

Görtler vortices: are they amenable to local eigenvalue analysis?

A. BOTTARO ^{a*}, P. LUCHINI ^b

ABSTRACT. – It is often quoted that Görtler vortices cannot be described by a local eigenvalue analysis. In this work, by using the inverse of the Görtler number as a small expansion parameter, we derive an asymptotic sequence continuable to all orders which is similar, in principle, to the one that justifies the application of the Orr-Sommerfeld equation to two-dimensional boundary-layer instabilities. Existing local theories from the literature can be framed within the leading term of this expansion; however, none of the heuristically proposed non-parallel corrections fully captures the next higher term. We show that, when this term is included, locally computed growth rates quickly collapse onto those obtained from numerical simulations of the parabolic linear stability equations, with initial conditions applied at the leading edge. The Görtler number (or, equivalently, the downstream distance) beyond which this non-parallel local theory is found out to be accurate encloses the commonly recognized experimental range. The small Görtler number (short distance) effect of initial conditions is described in a companion paper. © Elsevier, Paris.

1. Introduction

The original analysis (by Görtler himself and others) of the curvature-excited streamwise vortices that have become known as Görtler vortices was based on a heuristic extension of the theory of parallel centrifugal instabilities of the Taylor and Dean type.

However, the possibility of applying a quasi-parallel analysis to Görtler vortices has been questioned by Hall (1983) and ever since, because of the difficulties in identifying a suitable scaling parameter that could allow the theory to be cast into the form of a proper asymptotic expansion continuable to all orders.

Our concern here is with formulating the local linear stability theory in a systematic manner, which could enable us to provide mathematically consistent non-parallel corrections, and so to obtain accurate pointwise information of amplification factors and mode shapes for the vortices of arbitrary spanwise wavenumber β at any given (large) streamwise distance from the leading edge without resorting to full numerical simulations of the parabolic boundary layer equations.

A wealth of parallel or quasi-parallel theories populates the literature; since they are comprehensively reviewed by Floryan (1991) and Saric (1994), we only cite chronologically the major contributions, with the preliminary remark that until 1983 all analyses used the local mode approach. Görtler (1941) started by considering the stability of a strictly parallel Blasius base flow along a curved surface of asymptotically large radius of curvature, and established the parameters and the equations ruling the neutral stability of the disturbances. Solutions to these equations were provided by him and, later, by Hämmerlin (1955, 1956) and Smith (1955). Wide discrepancies in the prediction of the neutral curve were already noted at the time. Herbert (1976) extended the formulation to

* Correspondence and reprints.

^a Institut de Mécanique des Fluides de Toulouse, UMR 5502, Université Paul-Sabatier, 118 route de Narbonne, 31062 Toulouse, France.
E-mail: bottaro@imft.fr

^b Dipartimento di Ingegneria Aerospaziale, Politecnico di Milano, Via Lambruschini 15, 20168 Milano, Italy.
E-mail: luchini@aero.polimi.it

allow for finite variable radius of curvature R and found a strong dependence of the neutral curve on R in the range of low spanwise wavenumbers. Floryan and Saric (1979, 1982) tried to deal with the non-parallelism of the base Blasius flow by including into the disturbance equations terms like U_X (streamwise derivative of the base flow) and V (wall-normal base velocity). They derived what became known as the *locally non-parallel stability equations*: results, however, were not substantially different from those of the locally parallel approximation of Görtler (1941), except for low β . In 1982 Hall developed a WKB expansion in the wall-normal direction, valid when the Görtler number G and wavenumber β tend simultaneously to infinity in such a way that $G = O(\beta^2)$, and thus showed that at the abscissa $X = 0.114R^{-2}(\nu/U_\infty)^{-5}\beta^{-8}$ there is a transition from amplification to damping of the leading Görtler mode. He thus obtained an analytic expression for the upper branch of the neutral curve, which is in excellent agreement with previously published numerical solutions for large β 's.

In 1983 Hall took a different look at the linear stability equations. These equations are a parabolic set of partial differential equations which do not – a priori – allow a separation of variables. He solved them by a streamwise marching numerical technique and pointed out that a unique neutral curve cannot be defined for the following reasons:

1. the definition of amplification factor depends on the physical quantity being monitored;
2. such a quantity (e.g., the disturbance energy) has a different streamwise development as function of the initial conditions imposed and the initial position at which the marching procedure is initiated.

For the initial conditions chosen by Hall, and for his definition of amplification factor, there was a poor agreement in neutral curve prediction with previous local theories. This seemed to settle the issue, although local theories kept on being used because of their practicality in delivering pointwise information, and capacity of rapidly providing N factors for transition predictions. Later on, it was shown by Day *et al.* (1990) that the agreement between marching and local theories is quite good sufficiently far downstream for some choices of the amplification factor. They argued that the “downstream asymptotic” behaviour could be approximately described by a local analysis. Goulpié *et al.* (1996) demonstrated that the disagreement between neutral curves reported by Hall originates from the choice of initial conditions used in the marching calculations. When initial conditions with the shape of streamwise vortices were chosen, good agreement was obtained between marching and local predictions, at least for X sufficiently large.

The present study takes a fresh look at local theories, with the hope of arriving at a consistent formulation. It seems, in fact, odd that local theories – while widely accepted and used, for example, in defining the Orr-Sommerfeld (O-S) problem for the Blasius flow – are so controversial when applied to the Görtler problem. A convenient way to derive local theories is to adopt a multiple-scale formulation. The first such analysis for the O-S problem was carried out by Bouthier (1972, 1973); to leading order he recovered the O-S equation, and through the equation at the next higher order he was able to provide a correction to the local results. From his analysis, non-parallelism seemed to have a strong effect on the neutral curve, acting as a destabilizing agent. Gaster (1974) performed a comparable analysis with an iterative method and concluded that neutral curves are almost unaffected by a non-parallel corrections, except near the critical conditions. Although these results were not supported by an agreement with the experiments available at the time, recent carefully controlled measurements by Klingmann *et al.* (1993) established that non-parallelism has a minor influence on the neutral conditions. The apparent disagreement between Bouthier (1972) and Gaster (1974) originates uniquely from the way the disturbance growth was measured (see, e.g., the discussion given by Bertolotti 1991).

In this work we set out to demonstrate that a local analysis can also be consistently formulated for the Görtler problem, and that non-parallel effects can be appropriately accounted for through a WKB expansion continuable to all orders, which includes at leading order some of the local theories briefly cited earlier.

2. Mathematical description of the instability

We consider the flow over a concave surface of constant, asymptotically large radius of curvature R . The equations that govern the spatial evolution of a small disturbance are:

$$(1.1) \quad u_X + v_Y + w_Z = 0,$$

$$(1.2) \quad Uu_X + uU_X + Vv_Y + vV_Y = u_{YY} + u_{ZZ},$$

$$(1.3) \quad Uv_X + uV_X + Vv_Y + vV_Y + 2G^2Uu + p_Y = v_{YY} + v_{ZZ},$$

$$(1.4) \quad Uw_X + Vw_Y + p_Z = w_{YY} + w_{ZZ},$$

with the Görtler number G defined by $G^2 = l\text{Re}/R$ and $\text{Re} = (U_\infty l/\nu)^{1/2} \gg 1$. Here the longitudinal coordinate X is scaled with the reference length l that appears in the definitions of the Görtler and Reynolds numbers, the longitudinal velocity U and its perturbation u are scaled with the free stream velocity U_∞ , the wall normal and spanwise coordinates Y, Z are scaled with $\delta = (\nu l/U_\infty)^{1/2}$, and the corresponding velocities V, v and w are scaled with $\text{Re}^{-1}U_\infty$. Appropriate boundary conditions are

$$(2.1) \quad u = v = w = 0 \quad \text{at } Y = 0, \text{ and}$$

$$(2.2) \quad u = w = p = 0 \quad \text{for } Y \rightarrow \infty.$$

Equations (1-2) were given for the first time by Floryan and Saric (1979). Details of their derivation can be found, e.g., in Bottaro and Luchini (1996). There is theoretical (Herron 1985, 1991) and experimental (Swearingen and Blackwelder 1987) evidence that the primary instability is stationary and hence time-derivative terms have been omitted. The base flow U, V is determined by the same two-dimensional boundary layer equations that hold on a flat plate.

Because of translational invariance along Z , equations (1) admit solutions of the form:

$$(3) \quad (u, v, w, p) = (\tilde{u}(X, Y) \cos \beta Z, \tilde{v}(X, Y) \cos \beta Z, \tilde{w}(X, Y) \sin \beta Z, \tilde{p}(X, Y) \cos \beta Z),$$

with β real spanwise wavenumber. A more compact set of disturbance equations can be obtained through elimination of the pressure term and w and reads (after dropping tildes)⁽⁰⁾:

$$(4.1) \quad u_{YY} - Vv_Y - Uu_X - [\beta^2 + U_X]u - U_Yv = 0,$$

$$(4.2) \quad v_{YYYY} - Vv_{YYY} - Uv_{XYY} - [2\beta^2 + V_Y]v_{YY} + (\beta^2V + U_{XY})v_Y + [\beta^2U + U_{YY}]v_X \\ + [\beta^4 + \beta^2V_Y + U_{XYY}]v + V_Xu_{YY} + 2U_Xu_{XY} + 2U_{XY}u_X \\ + [U_{XXY} + 2\beta^2G^2U + \beta^2V_X]u = 0,$$

⁽⁰⁾ Alternative formulations based on different dependent variables are possible, and a particularly convenient one which leads to a two-equation set on u and Ξ (a modified streamwise vorticity) is described and used in the companion paper by Luchini and Bottaro (1998), from now on referred to as LB.

with boundary conditions

$$(5) \quad u = v = v_Y = 0 \quad \text{at } Y = 0 \text{ and } Y \rightarrow \infty,$$

and initial conditions

$$(6) \quad u = u_0 \text{ and } v = v_0 \quad \text{at } X = X_0.$$

This form of the stability equations was first given by Hall (1983). The parameters governing this problem are the Görtler number G , the wavenumber β , and the initial position X_0 at which the perturbations are provided. Furthermore, the initial perturbation distributions u_0 and v_0 play a crucial role in the instability development in the proximity of X_0 ; as a consequence Hall showed that the concept of a unique neutral curve is not tenable. To monitor the development of the instability one typically defines a perturbation energy, for example

$$(7) \quad E(X) = \int u^2(X, Y) dY$$

and introduces a dimensionless growth rate $\tilde{\sigma}$ as

$$(8) \quad \tilde{\sigma} = \frac{X}{2E} \frac{dE}{dX}.$$

The position of “neutral” stability is defined by the value of X for which $\tilde{\sigma}$ vanishes: since a Görtler number $G_X = GX^{3/4}$ and a wave number $\beta_X = \beta X^{1/2}$ are associated to each X , a “neutral stability curve” in the (β_X, G_X) plane can be constructed. Goulpié *et al.* (1996) have, however, demonstrated that even the definition of energy has profound repercussions on the location of the neutral points; this supports Hall’s argument that a unique marginal curve can not be defined.

What *can* be uniquely defined in the (β_X, G_X) plane are the different isolines of fixed amplification rate. Such isolines are unique if G_X is large enough, and can be obtained indifferently from marching or local approaches (Day *et al.*, 1993). The possibility of having at any given large G_X (which will be quantified) accurate local information of growth rates and mode shapes for all β_X ’s is attractive because it allows local considerations of transition thresholds (e.g., calculations of N factors). In addition it sets the stage for the receptivity analysis described by LB which relies on the downstream existence of modes.

3. Marching solutions of the linearized stability equations

In order to demonstrate the existence of modes far enough downstream of the leading edge of the curved plate, and to establish the meaning of “far enough”, we give an example of solutions of the marching equations for a variety of initial conditions. Similar calculations have been performed by Day *et al.* (1990).

Hall’s equations (4-6) are marched from $X = X_0 = 0$ to $X = X_f = 1$, where the value of G is set to 10. Notice that in this case the length scale is based on the final point, rather than the point from which the marching is initiated as is usually done when the computation is not initiated at $X = 0$. The parabolic set of equations is solved for three values of β (0.5, 1 and 2), with three different initial conditions at X_0 . In figure 1 we report the evolution of the energy (on the left) and of $\tilde{\sigma}$ (right) with X . The value attained by the energy at $X = X_f$ is clearly a function of the initial condition; the curves displaying the amplification factors, on the other hand, coalesce when $X > 0.7$ ($G_X > 7.6$) for each β . Likewise, the disturbance shapes collapse onto

one another. This indicates conclusively the existence of *modes* (and hence the applicability of the parallel flow approximation, Timoshin 1990) for Görtler numbers larger than about seven.

Incidentally, it should be pointed out that seven is not an extraordinarily large number, and that recent nonlinear calculations (Bottaro *et al.*, 1996) and experiments (Pexieder 1996) – both performed in a fairly standard disturbance environment – show that the quasi-exponential growth characteristic of the linear amplification phase persists up to G_X of about fifteen.

4. Multiple-scale analysis

A multiple-scale approximation can be set up when the disturbance behaves as a fast exponential $\exp[\varphi(X)/\varepsilon]$. The scale factor ε can be determined by imposing that the dominant terms in equations (1) be of comparable magnitudes. For example, by imposing that $vU_Y = O(Uu_X)$ from equation (1.2) we obtain that $v \approx u/\varepsilon$; similarly by imposing that $Uv_X = O(G^2Uu)$ (equation 1.3) we find that $v \approx G^2\varepsilon u$, so that the scaling parameter $\varepsilon = G^{-1}$ emerges clearly. Asymptotic expansions can then be set up in the form:

$$(9.1) \quad u = e^{\varphi(X)/\varepsilon} \varepsilon [u_0(X, Y) + \varepsilon u_1(X, Y) + \dots],$$

$$(9.2) \quad v = e^{\varphi(X)/\varepsilon} [v_0(X, Y) + \varepsilon v_1(X, Y) + \dots].$$

Correspondingly, the derivative of, say, v with respect to X shall be given by a series of the form

$$(10) \quad v_X = e^{\varphi(X)/\varepsilon} [\varepsilon^{-1} \sigma v_0 + (v_{0.X} + \sigma v_1) + \varepsilon (v_{1.X} + \sigma v_2) + \dots],$$

with the definition $\sigma(X) = d\varphi/dX$. On inserting these expressions into equations (4) and collecting like powers of ε , the following hierarchy of equations is obtained:

$O(\varepsilon^{-1})$:

$$(11.1) \quad -\sigma U v_{0.YY} + (\beta^2 U + U_{YY}) \sigma v_0 + 2\beta^2 U u_0 = 0,$$

$O(1)$:

$$(11.2) \quad -\sigma U u_0 - U_Y v_0 = 0,$$

$$(11.3) \quad -\sigma U v_{1.YY} + (\beta^2 U + U_{YY}) \sigma v_1 + 2\beta^2 U u_1 = -v_{0.YYY} + G_1(Y) + U v_{0.XYY} - (\beta^2 U + U_{YY}) v_{0.X},$$

$O(\varepsilon)$:

$$(11.4) \quad -\sigma U u_1 - U_Y v_1 = -u_{0.YY} + \beta^2 u_0 + G_2(Y) + U u_{0.X},$$

etc.

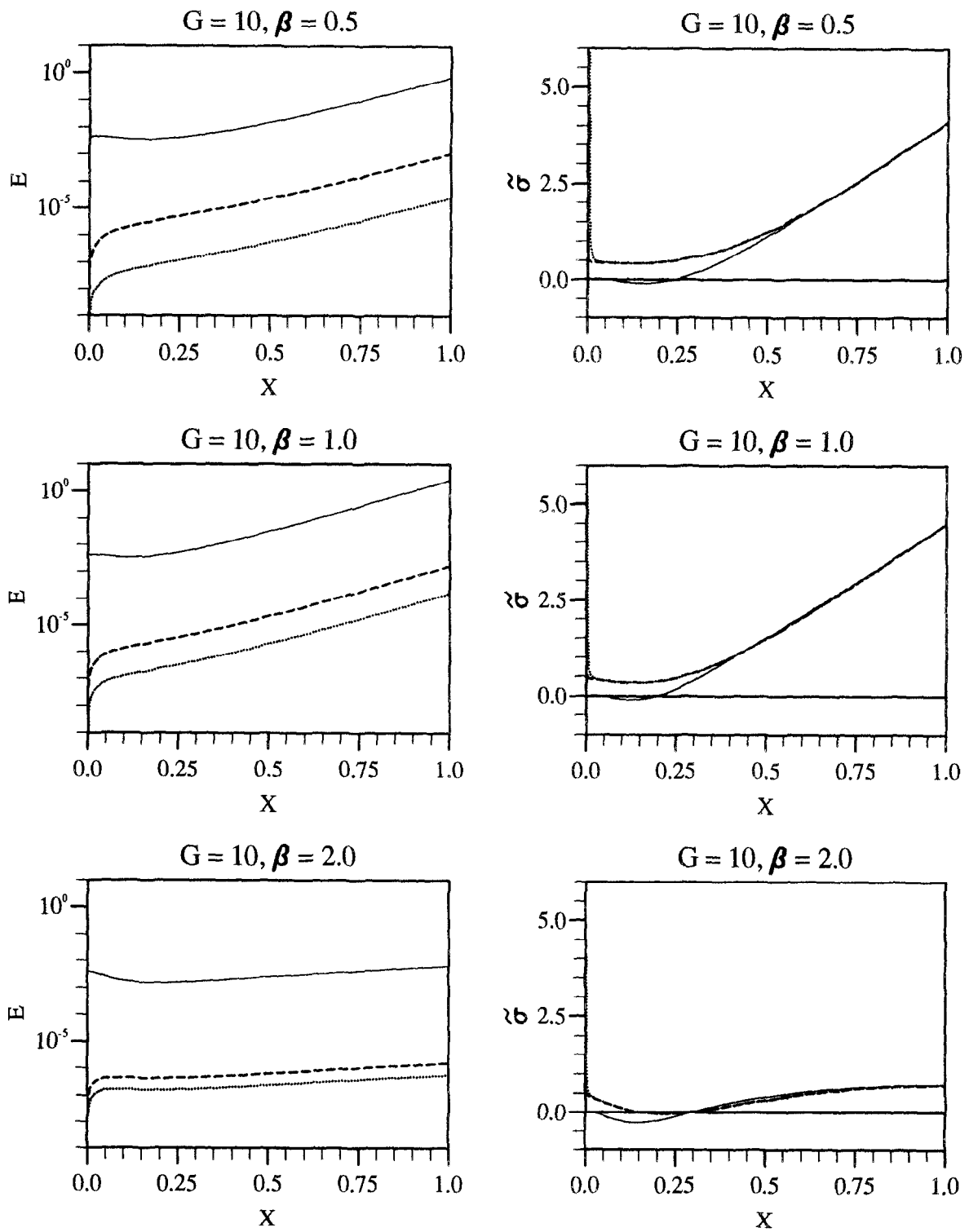


Fig. 1. - Left: energy curves as function of X for $G = 10$ and, from the top, $\beta = 0.5, 1$ and 2 , from the marching equations.

Right: corresponding amplification factors. The three initial conditions chosen are:

— $u = y \exp(-4y), \psi = w_y - v_z = 0;$

..... $u = 1$ on the first point above the wall, $u = 0$ everywhere else; $\psi = 0;$

- - - - $\psi = 10^{10}$ on the first point above the wall, $\psi = 0$ everywhere else; $u = 0.$

with

$$(12.1) \quad G_1(Y) = Vv_{0,YYY} + (2\beta^2 + V_Y)v_{0,YY} - (\beta^2V + U_{XY})v_{0,Y} \\ - (\beta^4 + \beta^2V_Y + U_{XY})v_0 - 2\sigma U_X u_{0,Y} - 2\sigma U_{XY}u_0$$

and

$$(12.2) \quad G_2(Y) = Vu_{0,Y} + U_Xu_0.$$

However, a difficulty, common to many other boundary layer perturbation expansions, is encountered: the expansion given above is not uniformly valid in Y , because the highest Y -derivatives are lost at leading order and with them disappears the possibility of enforcing a certain number of boundary conditions. Exactly the same difficulty turns up, for instance, in the derivation of the O-S equation as applied to non-parallel problems (Bouthier 1972, 1973; Gaster 1974; Saric and Nayfeh 1975; Itoh 1986), and, in fact, in almost all classical applications of singular perturbation theory (see, e.g., Van Dyke 1975). In general, when an asymptotic expansion is not uniformly valid, there is no possibility of arranging it as a series of powers of the expansion parameter ε (in Van Dyke's words) which are of smaller and smaller orders without being simple powers. Such a sequence is evidently not unique, because a higher-order function of ε can be summed into a lower-order one without disturbing the ordering of the sequence. Several examples are given in the introduction of Van Dyke's book.

One possibility to determine an asymptotic sequence from the singular perturbation expansion of a differential-equation problem – whose straightforward expansion in a power series would lead to lowering the order of the equations – is the multiple-deck approach. It consists in distinguishing the spatially limited zones in which the regular expansion loses validity and treating them separately after rescaling the dependent and independent variables, constructing then a compound solution (which in general does not behave any more as a simple power of the expansion parameter) through the theory of matched asymptotic expansions. An example of this way of proceeding is Tollmien's classical study of the stability of the boundary layer to two-dimensional perturbations, obtained by matching the solution of the Rayleigh equation with a separate treatment of the near-wall and critical-layer regions (e.g. Drazin and Reid 1981). This is the preferred route when an exact or approximate analytical solution can be found for each of the distinguished zones.

In the present problem, an example of the multiple-deck technique is provided by Hall (1982). By applying a WKB approximation in the Y -direction, he found that in the limit of large β the eigenfunctions are only appreciably nonzero in the vicinity of the point where UU_Y is maximum. Therefore the problem was solved by analysing a rescaled neighbourhood of this point independently of the wall (which is pushed away to minus infinity). Hall was thus able to determine the large- β behaviour of the neutral Görtler number analytically as

$$(13) \quad G^2 = 2.95\beta^4[1 + 0.96\beta^{-1} + O(\beta^{-2})].$$

Unfortunately, for the β of maximum amplification – the one expected to be observed in an experiment – Hall's (1982) approximation is not applicable.

On the other hand, when a numerical solution is sought, it is equivalent and much more convenient to construct a uniformly valid approximate differential equation by combining together all the terms of the original problem that become dominant in at least some zones. Once it is accepted that the result cannot be a power series but only an asymptotic sequence of *a priori* unspecified functions, it does not hurt, of course, if a non-dominant term (one that is of a higher order in the formal expansion in powers of the parameter) appears, and therefore the differential equation, just as the asymptotic sequence itself, is not unique. An example of this approach is the application of the Orr-Sommerfeld equation to boundary layer problems (see, e.g., Gaster's 1974 discussion of the order of magnitude of different terms). It may be instructive to identify two "extreme" formulations.

The first model does not contain V nor X -derivatives of the base flow in the equations at leading order (equations 14.1-2) and is analogous to Görtler's original formulation. The equations at higher order (equations 14.3-4) provide a correction on the leading order modes. We call this *model 1* and it is given by:

model 1:

$$(14.1) \quad -\sigma U u_0 - U_Y v_0 + \varepsilon(u_{0,YY} - \beta^2 u_0) = 0,$$

$$(14.2) \quad -\sigma U v_{0,YY} + (\beta^2 U + U_{YY})\sigma v_0 + 2\beta^2 U u_0 + \varepsilon v_{0,YYYY} = 0,$$

$$(14.3) \quad -\sigma U u_1 - U_Y v_1 + \varepsilon(u_{1,YY} - \beta^2 u_1) = U u_{0,X} + G_2(Y),$$

$$(14.4) \quad -\sigma U v_{1,YY} + (\beta^2 U + U_{YY})\sigma v_1 + 2\beta^2 U u_1 + \varepsilon v_{1,YYYY} = U v_{0,XY} - (\beta^2 U + U_{YY})v_{0,X} + G_1(Y).$$

In the second model we include all the terms that can possibly be promoted from the next higher order. The resulting equations (14.5-6) are *almost* the same as those given by Floryan and Saric (1982) ⁽¹⁾:

model 2:

$$(14.5) \quad -\sigma U u_0 - U_Y v_0 + \varepsilon[u_{0,YY} - \beta^2 u_0 - G_2(Y)] = 0,$$

$$(14.6) \quad -\sigma U v_{0,YY} + (\beta^2 U + U_{YY})\sigma v_0 + 2\beta^2 U u_0 + \varepsilon[v_{0,YYYY} - G_1(Y)] = 0,$$

$$(14.7) \quad -\sigma U u_1 - U_Y v_1 + \varepsilon[u_{1,YY} - \beta^2 u_1 - V u_{1,Y} - U_X u_1] = U u_{0,X},$$

$$(14.8) \quad -\sigma U v_{1,YY} + (\beta^2 U + U_{YY})\sigma v_1 + 2\beta^2 U u_1 + \varepsilon[v_{1,YYYY} - V v_{1,YY} - (2\beta^2 + V_Y)v_{1,YY} + (\beta^2 V + U_{XY})v_{1,Y} + (\beta^4 + \beta^2 V_Y + U_{XY})v_1 + 2\sigma U_X u_{1,Y} + 2\sigma U_{XY} u_1] = U v_{0,XY} - (\beta^2 U + U_{YY})v_{0,X}.$$

Equations (14.5-6) are sometimes called the local, non-parallel equations. There is however no non-parallel correction involved since the eigenvalue σ is calculated to the same order of approximation by equations (14.1-2) and (14.5-6).

Two additional formulations arise if the base flow is expressed in similarity coordinates X and $\eta = Y/X^{1/2}$. Since $\frac{\partial}{\partial X} \Big|_Y = \frac{\partial}{\partial X} \Big|_\eta - \frac{Y}{2X} \frac{\partial}{\partial Y} \Big|_X$, the adoption of similarities variables is tantamount to subtracting an $O(\varepsilon)$ Y -derivative term from the order-0 equations and adding the same term to the order-1 equations. We thus obtain from models 1 and 2:

⁽¹⁾ The difference with Floryan and Saric is due to the inverted order of normal mode substitution and differentiation with respect to X . Local results obtained with Floryan and Saric's equations are indistinguishable to plotting accuracy from those obtained with the equations proposed here for all β 's (see also Bottaro *et al.*, 1996).

model 3:

$$(14.9) \quad -\sigma U u_0 - U_Y v_0 + \varepsilon \left(u_{0,YY} - \beta^2 u_0 + \frac{Y}{2X} U u_{0,Y} \right) = 0,$$

$$(14.10) \quad -\sigma U v_{0,YY} + (\beta^2 U + U_{YY}) \sigma v_0 + 2\beta^2 U u_0 \\ + \varepsilon \left[v_{0,YYY} + \frac{Y}{2X} U v_{0,YYY} + \frac{1}{X} v_{0,Y} - (\beta^2 U + U_{YY}) \frac{Y}{2X} v_{0,Y} \right] = 0,$$

$$(14.11) \quad -\sigma U u_1 - U_Y v_1 + \varepsilon \left(u_{1,YY} - \beta^2 u_1 + \frac{Y}{2X} U u_{1,Y} \right) = U u_{0,X} + \frac{Y}{2X} U u_{0,Y} + G_2(Y),$$

$$(14.12) \quad -\sigma U v_{1,YY} + (\beta^2 U + U_{YY}) \sigma v_1 + 2\beta^2 U u_1 \\ + \varepsilon \left[v_{1,YYY} + \frac{Y}{2X} U v_{1,YYY} - (\beta^2 U + U_{YY}) \frac{Y}{2X} v_{1,Y} \right] = U v_{0,XY} \\ + \frac{Y}{2X} U v_{0,YY} + \frac{1}{X} v_{0,Y} - (\beta^2 U + U_{YY}) \left(v_{0,X} + \frac{Y}{2X} v_{0,Y} \right) + G_1(Y),$$

model 4:

$$(14.13) \quad -\sigma U u_0 - U_Y v_0 + \varepsilon \left[u_{0,YY} - \beta^2 u_0 + \frac{Y}{2X} U u_{0,Y} - G_2(Y) \right] = 0,$$

$$(14.14) \quad -\sigma U v_{0,YY} + (\beta^2 U + U_{YY}) \sigma v_0 + 2\beta^2 U u_0 + \varepsilon \left[v_{0,YYY} \\ + \frac{Y}{2X} U v_{0,YYY} + \frac{1}{X} v_{0,Y} - (\beta^2 U + U_{YY}) \frac{Y}{2X} v_{0,Y} - G_1(Y) \right] = 0,$$

$$(14.15) \quad -\sigma U u_1 - U_Y v_1 + \varepsilon \left[u_{1,YY} - \beta^2 u_1 + \frac{Y}{2X} U u_{1,Y} - V u_{1,Y} - U_X u_1 \right] = U u_{0,X} + \frac{Y}{2X} U u_{0,Y},$$

$$(14.16) \quad -\sigma U v_{1,YY} + (\beta^2 U + U_{YY}) \sigma v_1 + 2\beta^2 U u_1 + \varepsilon \left[v_{1,YYY} + \frac{Y}{2X} U v_{1,YYY} \\ - (\beta^2 U + U_{YY}) \frac{Y}{2X} v_{1,Y} - V v_{1,YY} - (2\beta^2 + V_Y) v_{1,Y} + (\beta^2 V + U_{XY}) v_{1,Y} \\ + (\beta^4 + \beta^2 V_Y + U_{XY}) v_1 + 2\sigma U_X u_{1,Y} + 2\sigma U_{XY} u_1 \right] \\ = U v_{0,XY} + \frac{Y}{2X} U v_{0,YY} + \frac{1}{X} v_{0,Y} - (\beta^2 U + U_{YY}) \left(v_{0,X} + \frac{Y}{2X} v_{0,Y} \right).$$

The use of similarity variables may be advantageous because the point of maximum UU_Y near which the eigenfunctions are located according to Hall (1982) has a fixed position in similarity variables. Therefore the eigenfunctions can be expected to change more slowly with X if expressed in such coordinates. Itoh (1986, 1994) also pointed out the advantage inherent in the use of η for the analysis of non-parallel effects in the stability of boundary layers.

In the just derived hierarchies, each pair of leading order equations like (14.1-2) forms a system of homogeneous ordinary differential equations for the unknowns u_0 and v_0 which must be solved with σ in the role of eigenvalue. The coefficients of these equations depend parametrically on X and so will the solution. The X -derivatives of this leading-order solution appear as known right-hand-sides in equations (14.3-4), which must be solved as ordinary differential equations in the unknowns u_1 and v_1 , and so on goes the sequence. In singular perturbation problems of the multiple-scale type, care must be taken of secular behaviour by imposing a solvability condition (Fredholm alternative). Suppose the leading order system has been solved, thus providing for a certain abscissa X a set of normalized direct eigenfunctions $u_d(X, Y)$, $v_d(X, Y)$ and adjoint eigenfunctions $u_a(X, Y)$, $v_a(X, Y)$, together with the eigenvalue $\sigma(X)$. The actual leading-order solution will be expressed by

$$(15) \quad (u_0, v_0) = A(X)[u_d(X, Y), v_d(X, Y)],$$

where $A(X)$ is a yet undetermined amplitude function. Substituting (15) into the solvability condition we obtain:

$$(16) \quad c_1 A_X + c_2 A = 0,$$

with $i = 1, 2, 3, 4$ for each of the four formulations proposed, with

$$(17.1) \quad c_1 = \int_0^\infty \{ [Uv_{d.YY} - (\beta^2 U + U_{YY})v_d]v_a + Uu_d u_a \} dY,$$

model 1:

$$(17.2) \quad c_{21} = \int_0^\infty \{ [Uv_{d.XYY} - (\beta^2 U + U_{YY})v_{d.X} + Vv_{d.YYY} + (2\beta^2 + V_Y)v_{d.YY} - (\beta^2 V + U_{XY})v_{d.Y} - (\beta^4 + \beta^2 V_Y + U_{XYY})v_d - 2\sigma U_X u_{d.Y} - 2\sigma U_{XY} u_d]v_a + [Uu_{d.X} + Vu_{d.Y} + U_X u_d]u_a \} dY,$$

model 2:

$$(17.3) \quad c_{22} = \int_0^\infty \{ [Uv_{d.XYY} - (\beta^2 U + U_{YY})v_{d.X}]v_a + Uu_{d.X} u_a \} dY,$$

model 3:

$$(17.4) \quad c_{23} = \int_0^\infty \{ Uv_{d.XYY} - (\beta^2 U + U_{YY})v_{d.X} + \left(V + \frac{Y}{2X} U \right) v_{d.YYY} + \left(2\beta^2 + V_Y + \frac{1}{X} \right) v_{d.YY} - \left[\frac{Y}{2X} (\beta^2 U + U_{YY}) + (\beta^2 V + U_{XY}) \right] v_{d.Y} - (\beta^4 + \beta^2 V_Y + U_{XYY})v_d - 2\sigma U_X u_{d.Y} - 2\sigma U_{XY} u_d \} v_a + [Uu_{d.X} + \left(V + \frac{Y}{2X} U \right) u_{d.Y} + U_X u_d] u_a dY,$$

model 4:

$$(17.5) \quad c_{24} = \int_0^\infty \left\{ \left[U v_{d,XY Y} + \frac{Y}{2X} U v_{d,YY Y} + \frac{1}{X} v_{d,YY} - (\beta^2 U + U_{YY}) \left(v_{d,X} + \frac{Y}{2X} v_{d,Y} \right) \right] v_a + \left(U u_{d,X} + \frac{Y}{2X} U u_{d,Y} \right) u_a \right\} dY.$$

Hence:

$$(18) \quad A(X) = A(0) \exp \left(- \int_0^X \frac{c_{2i}(X)}{c_{1i}(X)} dX \right).$$

Without consideration of $A(x)$ the locally scaled amplification factor of the instability (defined by eq. 8) would be

$$(19.1) \quad \tilde{\sigma}(X) = \frac{X \sigma(X)}{\varepsilon}.$$

This is the *geometrical-optics* approximation. With $A(X)$ included it becomes

$$(19.2) \quad \tilde{\sigma}(X) = X \left[\frac{\sigma(X)}{\varepsilon} - \frac{c_{2i}(X)}{c_{1i}(X)} \right],$$

with the normalization $\int_0^\infty u_d^2 dY = 1$ for the direct eigenfunctions. This is the *physical-optics* approximation. The form of the normalization has been chosen so that the amplification factor calculated from eq. (19.2) is the same as that defined in eq. (8). However, it can be verified that if a different normalization were adopted eq. (18) would yield a different factor $A(x)$ such that the product $A(x)u_d(X, Y)$ is unchanged.

A comment is necessary concerning the order of the approximations involved. The amplification factor $\tilde{\sigma}$ is itself a quantity $O(\varepsilon^{-1})$ with an expansion of the form

$$(20) \quad \tilde{\sigma} = \tilde{\sigma}_{-1} \varepsilon^{-1} + \tilde{\sigma}_0 \varepsilon^0 + \tilde{\sigma}_1 \varepsilon^1 + \dots.$$

The geometrical-optics approximation halts at calculating the term $\tilde{\sigma}_{-1}$. The physical-optics (slightly non-parallel) approximation includes the term $\tilde{\sigma}_0$. To this end it is irrelevant whether a few $O(\varepsilon)$ terms are positioned in equations (14.1-2) than in equations (14.3-4); in particular those $O(\varepsilon)$ terms that do not contain derivatives with respect to X nor highest-order derivatives with respect to Y can be put in either place and still give a physical-optics approximations of $\tilde{\sigma}$ which is good to within an error $O(\varepsilon)$. Once the amplitude factor is corrected one can dispense with actually calculating the solution to the order-one equations, which only affects $\tilde{\sigma}$ at $O(\varepsilon)$.

5. Local versus marching results

The local stability equations (14) are solved on the finite domain $Y \in [0, Y_\infty]$ using a Chebychev collocation method. Zero disturbance conditions are applied on $Y = 0$ and $Y = Y_\infty$. The quadratic transformation

$$(21) \quad Y = Y_\infty \{1 - [0.5(1 - \zeta)]^{0.5}\}$$

maps the computational variable $\zeta \in [-1, 1]$ onto real space. For the present computation we have established that $Y_\infty = 200$ and 50 Chebychev polynomials are needed to guarantee accurate results. The generalized eigenvalue problems (and their adjoints) are solved by a QZ algorithm from the NAG library, for X going from 1 (where G is taken to be equal to 1) to 120 ($G_X = 36.25$) in X -steps of 1.

The parabolic equations (4) are marched streamwise with a second-order finite-difference procedure. Up to one thousand streamwise steps are employed, to guarantee grid-independent results. In the vertical direction a fourth-order compact finite-difference scheme is used, with one hundred grid points. Hall's (1983) results were faithfully reproduced to provide confidence in the accuracy of the code.

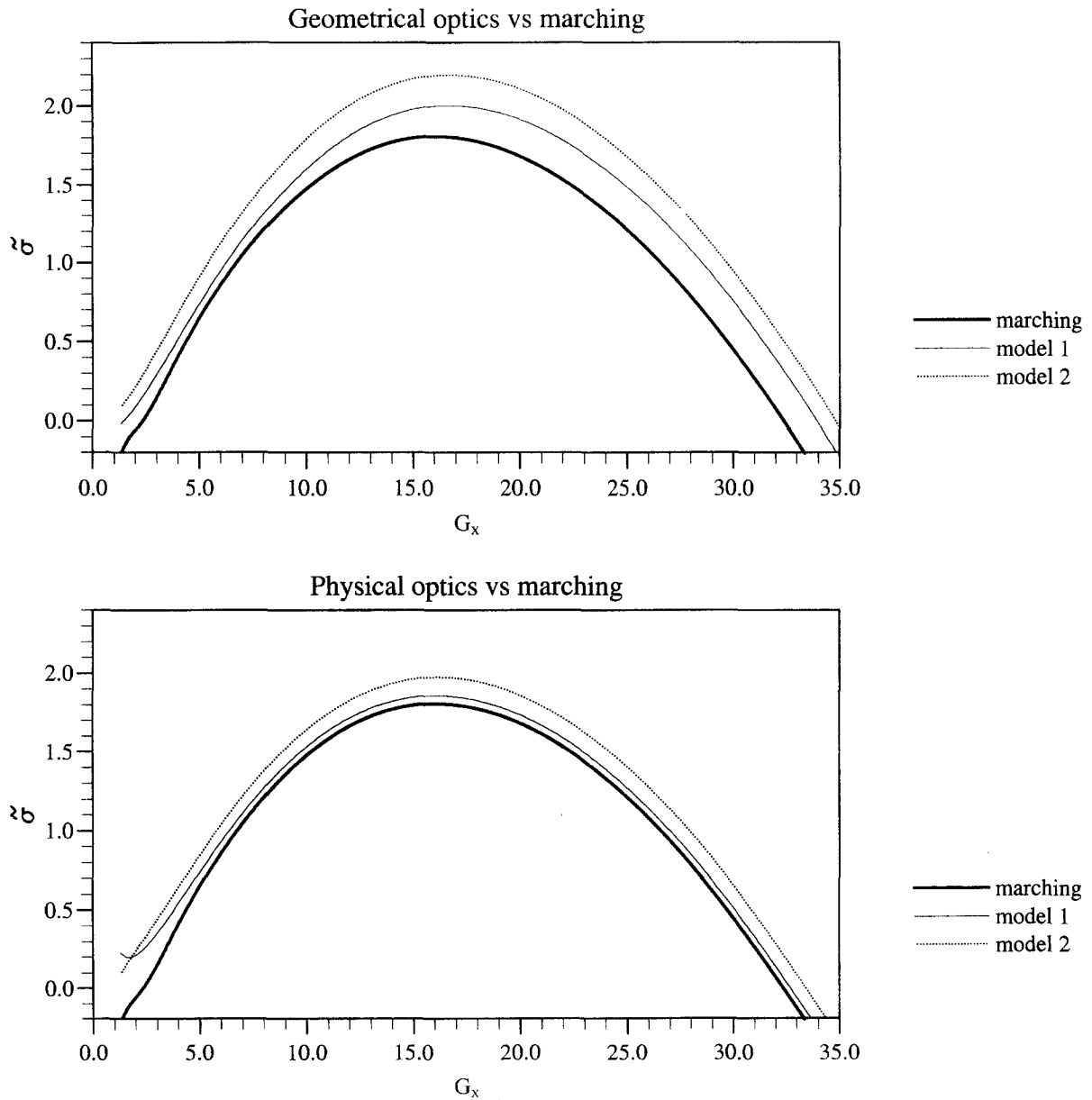


Fig. 2. - Strictly local results and quasi parallel corrections (cartesian variables) against exact solutions for $\Lambda = 62$.
The initial condition for the marching calculation is the second condition of figure 1.

In figures 2 and 3 the amplification factor is shown for two representative cases, $\Lambda = 62$ and 210, computed in cartesian coordinates, in comparison with the corresponding exact solutions drawn with thick solid lines. Λ is a dimensionless wavelength parameter, defined by

$$(22) \quad \Lambda = G \left(\frac{2\pi}{\beta} \right)^{3/2} .$$

It is a quantity which remains constant with X , and is frequently used in experiments and numerical simulations. The case $\Lambda = 62$ is interesting because it demonstrates clearly the initial growth of the instability, followed

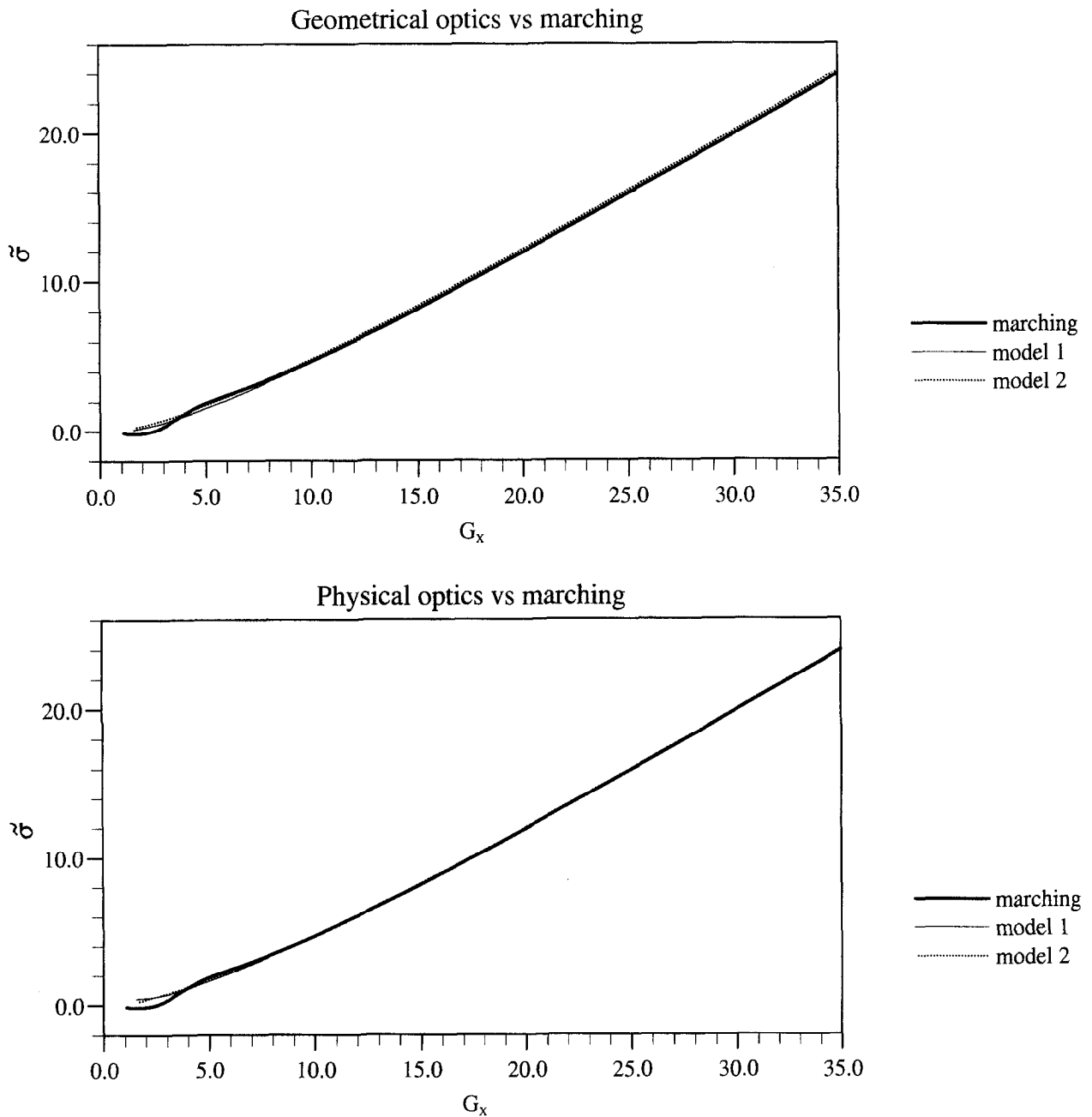


Fig. 3. - Same as figure 2 for $\Lambda = 210$.

by downstream decay. $\Lambda = 210$ corresponds to what is often quoted as the most linearly unstable wavelength for the Görtler problem (Floryan and Saric 1984). Reference solutions for both situations are available in the literature (see, e.g. Day *et al.*, 1990; Goulpié *et al.*, 1996), and both our marching and strictly parallel solutions perfectly match available results. The slightly non-parallel (physical-optics) estimates of the growth rates are, on the other side, new and lead to the following considerations:

1. for low Λ 's the agreement in $\tilde{\sigma}$ between the two models and the exact solution is improved when going from the strictly local to the slightly non-parallel solution, but there is no perfect match;
2. as Λ increases both the geometrical- and the physical-optics approximations produce improved accord with the solution, except for G_X less than about 7, where the influence of the inlet condition is felt;
3. model 1 produces always a better agreement with the exact solution than model 2.

Point one above seems to imply that the next approximation must be computed (the equations for $\tilde{\sigma}_0$ must be solved and the solvability condition for $\tilde{\sigma}_1$ must be imposed) in order to find a better agreement between the local and the marching estimates of $\tilde{\sigma}$. Luckily we do not need to go this far.

Figure 4 shows the geometrical- and physical-optics results for $\Lambda = 62$ when similarity variables are employed. While the strictly local results now underestimate the amplification factor, there is an almost perfect agreement between the physical-optics results of both models and the exact solution. With the increase of Λ the situation becomes even better, and a perfect match is found already between strictly local and marching results for G_X larger than 7. Incidentally, if the initial condition had been applied downstream of $X = 0$, its influence would have been felt beyond $G_X = 7$ and likewise the asymptotic collapse of growth rates would have occurred further downstream. This occurrence is, however, more of philosophical, rather than practical or physical, relevance. In figure 5 we give some examples of how the local u - and v -mode shapes compare to the exact solutions. The normalization adopted is that the maximum absolute value of v is equal to one. Results are plotted at $G_X = 15$ for $\Lambda = 62$ and 210. Note that a similarly good agreement as far as the eigenfunctions shape is concerned could also be found when cartesian variables were employed.

The final stability diagram is provided in figure 6. The lines of different amplification rate $\tilde{\sigma}$ have been obtained from physical-optics results of model 4. Instead of the conventional representation in the $\beta_X - G_X$ plane we have chosen to show results in the $\Lambda - G_X$ plane. This representation has the advantage of highlighting the physically interesting region of maximum amplification, namely that for which Λ is close to 200. The low- Λ leading order asymptotic result of equation (13) is also shown in the figure, and it is found to perfectly match the present neutral boundary. This part of parameter space is, however, of little interest because of the very low amplification factors present at every G_X for Λ less than about one hundred. A large body of experimental evidence (Bippes 1978, Swearingen and Blackwelder 1987, Pexieder 1996) reports observations of Görtler vortices with wavelengths Λ ranging from 200 to 600, unless even larger wavelengths are explicitly triggered by perturbations inherent to the installation (Aihara *et al.* 1985). The isolines shown in figure 6 are only a few percent away from the corresponding strictly local results of either one of the four models proposed here (for G_X larger than seven). The agreement between local and exact growth rates improves with the increase of Λ (cf. figures 2-4).

6. The failure of local theories at low G_X 's and the receptivity problem

The Görtler instability is initiated close to the X position where the concave curvature and the boundary layer thickness render G_X of order unity. Upstream of this position the effect of curvature is relatively irrelevant but there is already a three-dimensional algebraic instability of the flat plate boundary layer ($G_X \equiv 0$) related to the divergence of the Blasius flow (Luchini 1996). This instability applies to small spanwise wavenumber disturbances. So, vortex-like structures with low β and low G exist, even though local theories can not properly

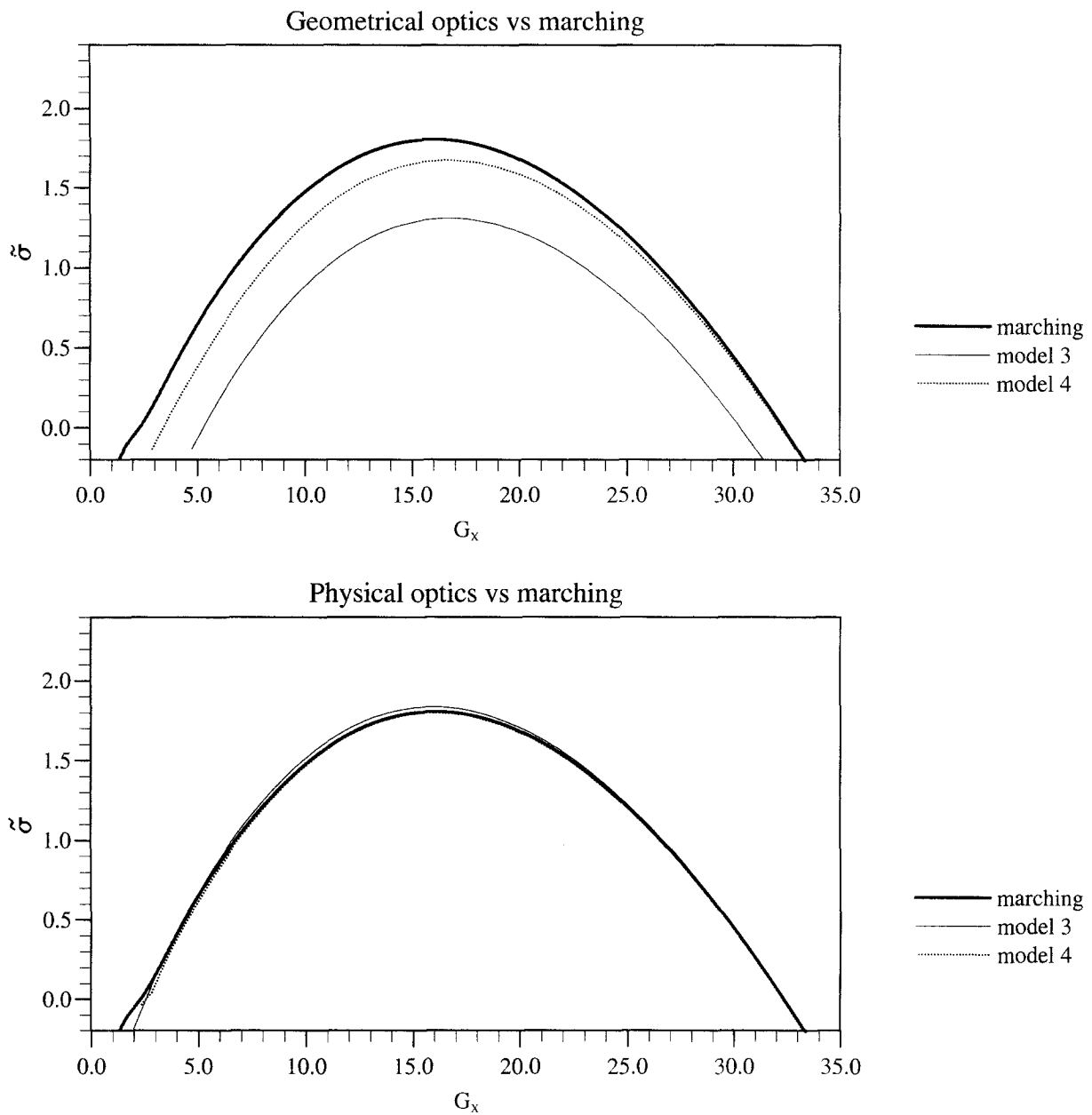


Fig. 4. – Same as figure 2 for $\Lambda = 62$, similarity variables.

account for them ⁽²⁾. Eventually curvature effects become important and the flat plate modes are amplified exponentially by the centrifugal instability mechanism.

For specified initial conditions one can solve for the early development of the vortices by marching the parabolic equations numerically, but there is no way to know in advance whether one rather than another

⁽²⁾ Le Cunff and Zebib (1996) noted that for $\beta_N \rightarrow 0$ the system of equations (14) becomes independent of the Görtler number and suggested that the length scales should be rethought. Luchini (1996) identified the new instability of the flat plate boundary layer exactly after scaling y and z differently from one another.

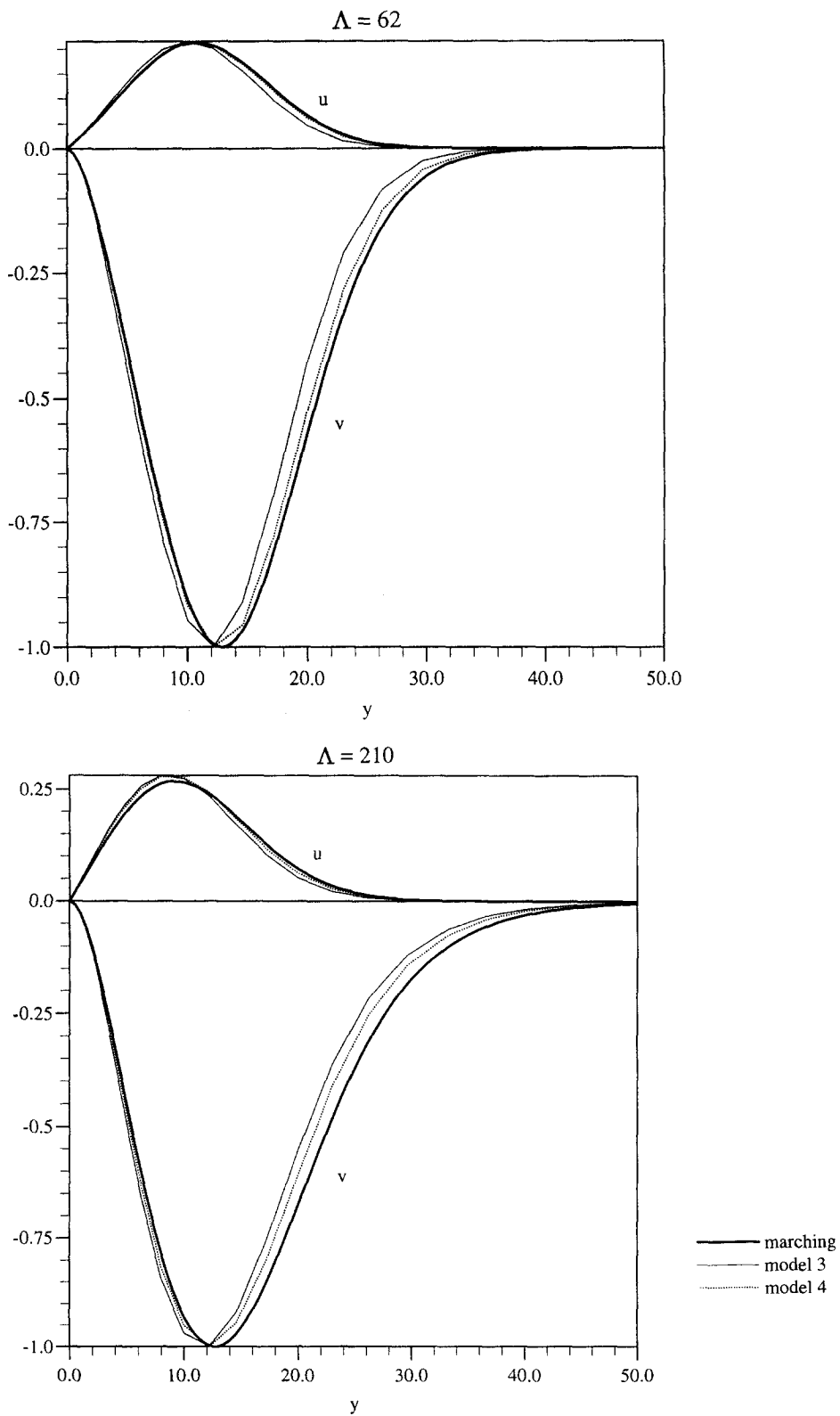


Fig. 5. — u (above) and v mode shapes against exact solutions, $G = 15$, $\Lambda = 62$ (upper) and 210 (lower).

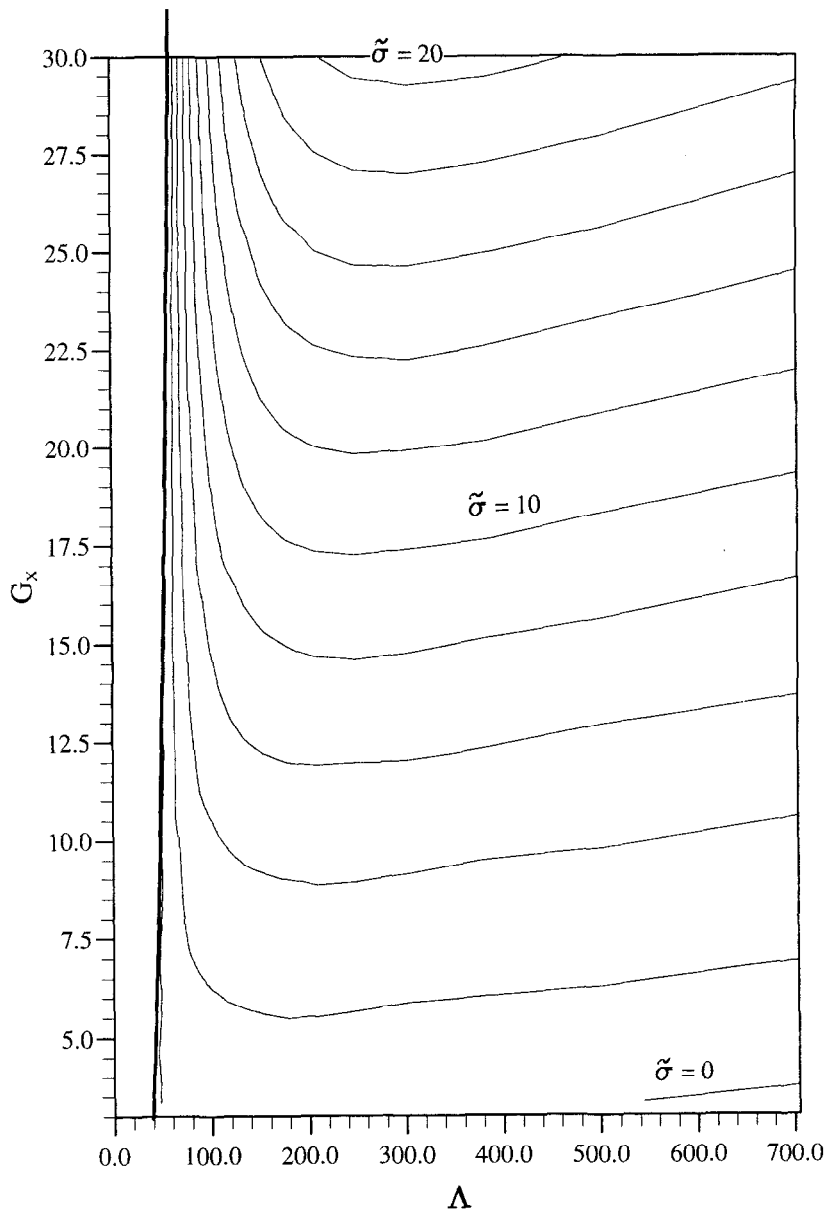


Fig. 6. – Curves of constant locally scaled growth factor in the $\Lambda - G_x$ plane. The leading order asymptotic neutral curve (equation 13) is given with a thick line.

type of initial conditions deserves being investigated. The important question of what initial and boundary conditions are most effective in exciting Görtler instabilities (the so-called receptivity problem) is the subject of the companion paper by LB.

7. Summary

The linear stability theory for the Görtler vortex problem has been outlined, the main characteristic of this problem as opposed to, e.g., the Dean problem, is the non-parallelism of the base flow; this entails the fact that the conventional normal mode approach must be abandoned and one should solve a parabolic system of partial

differential equations – equations (4.1-2) – subject to some initial and boundary conditions. These equations arise from a boundary-layer-type expansion in terms of Re^{-1} . Different choices of the initial conditions (and of the initial point from which the equations are spatially marched) cause different initial transients; the asymptotic behaviour for X sufficiently large is, however, independent of the initial stage. Such asymptotic structure can be described, for a wide range of spanwise wavenumbers β , with an approach which exploits the existence of two streamwise scales: one for the development of the base flow and one for the development of the vortices. The WKB expansion (in terms of the small parameter G^{-1}) of this problem has been outlined here for the first time, and its leading order approximation is the same that was heuristically employed by Görtler himself and several other authors; it permits, in a consistent manner, to take into account non-parallel effects in the evaluation of the amplification factor of the instability.

We have chosen to show four models that may result from the proposed expansion when the vertical nonuniformity is eliminated. A similar non-uniqueness in the formulation of the first order problem arises in the context of the OS equation (cf., e.g., Barry and Ross 1970 and Gaster 1974) but its implication for the Görtler instability has never been clearly addressed before. Here we have shown that for moderately large wavelengths (Λ larger than about 100) different possible models produce results in very good agreement with one another and with marching solutions. For small values of Λ , a formulation that employs similarity variables is particularly convenient, and physical-optics results collapse onto one another and onto the exact solution. Hence, the choice of the leading order model is really unimportant, as long as growth rates are corrected through the appropriate solvability condition.

The present modal analysis of the Görtler problem only applies sufficiently far downstream (where the local Görtler number is greater than about seven) and, just as any parallel or non-parallel local theory, calculates the amplitude of each mode only up to a (complex) multiplicative coefficient. The receptivity calculations of LB provide this coefficient.

Acknowledgements. – Some of the results described in this paper were orally communicated at the ERCOFTAC Workshop on Bypass Transition (Abisko, Sweden, 17-20 April 1996) and at the DFD Meeting of the American Physical Society (Syracuse, NY, U.S.A., 23-26 November 1996). AB acknowledges support of the Swiss National Fund, grant no. 2118-036035.92. PL has been supported by the Swiss Leonhard Euler Center for Fluid Mechanics, through several grants for extended visits at the Swiss Federal Institute of Technology, Lausanne (where the work was carried out during the first author's affiliation there), and by the Italian Ministry of University and Research. P.A. Monkewitz is thanked for helpful comments on the manuscript.

REFERENCES

- AIHARA Y., TOMITA Y., ITO A., 1985, Generation, development and distortion of longitudinal vortices in the boundary layer along concave and flat plates. In *Laminar-Turbulent Transition*, Kozlov V.V. editor, Springer, 447-454.
- BARRY M.D.J., ROSS M.A.S., 1970, The flat plate boundary layer. Part 2: The effect of increasing thickness on stability. *J. Fluid Mech.*, **43**, 813-818.
- BERTOLOTTI F.P., 1991, *Linear and Nonlinear Stability of Boundary Layers with Streamwise Varying Properties*. Ph.D. Thesis, The Ohio State University.
- BIPPES H., 1978, Experimental study of the laminar-turbulent transition of a concave wall in a parallel flow. *NASA TM 75243*.
- BOTTARO A., KLINGMANN B.G.B., ZEBIB A., 1996, Görtler vortices with system rotation. *Theor. Comp. Fluid Dyn.*, **8**, 325-347.
- BOTTARO A., LUCHINI P., 1996, The linear stability of Görtler vortices revisited. In *Mathematical Modeling and Simulation in Hydrodynamic Stability*, D.N. Riahi ed., World Scientific, 1-14.
- BOUTHIER M., 1972, Stabilité linéaire des écoulements presque parallèles, *J. de Mécanique*, **11**, 599-621.
- BOUTHIER M., 1973, Stabilité linéaire des écoulements presque parallèles. Partie II. La couche limite de Blasius. *J. de Mécanique* **12**, 75-95.
- DAY H. P., HERBERT Th., SARIC W.S., 1990, Comparing local and marching analysis of Görtler instability. *AIAA J.* **28**, 1010-1015.

- DRAZIN P.G., REID W.H., 1981, *Hydrodynamic Stability*, Cambridge University Press, Cambridge.
- FLORYAN J.M., 1991, On the Görtler instability of boundary layers. *Prog. Aerospace Sci.*, **28**, 235-271.
- FLORYAN J.M., SARIC W.S., 1979, Stability of Görtler vortices in boundary layers with suction, *AIAA paper* 79-1497.
- FLORYAN J.M., SARIC W.S., 1982, Stability of Görtler vortices in boundary layers, *AIAA J.*, **20**, 316-324.
- FLORYAN J.M., SARIC W.S., 1984, Wavelength selection and the growth of Görtler vortices, *AIAA J.*, **22**, 1529-1538.
- GASTER M., 1974, On the effects of boundary layer growth on flow stability, *J. Fluid Mech.*, **66**, 465-480.
- GÖRTLER H., 1941, Instabilität laminarer Grenzschichten an konkaven Wänden gegenüber gewissen dreidimensionalen Störungen. *ZAMM*, **21**, 250-252.
- GOULPIÉ P., KLINGMANN B.G.B., BOTTARO A., 1996, Görtler vortices in boundary layers with streamwise pressure gradient: Linear theory. *Phys. Fluids*, **8**, 451-459.
- HALL P., 1982, Taylor-Görtler vortices in fully developed or boundary layers flows: Linear theory, *J. Fluid Mech.*, **124**, 475-494.
- HALL P., 1983, The linear development of Görtler vortices in growing boundary layers, *J. Fluid Mech.*, **130**, 41-58.
- HÄMMERLIN G., 1955, Über das Eigenwertproblem der dreidimensionalen Instabilität laminar Grenzschichten, *J. Rat. Mech. Analysis*, **4**, 279-321.
- HÄMMERLIN G., 1956, Zur theorie der dreidimensionalen Instabilität laminarer Grenzschichten, *ZAMP*, **1**, 156-167.
- HERBERT Th., 1976, On the stability of the boundary layer along a concave wall. *Archives of Mechanics*, **28**, 1039-1055.
- HERRON I.H., 1985, Exchange of stabilities for Görtler flow, *SIAM J. Appl. Math.*, **45**, 775-779.
- HERRON I.H., 1991, Exchange of stabilities for flow along a concave wall, *SIAM J. Appl. Math.*, **22**, 1349-1353.
- ITOH N., 1986, The origin and subsequent development in space of Tollmien-Schlichting waves in a boundary layer. *Fluid Dyn. Research*, **1**, 119-130.
- ITOH N., 1994, Centrifugal instability of three-dimensional boundary layers along concave walls, *Trans. Japan Soc. Aero. Space Sci.*, **37**, 125-138.
- KLINGMANN B.G.B., BOIKO A.V., WESTIN K.J.A., KOZLOV V.V., ALFREDSSON P.H., 1993, Experiments on the stability of Tollmien-Schlichting waves. *Eur. J. Mech. B/Fluids*, **12**, 493-514.
- LE CUNFF C., ZEBIB A., 1996, Nonlinear spatially developing Görtler vortices in curved wall jet flow, *Phys. Fluids*, **8**, 2375-2382.
- LUCHINI P., 1996, Reynolds number independent instability of the Blasius boundary layer, *J. Fluid Mech.*, **327**, 101-115.
- LUCHINI P., BOTTARO A., 1998, Görtler vortices: a backward-in-time approach to the receptivity problem, *J. Fluid Mech.*, **363**, 1-23 Referred to herein as LB.
- NAYFEH A. H., 1973, *Perturbation Methods*, John Wiley and Sons, New York.
- PEXIEDER A., 1996, Effects of system rotation on the centrifugal instability of the boundary layer on a curved wall: An experimental study. *EPFL Ph. D. Thesis No. 1507*.
- SARIC W.S., 1994, Görtler vortices, *Annu. Rev. Fluid Mech.*, **26**, 379-409.
- SARIC W.S., NAYFEH A. H., 1975, Nonparallel stability of boundary layers flows, *Phys. Fluids*, **18**, 945-950.
- SMITH A.M.O., 1955, On the growth of Taylor-Görtler vortices along highly concave walls, *Q. Appl. Math.*, **13**, 233-262.
- SWEARINGEN J.D., BLACKWELDER R.F., 1987, The growth and breakdown of streamwise vortices in the presence of a wall, *J. Fluid Mech.*, **182**, 255-290.
- TIMOSHIN S.N., 1990, Asymptotic analysis of a spatially unstable Görtler vortex spectrum, *Izv. Akad. Nauk. SSSR, Mekh. Zhidk. Gaza* **1**, 32-41.
- VAN DYKE M., 1975, *Perturbation Methods in Fluids Mechanics*. The Parabolic Press, Stanford.

(Received 3 February 1998;

Revised 25 July 1998;

accepted 4 September 1998.)