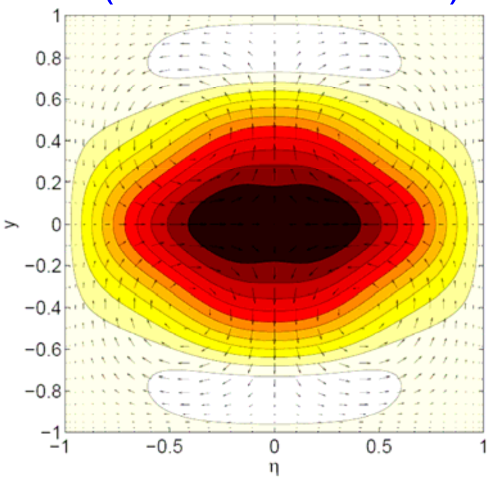
upper branch solution

Global optimal perturbations

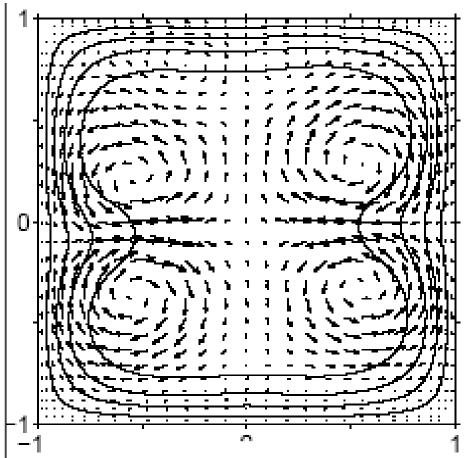


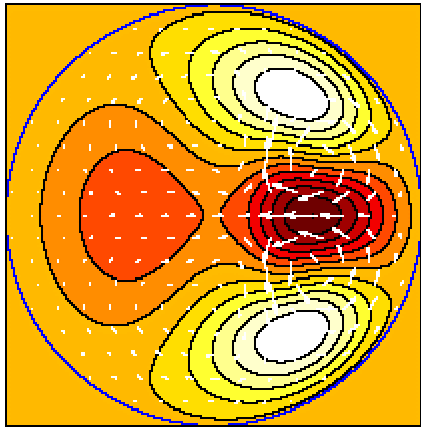


saddle node state
(Wedin *et al*. 2007)



“edge states” ?





Partial conclusions 2

- Global optimals (4-vortex structure) resemble “edge states” (just a coincidence?).

- A *saddle point* appears in phase space to structure the turbulence; after leaving the unstable manifold of the saddle, the trajectory loops in phase space a few times, with the flow spending most of the time in the vicinity of the saddle, until the flow eventually relaminarizes.

- The lifetime depends on the box size.

HYDRODYNAMIC STABILITY THEORY

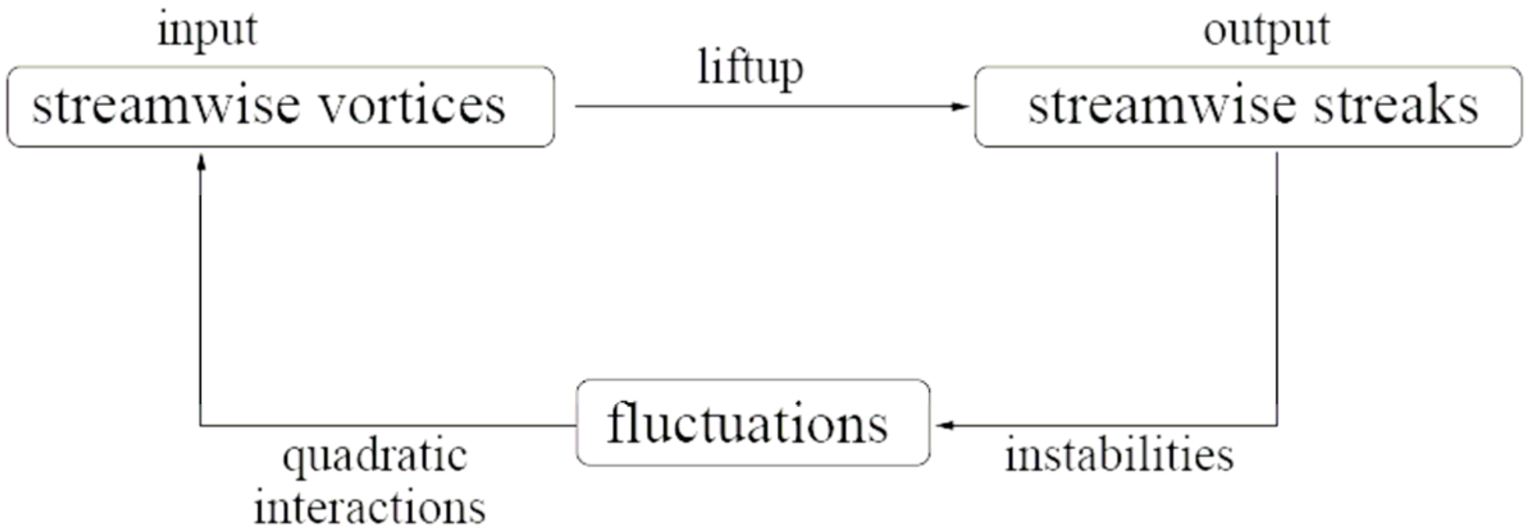
- Distorted base flow: **minimal defects**
  - Bottaro, Corbett & Luchini (2003)
  - Biau & Bottaro (2004)
  - Ben-Dov & Cohen (2007)

The growth of instabilities on top of a distorted base flow (related to dynamical uncertainties and/or poorly modeled terms) is related to structured operator's perturbations, *i.e.* to the structured pseudospectrum of the linear stability operator.

$[L(U, \omega; \alpha, \beta, Re) + \Delta]\, v = 0$ Unstructured perturbations

$[L(U + \Delta U, \omega; \alpha, \beta, Re)]\, v = 0$ Structured perturbations

Input-output framework



Instead of optimising the input (the vortex),
we optimise the output (the streak)!

SENSITIVITY ANALYSIS

The linear stability equation is:

$$L(U)q = \omega q$$

With a base flow defect δU:

$$U + \delta U \Rightarrow \begin{cases} \omega = \omega + \delta\omega \\ q = q + \delta q \end{cases}$$

$$\Rightarrow L(U + \delta U)(q + \delta q) = (\omega + \delta\omega)(q + \delta q)$$

SENSITIVITY ANALYSIS

we use the adjoint function defined by:

$$< q^{\dagger},\ (L - \omega)q > = < q,\ (L^{\dagger} - \omega^{*})q^{\dagger} > = 0$$

then we obtain the sensitivity function G_U:

$$\begin{aligned} \delta\omega &= < q^{\dagger},\ L(\delta U)q > \\ &= < G_U,\ \delta U > \end{aligned}$$

with normalization:

$$< q^{\dagger},\ q > = 1$$

MINIMAL DEFECT

- maximization of the growth rate ω_i
- constraint on the energy of the defect $\epsilon = \int_{yz} (U - U_0)^2 \; dydz$.

Lagrange functional:

$$\mathcal{L} = \omega_i - \lambda \left(\int_{yz} (U - U_0)^2 \; dydz - \epsilon \right)$$

Which leads to the optimization loop:

$$\Rightarrow \begin{cases} \lambda^{n+1} = \sqrt{\frac{1}{4\epsilon} \int_{yz} imag(G_U^n)^2 \; dydz} \\ \delta U^{n+1} = \frac{1}{2\lambda} imag(G_U^n) \end{cases}$$

direct problem:

$$
\begin{aligned}
i\alpha u + v_y + w_z &= 0, \\
u_t + i\alpha U u + v U_y + w U_z &= -i\alpha p + \frac{1}{Re_\tau}(-\alpha^2 u + u_{yy} + u_{zz}), \\
v_t + i\alpha U v &= -p_y + \frac{1}{Re_\tau}(-\alpha^2 v + v_{yy} + v_{zz}), \\
w_t + i\alpha U w &= -p_z + \frac{1}{Re_\tau}(-\alpha^2 w + w_{yy} + w_{zz}).
\end{aligned}
$$

with $u = v = w = 0$ on the walls

adjoint problem:

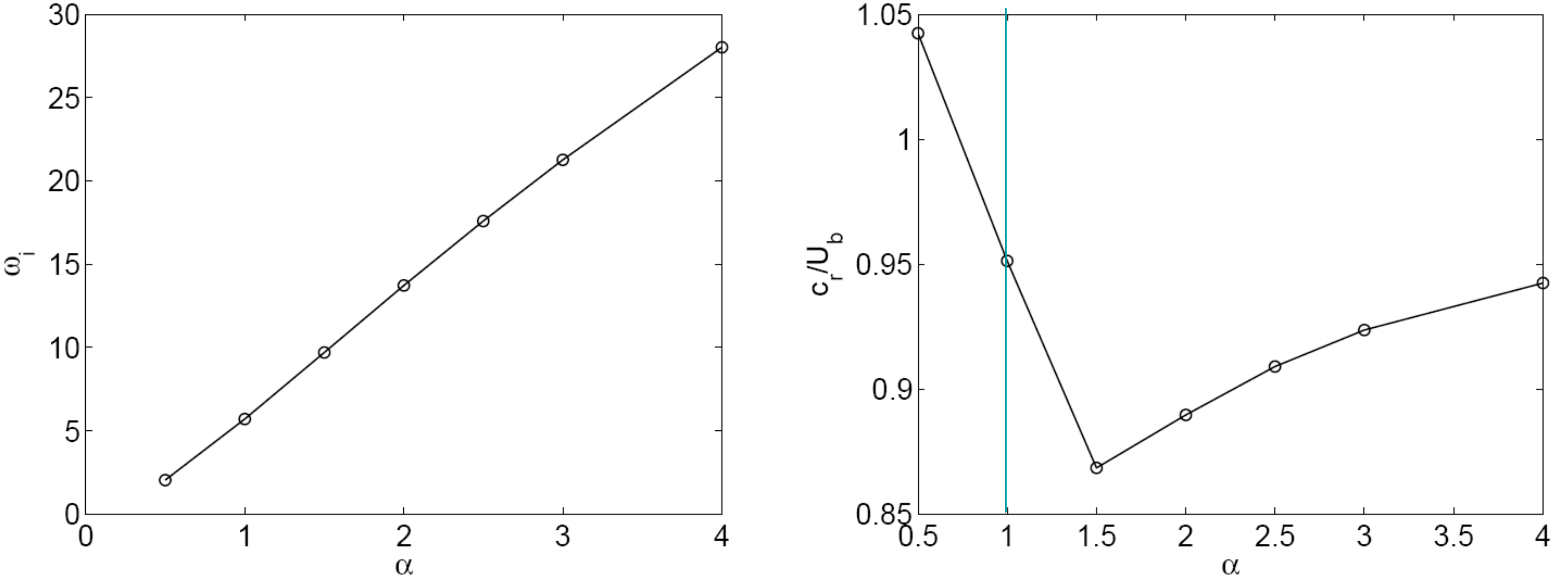
$$
\begin{aligned}
i\alpha u^\dagger + v_y^\dagger + w_z^\dagger &= 0, \\
-u_t^\dagger - i\alpha U u^\dagger &= -i\alpha p^\dagger + \frac{1}{Re_\tau}(-\alpha^2 u^\dagger + u_{yy}^\dagger + u_{zz}^\dagger), \\
-v_t^\dagger - i\alpha U v^\dagger + u^\dagger U_y &= -p_y^\dagger + \frac{1}{Re_\tau}(-\alpha^2 v^\dagger + v_{yy}^\dagger + v_{zz}^\dagger), \\
-w_t^\dagger - i\alpha U w^\dagger + u^\dagger U_z &= -p_z^\dagger + \frac{1}{Re_\tau}(-\alpha^2 w^\dagger + w_{yy}^\dagger + w_{zz}^\dagger).
\end{aligned}
$$

with $u^\dagger = v^\dagger = w^\dagger = 0$ on the walls.

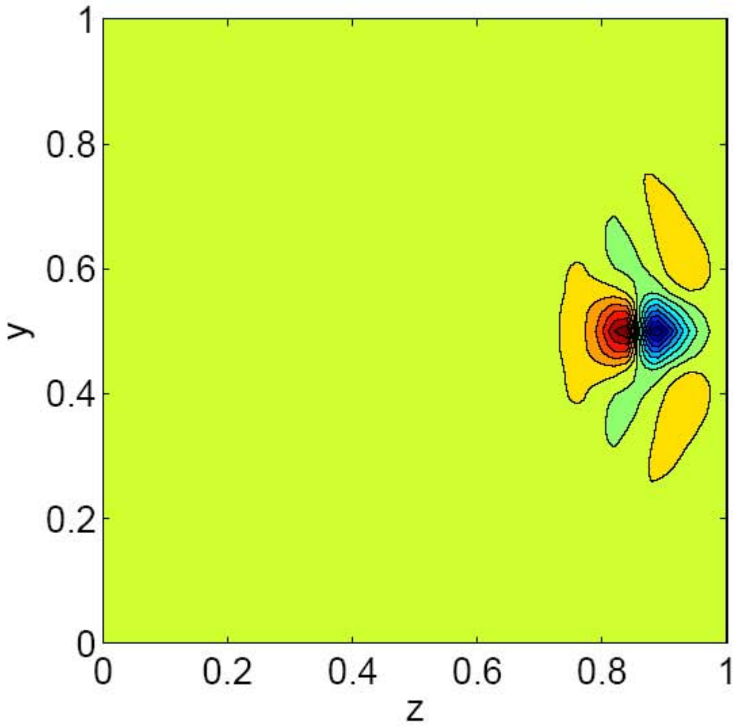
We do not look directly at the spectrum of temporal eigenvalues of the direct and adjoint equations, instead we iterate the PDE's for long time, until the leading eigenvalue emerges.

Advantage: if we decided to iterate for a fixed (short) time, we could find the optimal defect for transient growth …

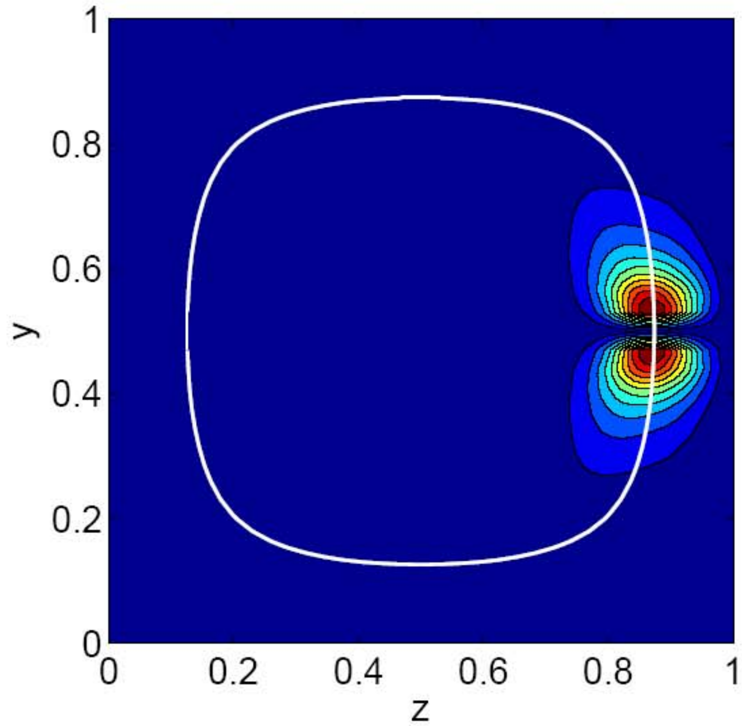
Minimal defects for ε = 1, Re_τ = 150



(ε = 1 corresponds to 0.16 of the laminar flow energy)



Minimum defect for $\alpha = 1$:
modulus of the streamwise velocity defect



Unstable eigenmode:
streamwise vel. perturbation

(the white line corresponds to $u = U_b$)

Partial conclusions 3

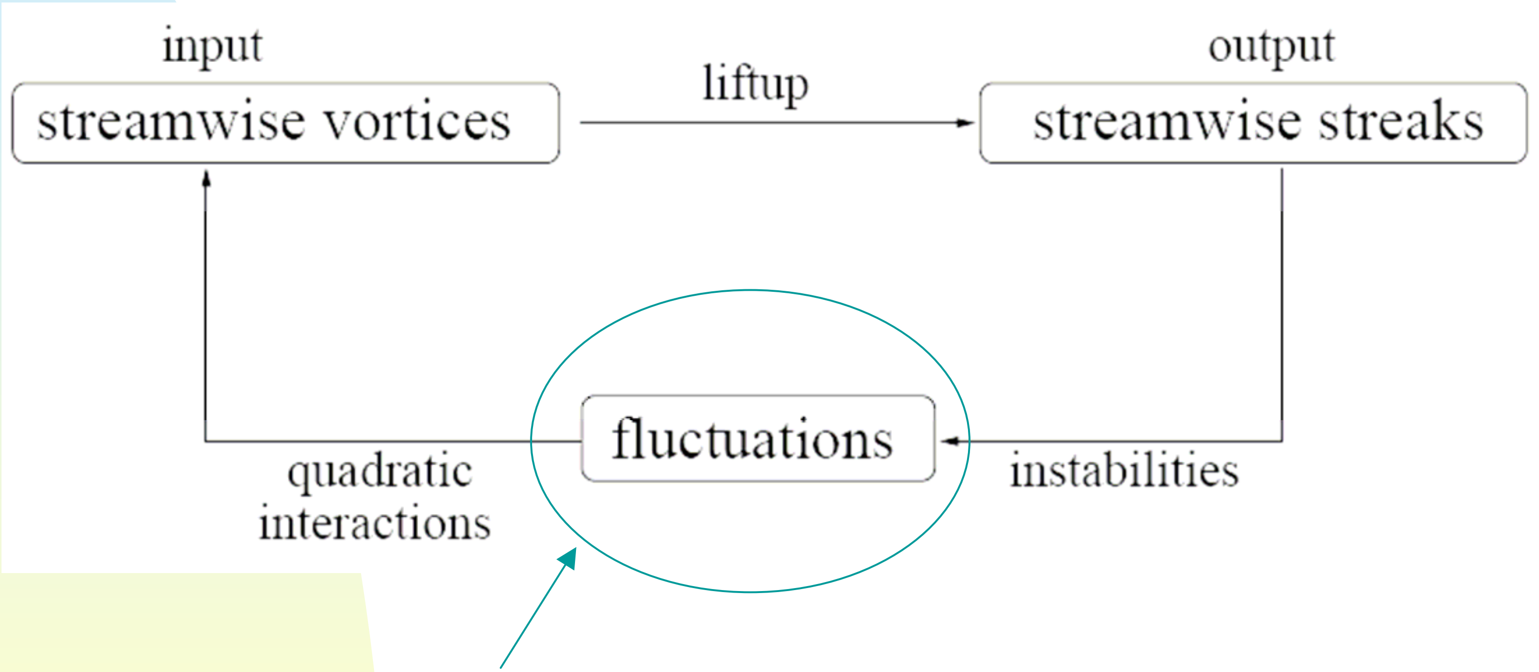
- Minimal defects represent the optimisation of the output (rather than the input, as customarily done in optimal perturbation analysis)

- They can lead to transition as efficiently as **sub**optimal perturbations

- Both minimal defects and suboptimals are more efficient for initial α's larger than zero (i.e. when the disturbances have small longitudinal dimensions)

- We have not closed the cycle yet ... <u>we need to include the feedback</u> !

Self sustained process



A NEW OPTIMISATION APPROACH:
i.e. maximising the feedback

The initial process of transition:

1. Algebraic growth of a $O(\varepsilon)$ travelling wave
2. Generation of weak streamwise vortices $O(\varepsilon^2)$ by quadratic interactions
3. Production of a strong streak $O(\varepsilon^2 Re)$ by lift-up
4. Wavy instability of the streak to close the loop

$$\begin{Bmatrix} U_0(y,z) \\ 0 \\ 0 \\ P_0(x) \end{Bmatrix} + \begin{Bmatrix} U(y,z,t) \\ V(y,z,t) \\ W(y,z,t) \\ P(y,z,t) \end{Bmatrix} + \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \\ p(x,y,z,t) \end{Bmatrix}$$

Inserting into the Navier-Stokes equations we obtain equations for

STREAMWISE-AVERAGED FLOW

$$\begin{aligned}
V_y + W_z &= 0, \\
U_t + V(U_0+U)_y + W(U_0+U)_z &= \frac{1}{Re}(U_{yy}+U_{zz}) - \left(\overline{vu}|_y + \overline{wu}|_z\right), \\
V_t &= -P_y + \frac{1}{Re}(V_{yy}+V_{zz}) - \left(\overline{vv}|_y + \overline{wv}|_z\right), \\
W_t &= -P_z + \frac{1}{Re}(W_{yy}+W_{zz}) - \left(\overline{vw}|_y + \overline{ww}|_z\right),
\end{aligned}$$

plus **FLUCTUATIONS**

$$\begin{aligned}
u_x + v_y + w_z &= 0, \\
u_t + (U_0+U)u_x + v(U_0+U)_y + w(U_0+U)_z &= -p_x + \frac{1}{Re}(u_{xx}+u_{yy}+u_{zz}), \\
v_t + (U_0+U)v_x &= -p_y + \frac{1}{Re}(v_{xx}+v_{yy}+v_{zz}), \\
w_t + (U_0+U)w_x &= -p_z + \frac{1}{Re}(w_{xx}+w_{yy}+w_{zz}),
\end{aligned}$$

Goal: maximise the feedback of the wave onto the rolls, i.e. maximise Reynolds stress terms or, more simply, the energy GAIN of the wave over a fixed time interval:

G = e(T)/e(0)

with
$$e(t) = \frac{1}{2} \int_{xyz} (u^2 + v^2 + w^2)\ dx\ dy\ dz$$

Tool: the usual Lagrangian optimisation, with adjoints

ADJOINT EQUATIONS:

$$\left\{\begin{aligned} V_y^\dagger + W_z^\dagger &= 0, \\ -U_t^\dagger &= \frac{1}{Re}(U_{yy}^\dagger + U_{zz}^\dagger) \; + \; G_U, \\ -V_t^\dagger + U^\dagger (U_0 + U)_y &= -P_y^\dagger + \frac{1}{Re}(V_{yy}^\dagger + V_{zz}^\dagger), \\ -W_t^\dagger + U^\dagger (U_0 + U)_z &= -P_z^\dagger + \frac{1}{Re}(W_{yy}^\dagger + W_{zz}^\dagger), \end{aligned}\right.$$

$$\left\{\begin{aligned} u_x^\dagger + v_y^\dagger + w_z^\dagger &= 0, \\ -u_t^\dagger - (U_0 + U) u_x^\dagger &= -p_x^\dagger + \frac{1}{Re}(u_{xx}^\dagger + u_{yy}^\dagger + u_{zz}^\dagger) \; + \; g_u, \\ -v_t^\dagger - (U_0 + U) v_x^\dagger + u^\dagger (U_0 + U)_y &= -p_y^\dagger + \frac{1}{Re}(v_{xx}^\dagger + v_{yy}^\dagger + v_{zz}^\dagger) \; + \; g_v, \\ -w_t^\dagger - (U_0 + U) w_x^\dagger + u^\dagger (U_0 + U)_z &= -p_z^\dagger + \frac{1}{Re}(w_{xx}^\dagger + w_{yy}^\dagger + w_{zz}^\dagger) \; + \; g_w, \end{aligned}\right.$$

with **SENSITIVITY FUNCTIONS**

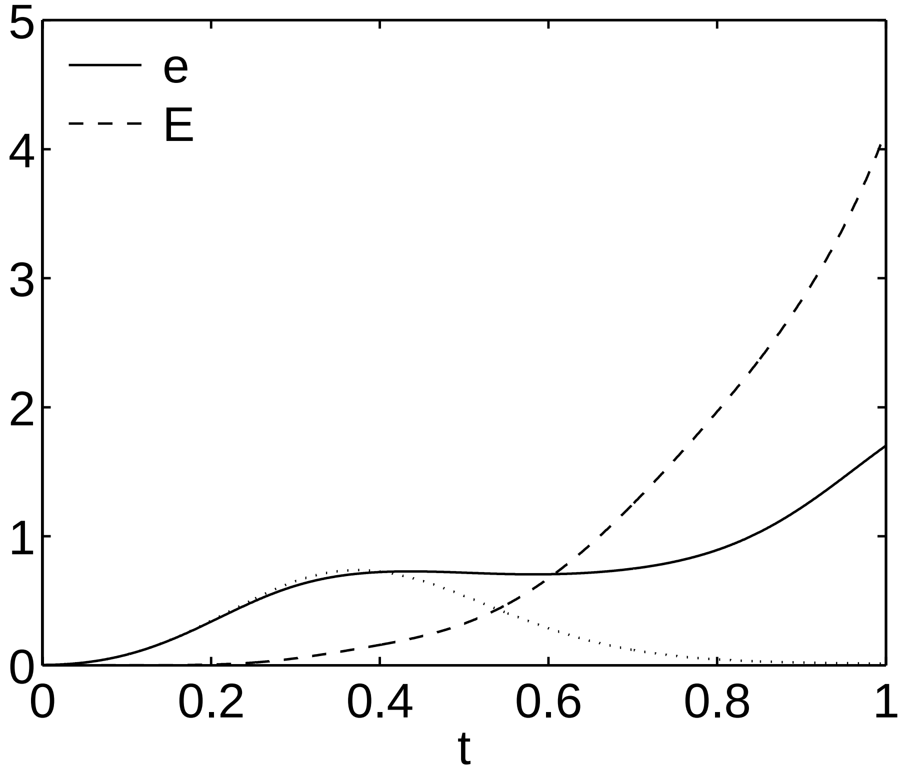
$$\begin{aligned} G_U &= -u^\dagger (u_x + v_x + w_x)_x + (u^\dagger v)_y + (u^\dagger w)_z, \\ g_u &= -U_y^\dagger v - U_z^\dagger w, \\ g_v &= -U_y^\dagger u - 2V_y^\dagger v + w(V_z^\dagger + W_y^\dagger), \\ g_w &= -U_z^\dagger u - 2W_z^\dagger w + v(V_z^\dagger + W_y^\dagger). \end{aligned}$$

Ansatz for the fluctuations:

$$q(x,y,z,t) = \tilde{q}(y,z,t)e^{+i\alpha x} + \tilde{q}^*(y,z,t)e^{-i\alpha x}$$

acceptable as long as in the first stage of the process the large scale coherent wave is but mildly affected by the behaviour of features occurrring at smaller space-time scales

Optimisation result for $Re_\tau = 150$, $\alpha = 1$, $T = 1$, $e_0 = 3x10^{-3}$



$$E(t) = \frac{1}{2}\int_{yz} (U^2 + V^2 + W^2)\ dy\ dz$$

Energy of the defect

$$e(t) = \frac{1}{2}\int_{xyz} (u^2 + v^2 + w^2)\ dx\ dy\ dz$$

Energy of the wave

Parametric study

Wave

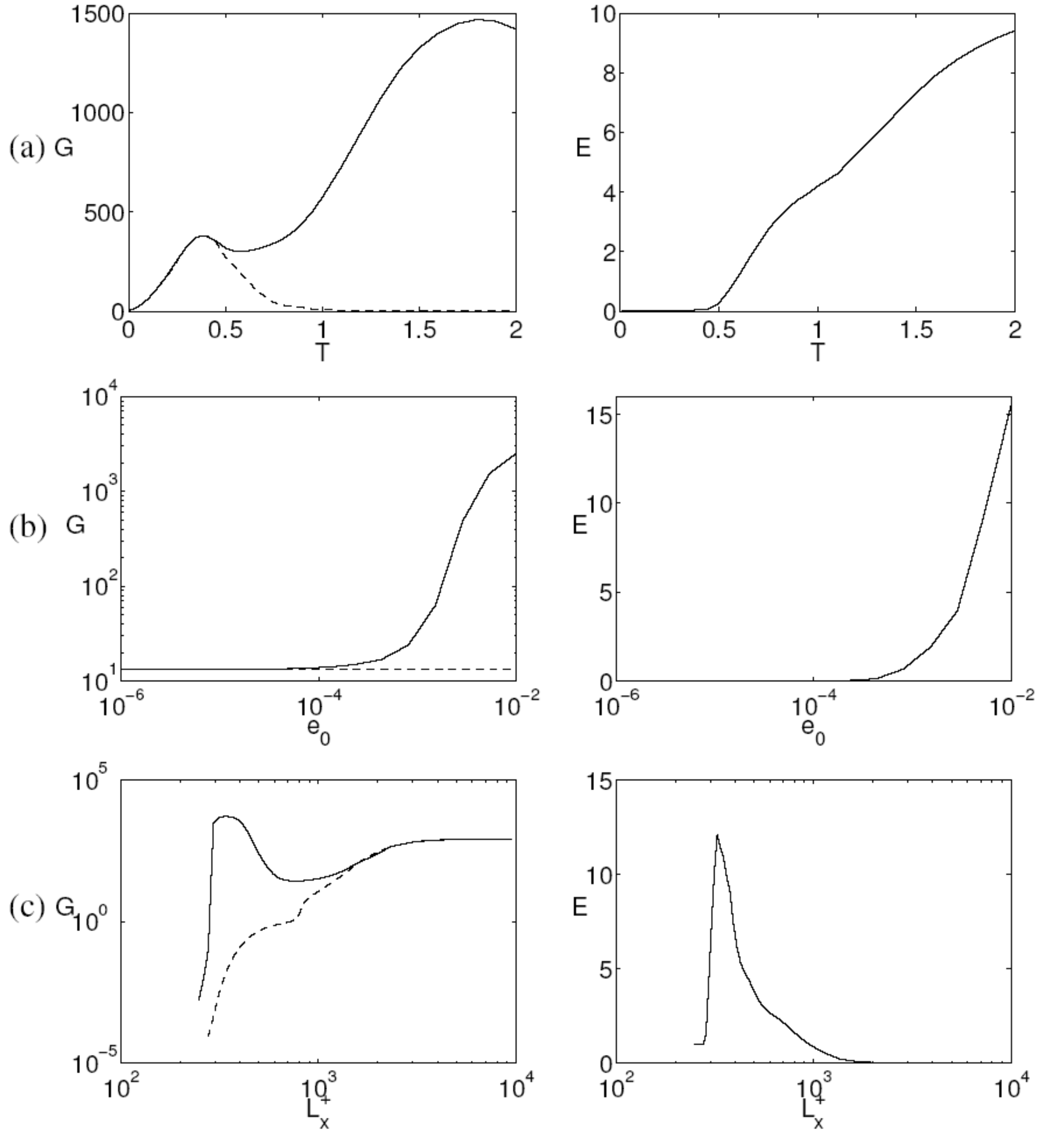


Figure 3. Parametric study. (a) $\alpha = 1$; $e_0 = 3 \times 10^{-3}$. (b) $\alpha = 1$; $T = 1$. (c) $T = 1$; $e_0 = 10^{-3}$. The dashed lines represent the linear case, in the absence of mean flow defect.

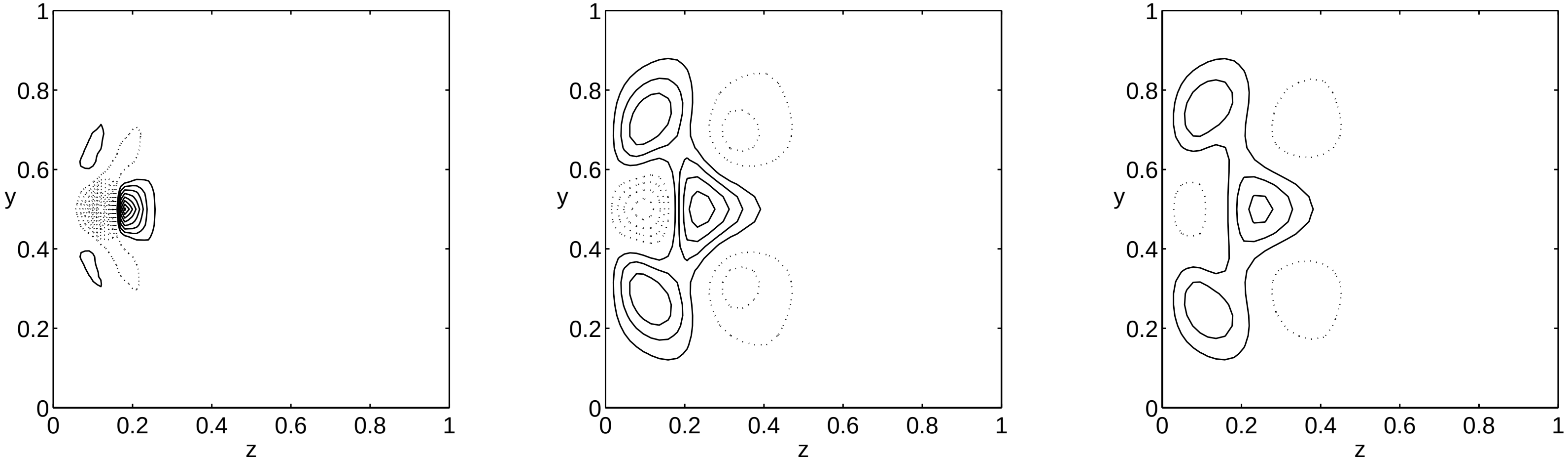
Defect

Two comments:

α small → defect is not created
(<u>classical optimal perturbation are of little use</u>)

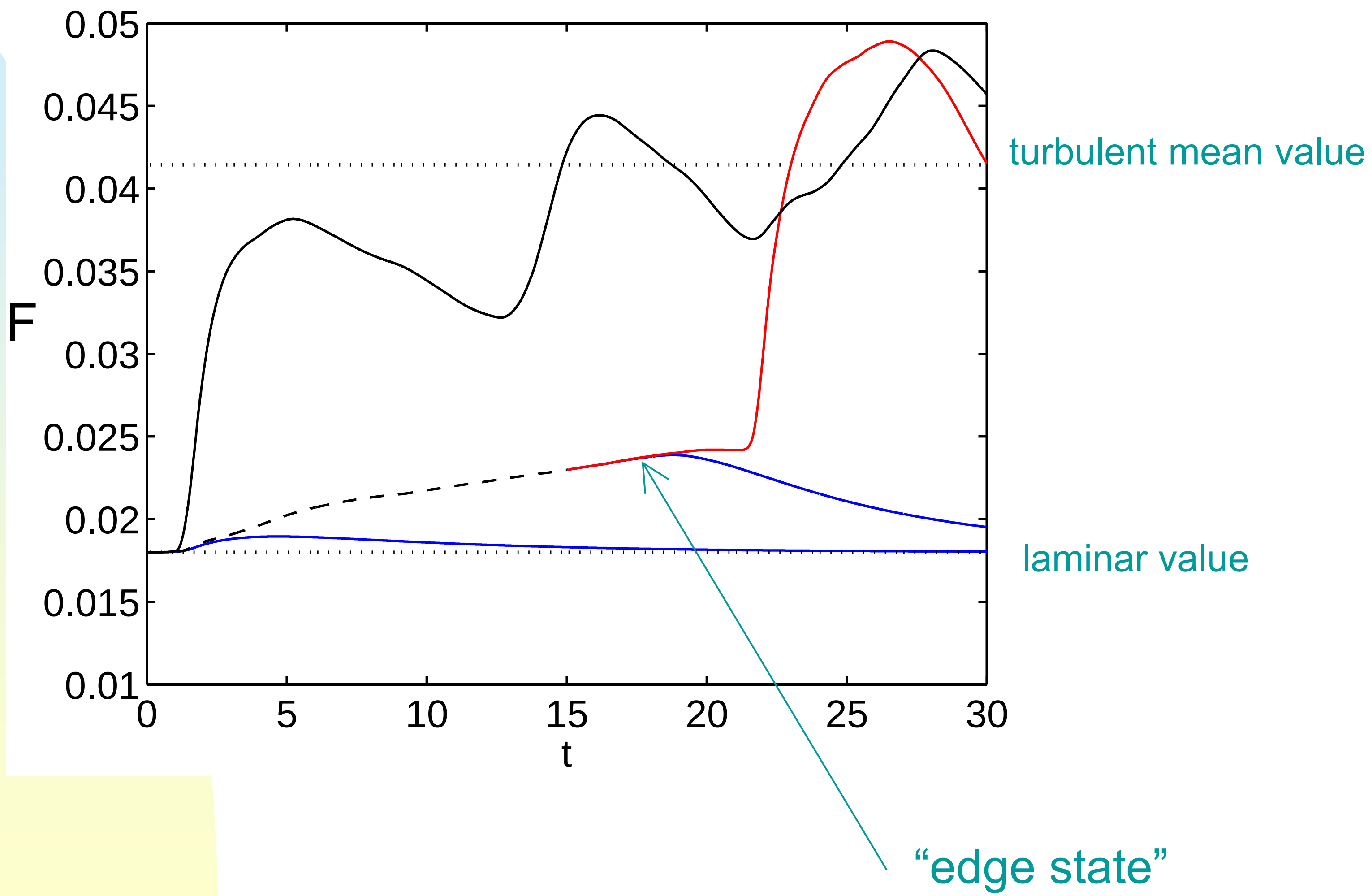
α large → cut off length → “minimal channel”

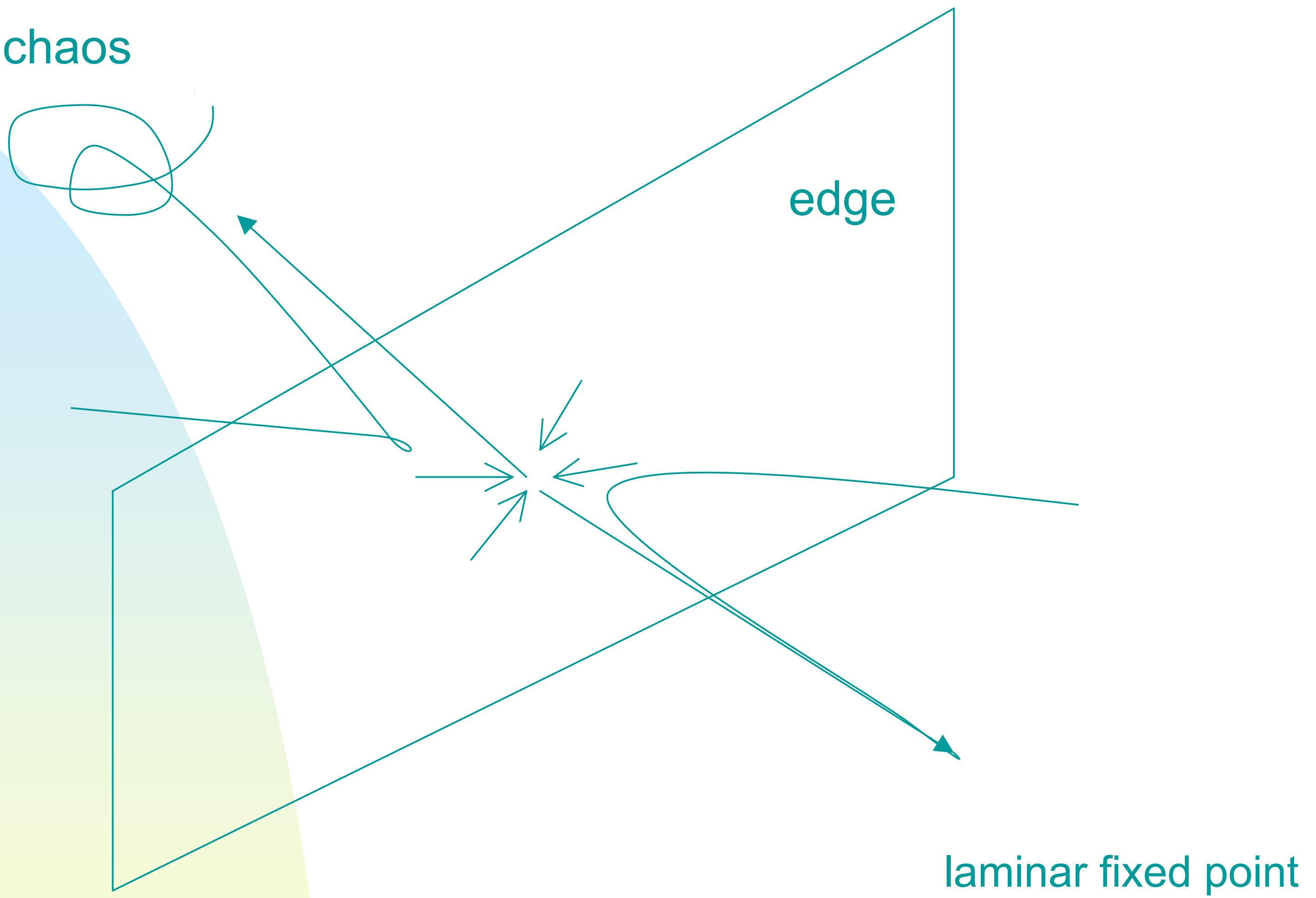
Comparison between the minimal defect (left) and the optimal streak at t = 0.6 (centre)



DNS at t = 0.6

Evolution of the skin friction in time





THE EDGE MEDIATES LAMINAR-TURBULENT TRANSITION,
it is the stable manifold of the hyperbolic fixed point

The edge state solution

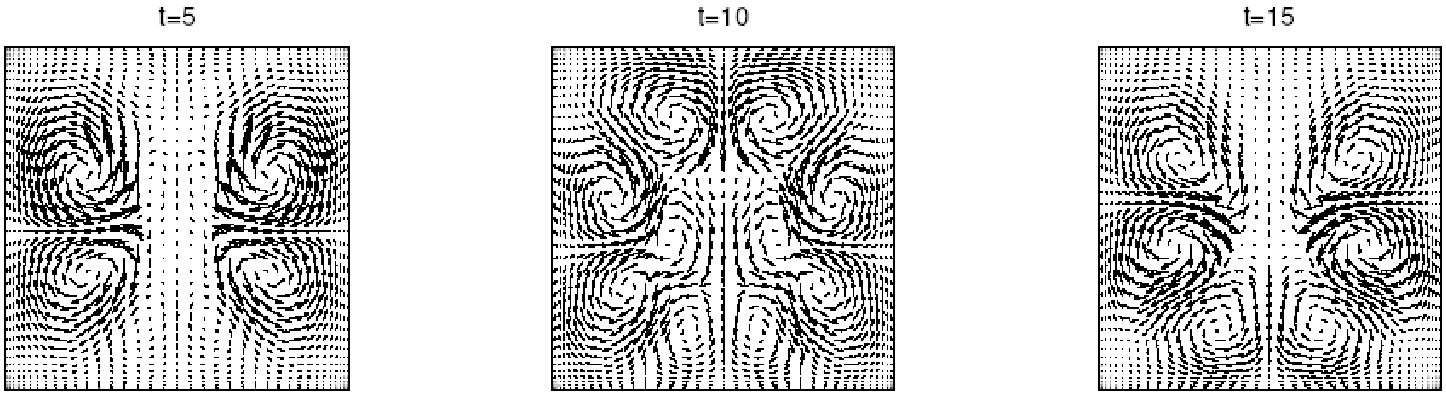
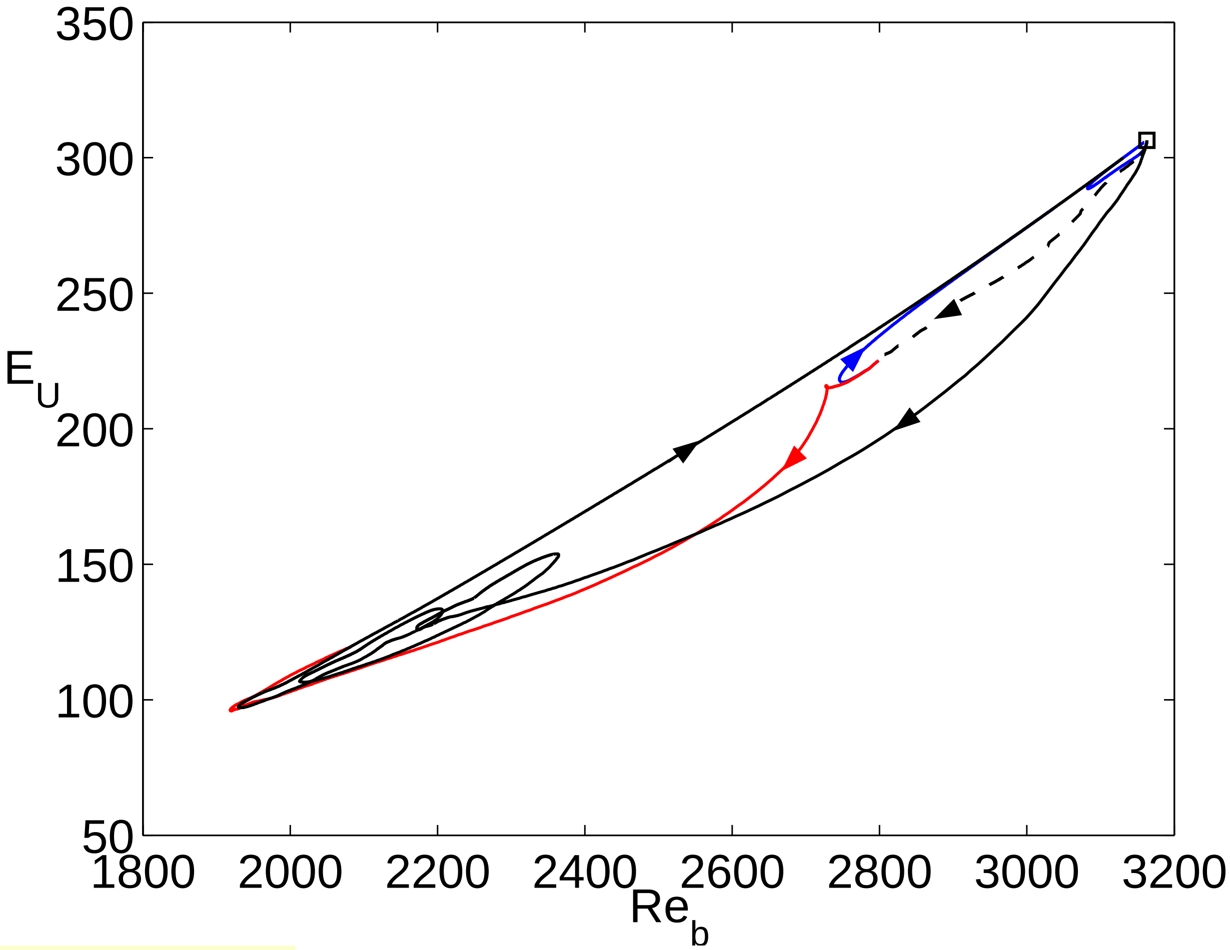
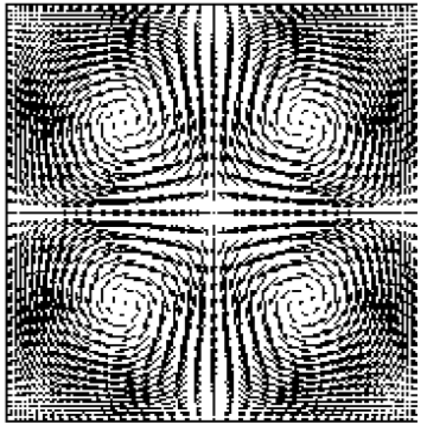


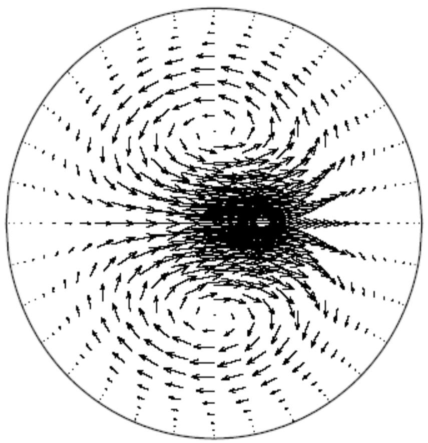
Figure 6. Instantaneous secondary flows of the solution on the edge.

Simplified phase diagram

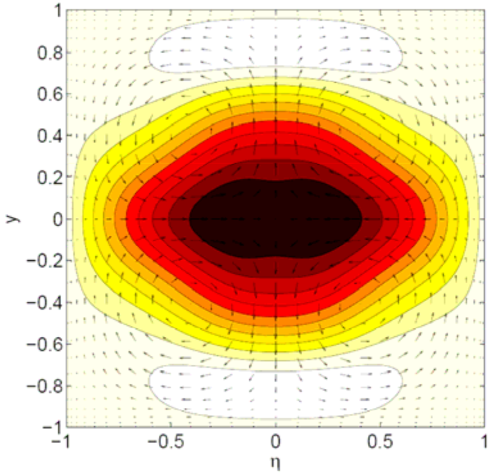


Global optimal perturbations

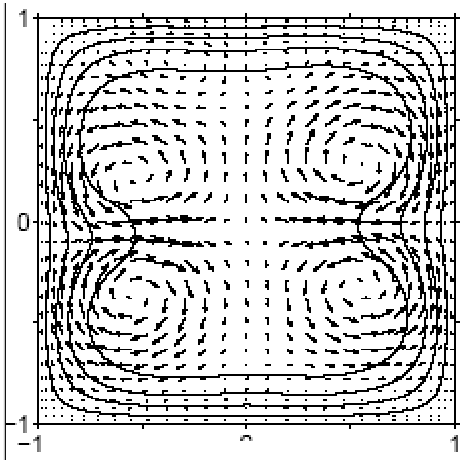


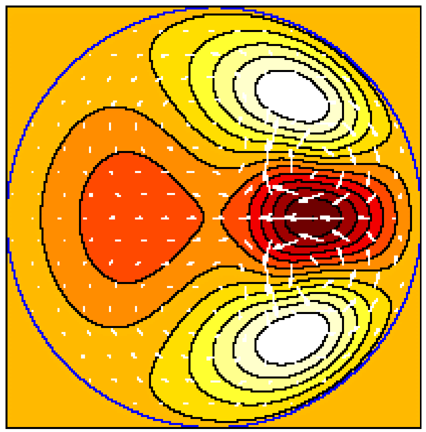


saddle node state
(Wedin *et al*. 2007)



“edge states”





Partial conclusions 4

- A model has been developed which closes the loop and combines transient growth, exponential amplification, defects, the "exact coherent states" and the SSP
- In the very initial stages of transition we need growth of a wave and feedback onto the mean flow, to create a vortex. The lift-up then generates the streaks, which break down and re-generate the wave.
- The initial conditions in the form of elongated streaks (the optimal disturbance) is inefficient at yielding transition because it fails at modifying the mean flow
- Our new simplified model yields a flow which sits initially on a trajectory directed towards the saddle point on the edge surface; direct simulations confirm the suitability of the proposed model
- The basic flow structure of the edge state is a vortex pair, with upwash near a wall bisector
- LINEAR equations can go a long way !!!