ADJOINT HOMOGENIZATION

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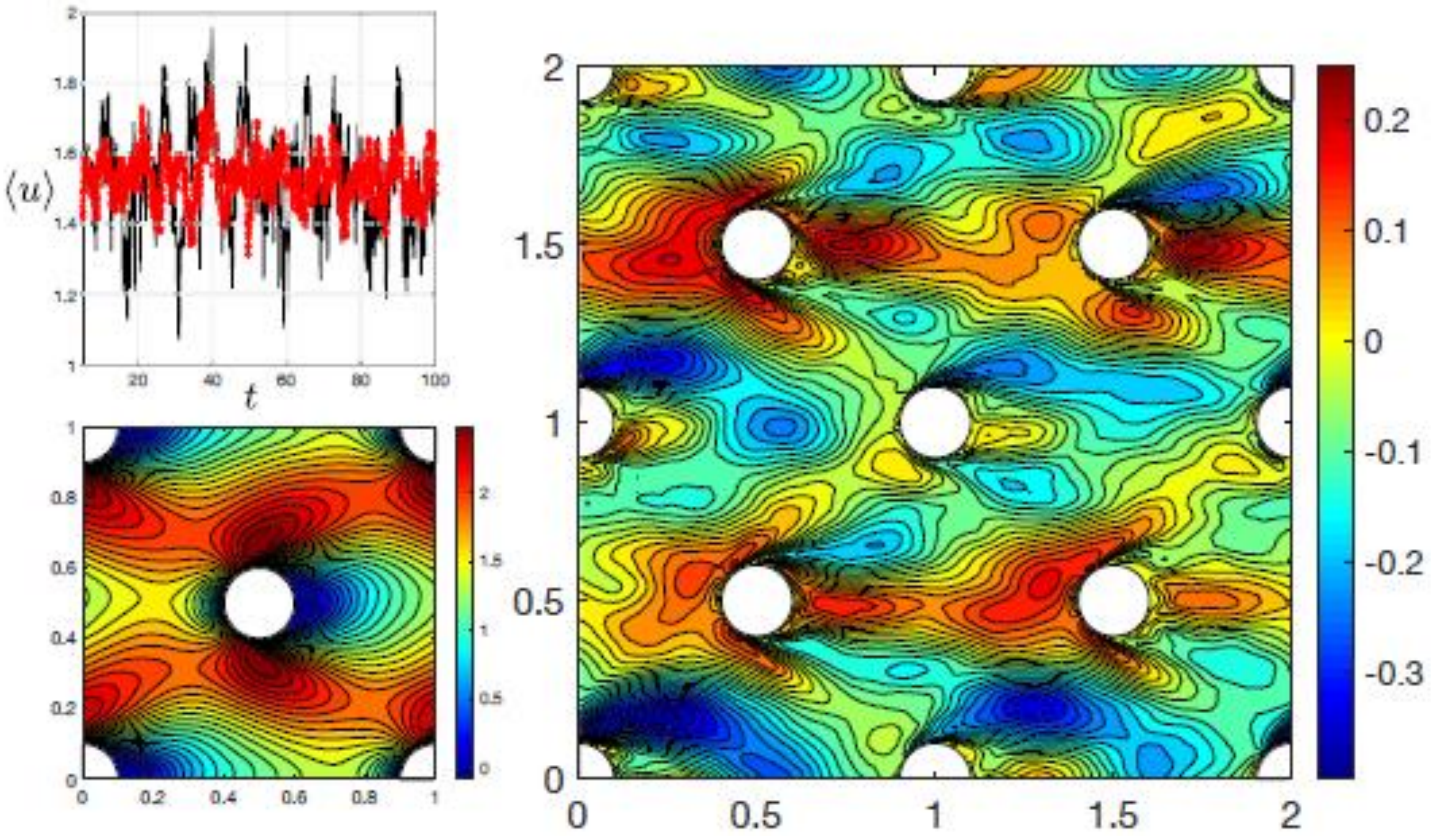
1. Porous media



$$\boxed{\frac{\Delta P}{L} = \mathcal{O}\left(\frac{\mu \underline{\mathrm{u}}}{l^2}\right)}$$

$$\left\{ \begin{array}{c} \dfrac{\partial u_i}{\partial x_i} = 0, \\ Re\left[\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j}\right] = -\dfrac{1}{\epsilon}\dfrac{\partial p}{\partial x_i} + \dfrac{\partial^2 u_i}{\partial x_j^2} + f_i \end{array} \right.$$

$$\epsilon = l/L \ll 1 \qquad Re = \frac{\rho \underline{\mathrm{u}} l}{\mu}$$

$$\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \epsilon \frac{\partial}{\partial X_i}$$

$$g(t, x_i, X_i) = g^{(0)}(t, x_i, X_i) + \epsilon\, g^{(1)}(t, x_i, X_i) + \ldots$$

$$
\left\{
\begin{aligned}
&\frac{\partial u_i^{(0)}}{\partial x_i} = 0, \\
&\frac{\partial u_i^{(1)}}{\partial x_i} + \frac{\partial u_i^{(0)}}{\partial X_i} = 0, \\
&\frac{\partial p^{(0)}}{\partial x_i} = 0, \\
&Re\left[\frac{\partial u_i^{(0)}}{\partial t} + u_j^{(0)}\frac{\partial u_i^{(0)}}{\partial x_j}\right] = -\frac{\partial p^{(1)}}{\partial x_i} - \frac{\partial p^{(0)}}{\partial X_i} + \frac{\partial^2 u_i^{(0)}}{\partial x_j^2} + f_i^{(0)}
\end{aligned}
\right.
$$

$$
\Rightarrow \quad p^{(0)} = p^{(0)}(t, X_i)
$$

$$\langle a \rangle = \frac{1}{\mathcal{V}} \int_{\mathcal{V}_f} a \, \mathrm{d}\mathcal{V},$$

$$(b, c) = \frac{1}{\mathcal{V}} \int_{\mathcal{V}_f} b \, c \, \mathrm{d}\mathcal{V},$$

Lagrange-Green identity

$$\int_{\mathcal{V}} a \frac{\partial b}{\partial x_i} \mathrm{d}\mathcal{V} = \int_{\partial \mathcal{V}} a \, b \, n_i \, \mathrm{d}A - \int_{\mathcal{V}} \frac{\partial a}{\partial x_i} b \, \mathrm{d}\mathcal{V}$$

Re

$$0 = \int_0^T \int_{V_f} u_i^{\dagger} \left[- \left(\frac{\partial u_i^{(0)}}{\partial t} + u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} \right) - \frac{\partial p^{(1)}}{\partial x_i} - \frac{\partial p^{(0)}}{\partial X_i} + \mu \frac{\partial^2 u_i^{(0)}}{\partial x_j^2} \right] + p_k^{\dagger} \frac{\partial u_i^{(0)}}{\partial x_i} \,\mathrm{d}V \,\mathrm{d}t$$

$$\int_0^T \left(\frac{\partial u_i^\dagger}{\partial x_i}, p^{(1)}\right) + \left(Re\left[\frac{\partial u_i^\dagger}{\partial t} + u_j^{(0)}\frac{\partial u_i^\dagger}{\partial x_j}\right] - \frac{\partial p^\dagger}{\partial x_i} + \frac{\partial^2 u_i^\dagger}{\partial x_j^2}, u_i^{(0)}\right) \mathrm{d}t$$

$$= \int_0^T \left(u_i^\dagger, \frac{\partial p^{(0)}}{\partial X_i} - f_i^{(0)}\right) \mathrm{d}t,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_i^\dagger, u_i^{(0)}) = 0.$$

Auxiliary problem:

$$\frac{\partial u_i^{\dagger(k)}}{\partial x_i} = 0; \quad -Re\left[\frac{\partial u_i^{\dagger(k)}}{\partial t} + u_j^{(0)}\frac{\partial u_i^{\dagger(k)}}{\partial x_j}\right] = -\frac{\partial p^{\dagger(k)}}{\partial x_i} + \frac{\partial^2 u_i^{\dagger(k)}}{\partial x_j^2} + \delta_{ki}$$

The resulting macroscopic equation is:

$$\int_0^T \langle u_k^{(0)} \rangle \, \mathrm{d}t = -\int_0^T \mathcal{K}_{ki}^{eff} \left[\frac{\partial p^{(0)}}{\partial X_i} - f_i^{(0)} \right] \mathrm{d}t,$$

with

$$\mathcal{K}_{ki}^{eff} = \langle u_i^{\dagger(k)} \rangle.$$

Auxiliary problem:

$$\frac{\partial u_i^{\dagger(k)}}{\partial x_i} = 0; \;\; -Re\left[\frac{\partial u_i^{\dagger(k)}}{\partial t} + u_j^{(0)}\frac{\partial u_i^{\dagger(k)}}{\partial x_j}\right] = -\frac{\partial p^{\dagger(k)}}{\partial x_i} + \frac{\partial^2 u_i^{\dagger(k)}}{\partial x_j^2} + \delta_{ki}$$

equivalent to:

$$\frac{\partial \mathcal{A}_{ki}^{\dagger}}{\partial x_i} = 0,$$

$$-\rho\left[\frac{\partial \mathcal{A}_{ki}^{\dagger}}{\partial t} + u_j^{(0)}\frac{\partial \mathcal{A}_{ki}^{\dagger}}{\partial x_j}\right] = -\frac{\partial p_k^{\dagger}}{\partial x_i} + \mu\frac{\partial^2 \mathcal{A}_{ki}^{\dagger}}{\partial x_j^2} + \delta_{ki}.$$

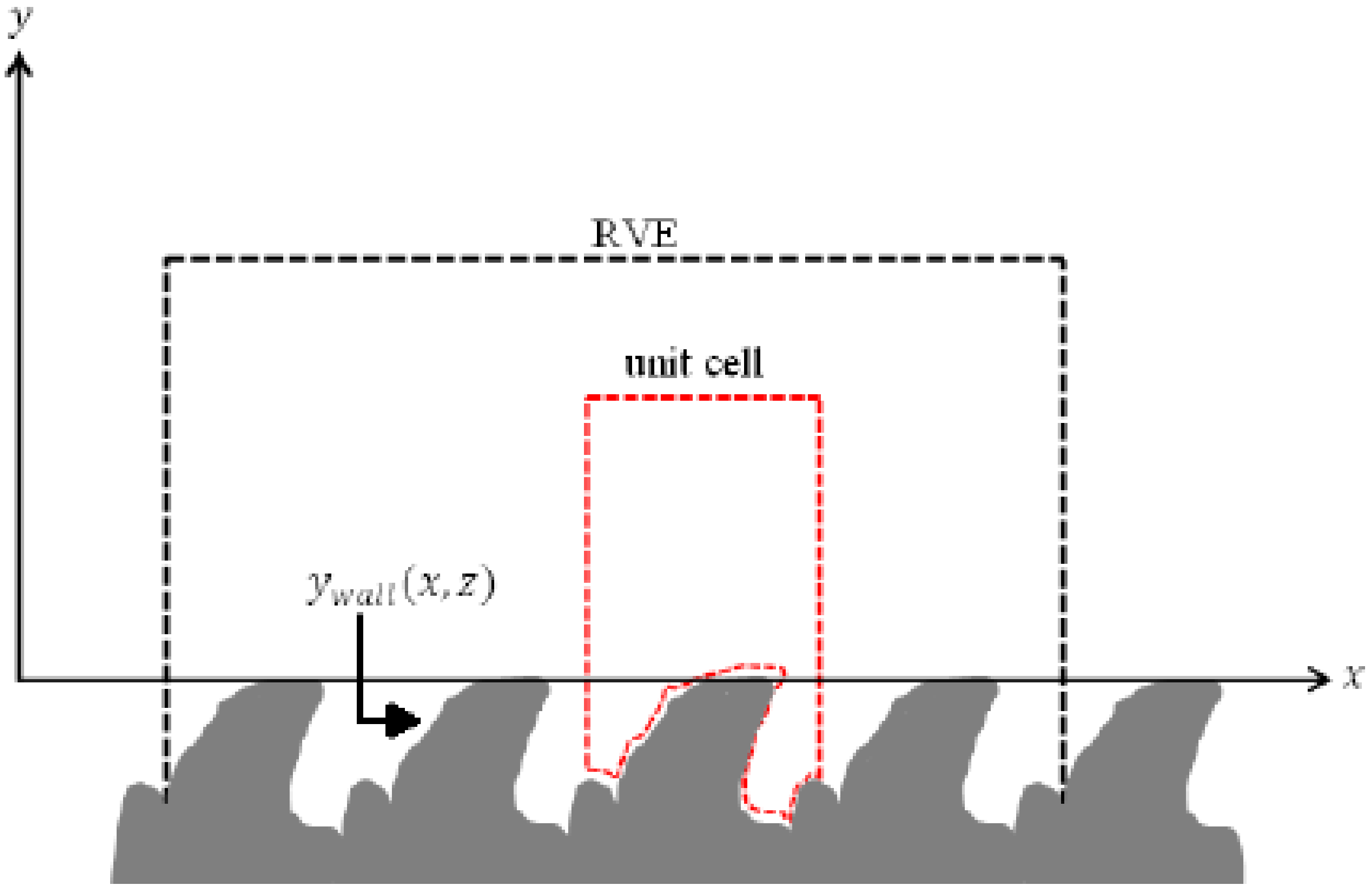
Same result as Whitaker's for the steady case:

$$\frac{\partial E_{ik}}{\partial x_i} = 0, \quad Re\, u_j^{(0)} \frac{\partial E_{ik}}{\partial x_j} = -\frac{\partial e_k}{\partial x_i} + \frac{\partial^2 E_{ik}}{\partial x_j^2} + \delta_{ik}$$

$$\mathcal{K}_{ik}^{eff} = \langle E_{ik} \rangle.$$

Same result as Lasseux et al. (JFM 2019) for the case of unsteady flows in porous media.

2. Flow over rough surfaces



$$\left\{ \begin{aligned} &\frac{\partial u_i}{\partial x_i} = 0, \\ &\mathcal{R}\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_j^2}, \qquad \mathcal{R} = \mathcal{U}\, l/\nu \end{aligned} \right.$$

Lagrange-Green identity: $0 = \int_0^T \left(p^{(0)}, \frac{\partial u_i^\dagger}{\partial x_i}\right) + \left(u_i^{(0)}, \left[\mathcal{R}\left(\frac{\partial u_i^\dagger}{\partial t} + u_j^{(0)} \frac{\partial u_i^\dagger}{\partial x_j}\right) - \frac{\partial p^\dagger}{\partial x_i} + \frac{\partial^2 u_i^\dagger}{\partial x_j^2}\right]\right) \mathrm{d}t \ + \mathrm{b.t.}$

\+ no-slip conditions for $u_i^\dagger$ at $y = y_{wall}$ and periodicity

$$\left\{\begin{array}{l} \dfrac{\partial u_i^\dagger}{\partial x_i} = 0, \\[2ex] -\mathcal{R}\left(\dfrac{\partial u_i^\dagger}{\partial t} + u_j^{(0)}\dfrac{\partial u_i^\dagger}{\partial x_j}\right) = -\dfrac{\partial p^\dagger}{\partial x_i} + \dfrac{\partial^2 u_i^\dagger}{\partial x_j^2}, \end{array}\right.$$

$$\text{b.t.} = \int_\Omega -\mathcal{R}\, v^{(0)} u_i^{(0)} u_i^\dagger - p^{(0)}\, v^\dagger + v^{(0)}\, p^\dagger + u_i^\dagger \frac{\partial u_i^{(0)}}{\partial y} - u_i^{(0)} \frac{\partial u_i^\dagger}{\partial y}\ \mathrm{d}x\, \mathrm{d}z = 0$$

at $y \to +\infty$

Choice for the adjoint boundary conditions at the outer edge of the RVE:

$$\frac{\partial u_i^{\dagger\,(k)}}{\partial y} = \delta_{ik}, \quad v^{\dagger\,(k)} := u_2^{\dagger\,(k)} = 0 \quad \text{at} \quad y \to +\infty$$

$$\lceil a \rceil = \frac{1}{\Omega} \int_{\Omega_f} a \, \mathrm{d}x \, \mathrm{d}z$$

$$\lim_{y \to +\infty} \lceil u^{(0)} \rceil := \lim_{y \to +\infty} \lceil u_1^{(0)} \rceil = \lim_{y \to +\infty} \lceil u^{\dagger\,(1)} \left(\frac{\partial u^{(0)}}{\partial y} - \mathcal{R}\, u^{(0)} v^{(0)} \right) \rceil +$$
$$+ \lceil w^{\dagger\,(1)} \left(\frac{\partial w^{(0)}}{\partial y} - \mathcal{R}\, v^{(0)} w^{(0)} \right) \rceil + \lceil p^{\dagger\,(1)} v^{(0)} \rceil;$$

and similarly for $\lim_{y \to +\infty} \lceil w^{(0)} \rceil$

First attempt:

$$U|_{Y\to 0^+} = \epsilon\lceil u^{(0)}\rceil|_{y\to+\infty}, \quad V|_{Y\to 0^+} = 0, \quad W|_{Y\to 0^+} = \epsilon\lceil w^{(0)}\rceil|_{y\to+\infty}$$

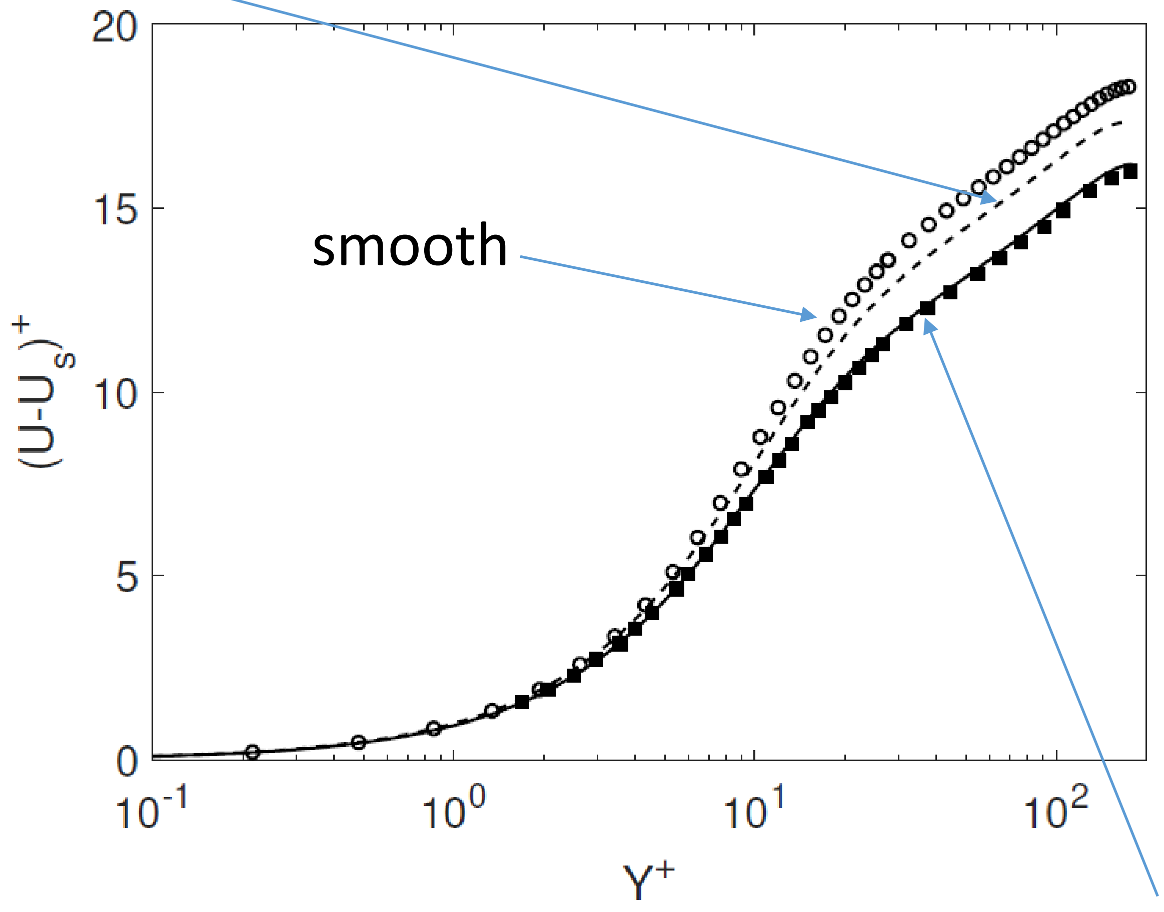
Flow in a channel, ***lower transitionally rough regime***: $\mathcal{R} \to 0$

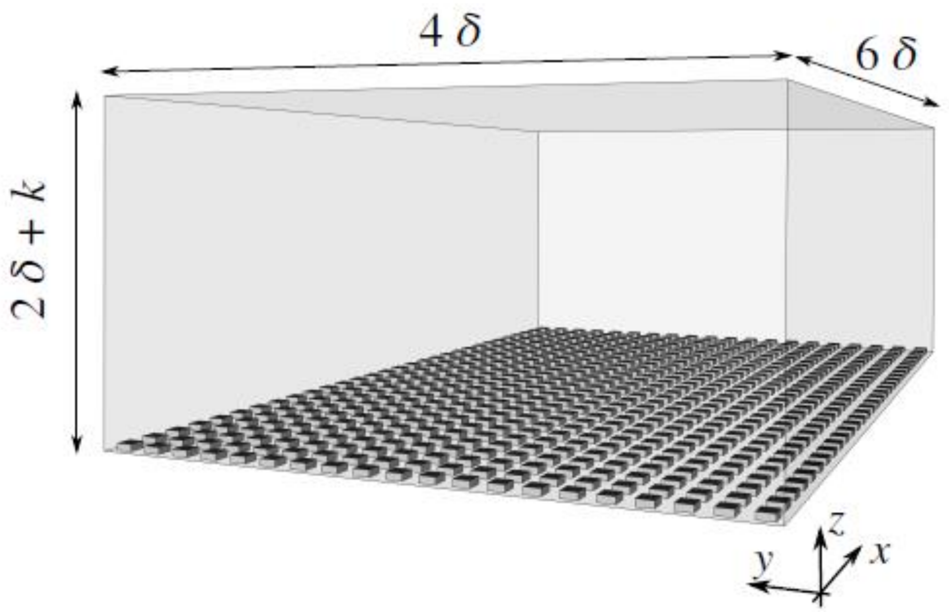
$$\begin{cases} \begin{pmatrix} U_s \\ W_s \end{pmatrix} = \epsilon\,\mathbf{\Lambda}\,\dfrac{\partial}{\partial Y}\begin{pmatrix} U \\ W \end{pmatrix}\Bigg|_{Y=0} \\ V\Big|_{Y=0} = 0 \end{cases}$$

$$\hat{\tau}^{\,total}_{lower\,wall} = \mu \left. \frac{\partial \overline{\hat{U}}}{\partial \hat{Y}} \right|_{\hat{Y}=0} - \rho \left. \overline{\hat{U}'\hat{V}'} \right|_{\hat{Y}=0}$$

slip, no wall transpiration

smooth





DNS by Lacis, Sudhakar & Bagheri (2018)

The transpiration velocity

$$\frac{\partial}{\partial Y}\begin{pmatrix} U \\ W \end{pmatrix}\Bigg|_{Y=0} = \epsilon^{-1}\,\mathbf{B}\begin{pmatrix} U_s \\ W_s \end{pmatrix} \qquad \mathbf{B} = \mathbf{\Lambda}^{-1} \text{ of components } b_{jk}$$

Linearity shows that $\mathbf{A}$ exists, such that: $$\begin{pmatrix} u^{(0)} \\ w^{(0)} \end{pmatrix} = \mathbf{A}\frac{\partial}{\partial Y}\begin{pmatrix} U \\ W \end{pmatrix}\Bigg|_{Y=0}$$

Then: $$\begin{pmatrix} u^{(0)} \\ w^{(0)} \end{pmatrix} = \epsilon^{-1}\,\mathbf{A}\,\mathbf{B}\begin{pmatrix} U_s \\ W_s \end{pmatrix} \Rightarrow u_i^{(0)} = \epsilon^{-1} a_{ij} b_{jk} U_{k,s}$$

The transpiration velocity

The dimensional continuity equation yields

$$\hat{V}\Big|_{\hat{Y}=0} = -\int_{\hat{Y}_{wall}}^{0}\left(\frac{\partial \hat{U}}{\partial \hat{X}} + \frac{\partial \hat{W}}{\partial \hat{Z}}\right)\mathrm{d}\hat{Y} = -\frac{\partial}{\partial \hat{X}}\int_{\hat{Y}_{wall}}^{0}\hat{U}\,\mathrm{d}\hat{Y} - \frac{\partial}{\partial \hat{Z}}\int_{\hat{Y}_{wall}}^{0}\hat{W}\,\mathrm{d}\hat{Y}$$

normalize with microscopic scales:

$$v^{(0)}\Big|_{y=0} + \epsilon v^{(1)}\Big|_{y=0} =$$

$$-\frac{\partial}{\partial x_i}\int_{y_{wall}}^{0} u_i^{(0)}\,\mathrm{d}y - \epsilon\left[\frac{\partial}{\partial x_i}\int_{y_{wall}}^{0} u_i^{(1)}\,\mathrm{d}y + \frac{\partial}{\partial X_i}\int_{y_{wall}}^{0} u_i^{(0)}\,\mathrm{d}y\right]$$

The transpiration velocity

$$v^{(0)}\big|_{y=0} = -\frac{\partial}{\partial x_i}\int_{y_{wall}}^{0} u_i^{(0)}\,\mathrm{d}y,$$

$$v^{(1)}\big|_{y=0} = -\frac{\partial}{\partial x_i}\int_{y_{wall}}^{0} u_i^{(1)}\,\mathrm{d}y - \frac{\partial}{\partial X_i}\int_{y_{wall}}^{0} a_{ij}\, b_{jk}\,(U_{k,s}/\epsilon)\,\mathrm{d}y.$$

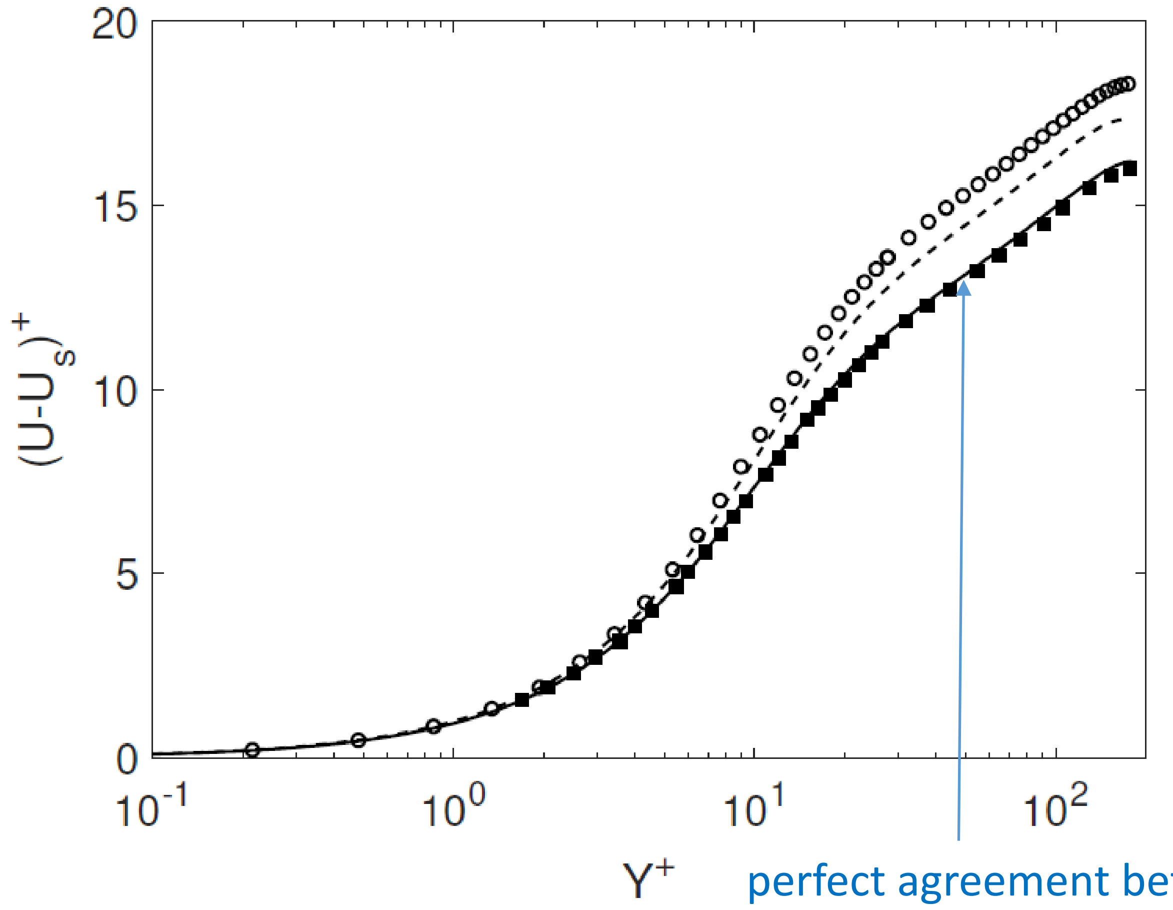
$$\lceil v^{(0)} \rceil = 0, \quad \lceil v^{(1)} \rceil = -\epsilon^{-1}\left[\int_{y_{wall}}^{0} \lceil a_{ij} \rceil\, b_{jk}\,\mathrm{d}y\right] \frac{\partial U_{k,s}}{\partial X_i}$$

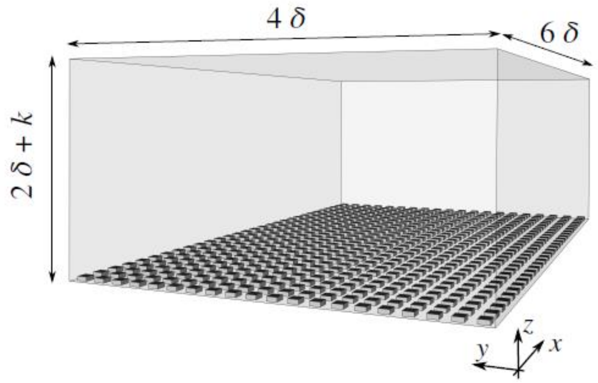
The transpiration velocity

$$V\Big|_{Y=0} = -\epsilon\, m_{ik} \frac{\partial U_{k,s}}{\partial X_i} = -\epsilon \left[m_{11} \frac{\partial U_s}{\partial X} + m_{13} \frac{\partial W_s}{\partial X} + m_{31} \frac{\partial U_s}{\partial Z} + m_{33} \frac{\partial W_s}{\partial Z} \right]$$

$$m_{ik} = \left[\int_{y_{wall}}^{0} \lceil a_{ij} \rceil \, \mathrm{d}y \right] b_{jk}$$

$$\hat{\tau}^{\,total}_{lower\,wall} = \mu \left.\frac{\partial \overline{\hat{U}}}{\partial \hat{Y}}\right|_{\hat{Y}=0} - \rho\, \overline{\hat{U}'\hat{V}'}\Big|_{\hat{Y}=0}$$



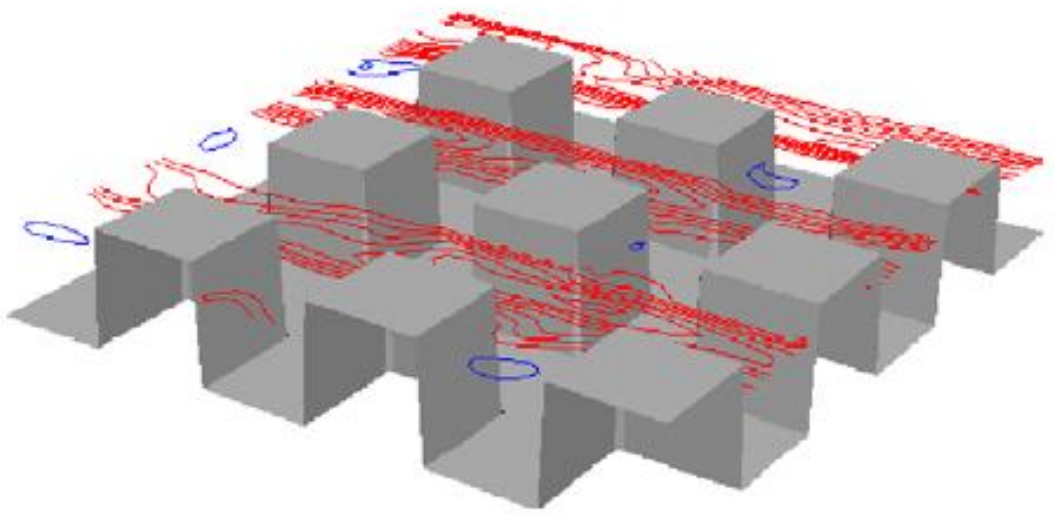


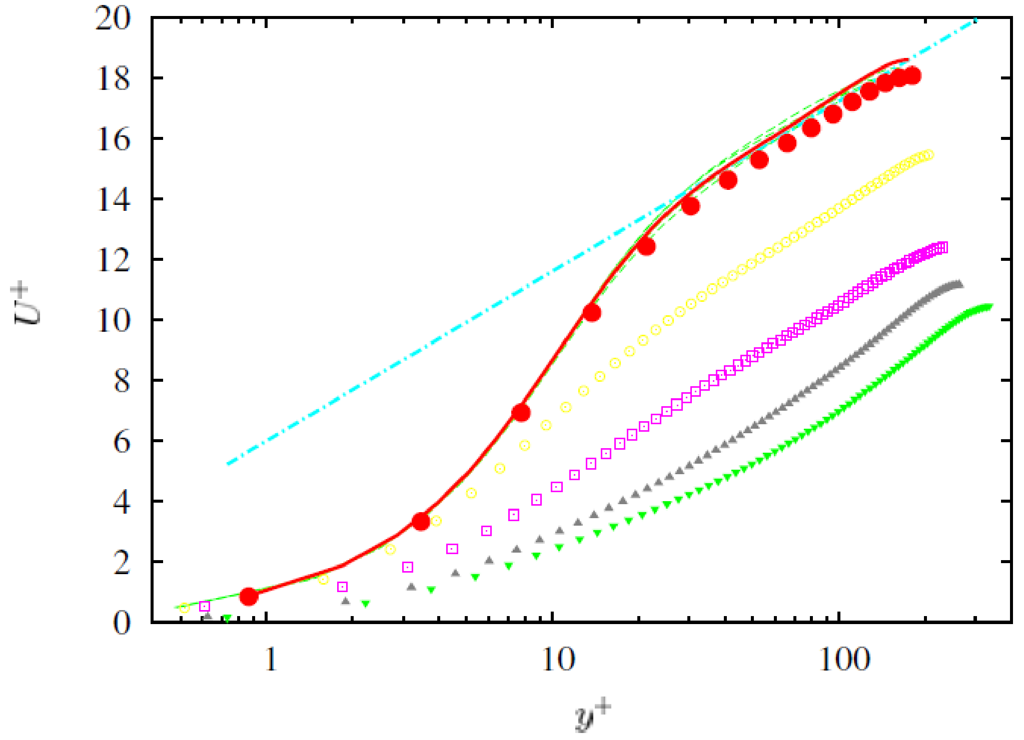
$$V\big|_{Y=0} = -\epsilon\, m_{11} \left[\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z}\right]\Bigg|_{Y=0}$$

$$V\big|_{Y=0} = \epsilon\, m_{11} \left.\frac{\partial V}{\partial Y}\right|_{Y=0}.$$

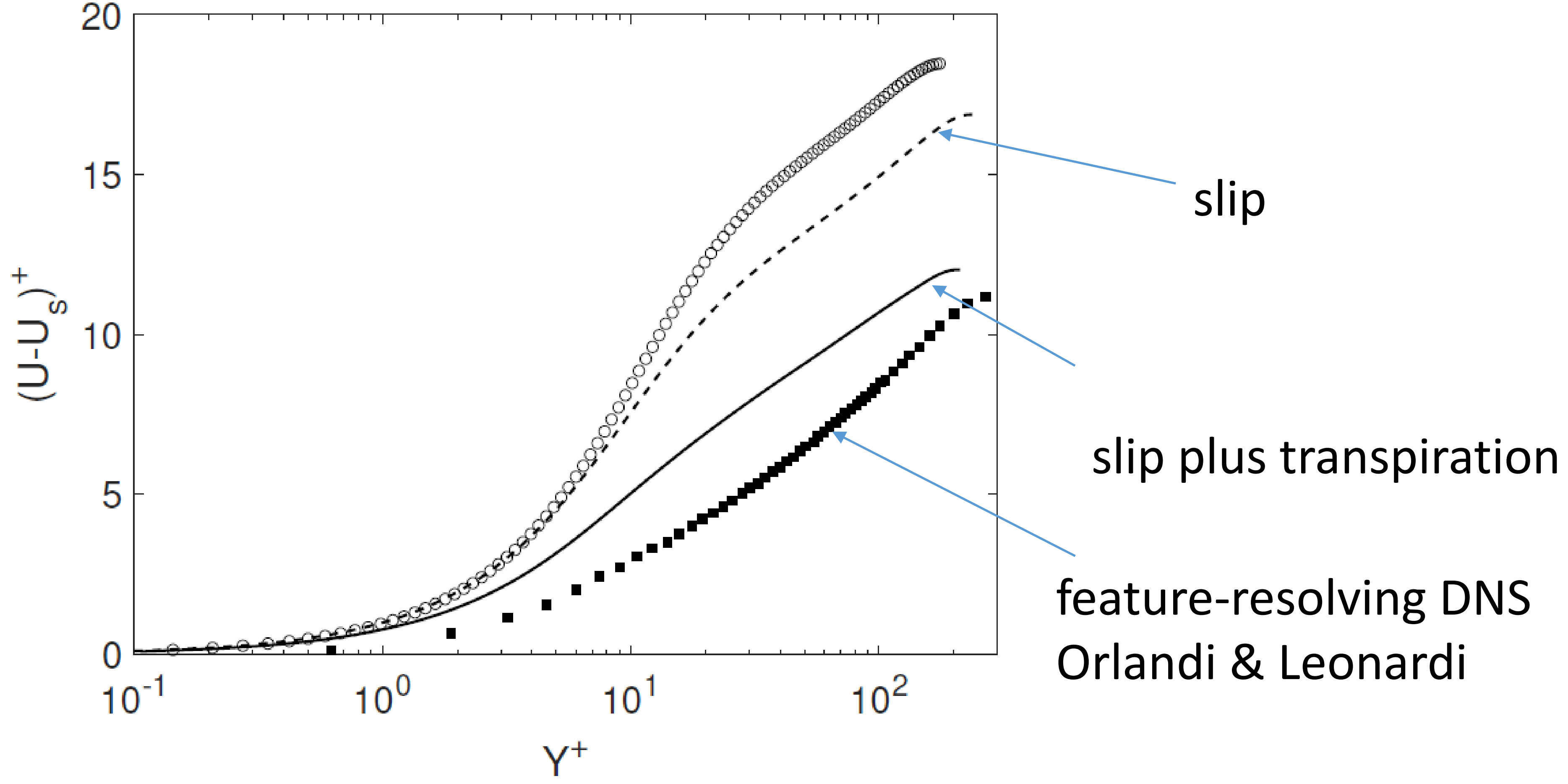
perfect agreement between the model and the feature-resolving DNS, in the lower transitionally rough regime

Upper transitionally rough regime





Orlandi & Leonardi (2006)


20
15
10
5
0
(U-U_s)^+
10^-1
10^0
10^1
10^2
Y^+
slip
slip plus transpiration
feature-resolving DNS
Orlandi & Leonardi

PERSPECTIVES

1. Can treat unsteady and turbulent flows in porous media
2. Can treat flow past rough walls (and interfaces, for example between a clear fluid region and a porous layer)

we can now parametrize any (sufficiently regular) rough wall with Λ and $\mathbf{M}$ and use such tensors to correlate the roughness pattern and amplitude to the *roughness function*.

This opens further perspectives for wall-control and optimization purposes, also using deformable, surface-based micro-actuators.