Local stability analysis of a coaxial jet at low Reynolds number

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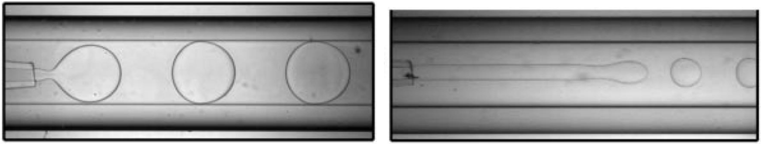
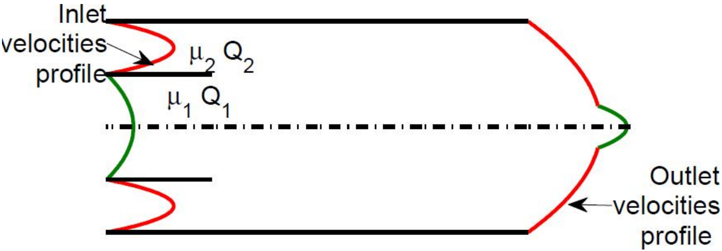
Goal of the project: explain the transition between dripping and jetting



Dripping and jetting regime, experimental study by Guillot



Local stability analysis in the developing region

Mathematical formulation of the problem

- $Re = 0$
- No gravity terms
- Axisymmetric domain
- Axisymmetric flow
- Stationary flow

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\nabla p - \mu \nabla^2 \boldsymbol{u} = 0,$$

with no-slip conditions at the wall and symmetry at the axis.

Interface conditions

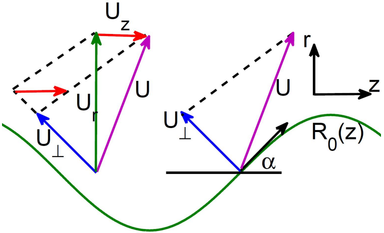
- Continuity of velocities at the interface

$$\boldsymbol{u}_1(\boldsymbol{x}_{int}) = \boldsymbol{u}_2(\boldsymbol{x}_{int}).$$

- Continuity of tangential stress and balance of normal stress

$$\boldsymbol{t}^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)\boldsymbol{n} = 0, \qquad \boldsymbol{n}^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)\boldsymbol{n} = \kappa\gamma\,\boldsymbol{n},$$

κ being the curvature in the meridian plane.



- Impermeability condition

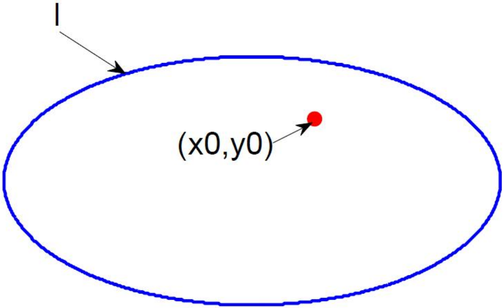
$$u_r \cos\alpha - u_z \sin\alpha = U_\perp = 0 \Longrightarrow u_r - u_z \frac{\partial R_0}{\partial z} = 0.$$

Boundary elements method

Formulation of the Stokes equations in term of boundary integral equations (Pozrikidis 1992)

$$\boldsymbol{u}(\boldsymbol{x}_0) = -\frac{1}{8\pi\mu}\oint_l \boldsymbol{G}(\boldsymbol{x}, x_0)\boldsymbol{f}(\boldsymbol{x})dl(\boldsymbol{x}) + \frac{1}{8\pi}\oint_l \boldsymbol{u}(\boldsymbol{x})\boldsymbol{T}(\boldsymbol{x}, x_0)\boldsymbol{n}(\boldsymbol{x})dl(\boldsymbol{x}),$$

- $\boldsymbol{G}$ and $\boldsymbol{T}$ are the Green's functions
- $\boldsymbol{f}$ is a vector representing the stresses and $\boldsymbol{u}$ the velocities

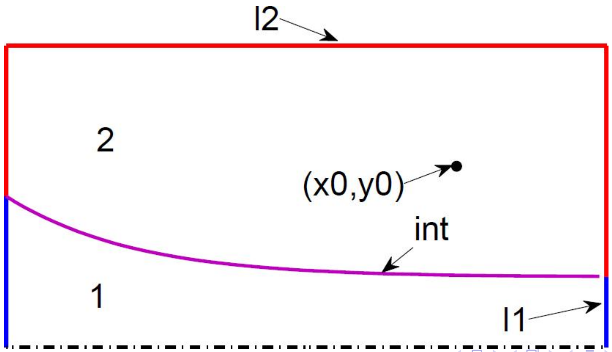


Boundary integral equation of a domain with interface

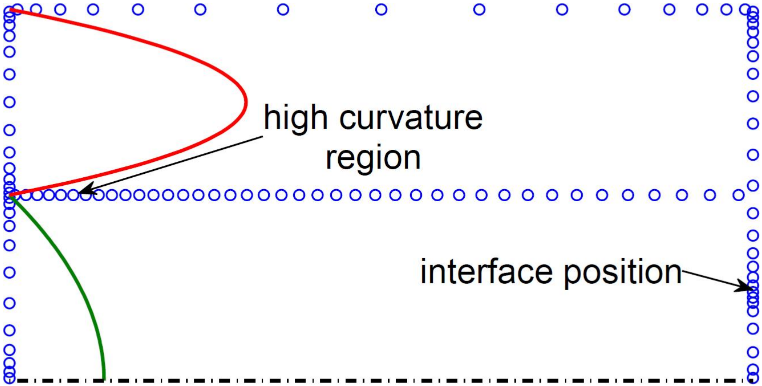
$$8\pi\mu_2 \boldsymbol{u}(\boldsymbol{x}_0) = -\int_l \boldsymbol{G}(\boldsymbol{x}_0,\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})dl + \mu_1 \int_{l_1} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{T}(\boldsymbol{x}_0,\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})dl +$$

$$\mu_2 \int_{l_2} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{T}(\boldsymbol{x}_0,\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})dl - \int_{int} \boldsymbol{G}(\boldsymbol{x}_0,\boldsymbol{x})\cdot\Delta\boldsymbol{f}(\boldsymbol{x})dl +$$

$$(\mu_2-\mu_1)\int_{int} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{T}(\boldsymbol{x}_0,\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})dl.$$



Discretization



Converge when $u_{r_{int}} < 10^{-3}$ and $u_{\perp}(int) < 10^{-3}$.

MOVIE 1

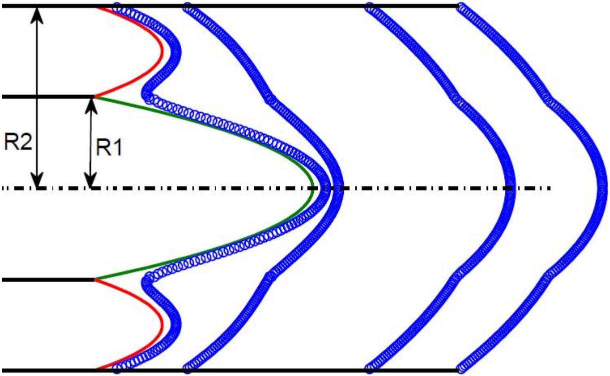
MOVIE 2

FreeFem simulation

Pressure field and axial velocity field

Local stability analysis

- Base flow taken at a given axial coordinate
- Non dimensional number: $Q = \frac{Q_1}{Q_2}$, $\lambda = \frac{\mu_1}{\mu_2}$, $Ka = \frac{d_z p R_2}{\gamma}$.



Linearize the equations adding small perturbations at the the base flow

$$\begin{pmatrix} \bar{P}_2 \\ \bar{P}_1 \\ \bar{U}_{r_1} \\ \bar{U}_{z_1} \\ \bar{U}_{r_2} \\ \bar{U}_{z_2} \\ \bar{R}_0 \end{pmatrix} = \begin{pmatrix} P_2 + \varepsilon p_2 \\ P_1 + \varepsilon p_1 \\ 0 + \varepsilon u_{r_1} \\ U_{z_1} + \varepsilon u_{z_1} \\ 0 + \varepsilon u_{r_2} \\ U_{z_2} + \varepsilon u_{z_2} \\ R + \varepsilon \eta \end{pmatrix} \qquad \varepsilon \ll 1$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\nabla p - \mu \nabla^2 \boldsymbol{u} = 0.$$

Boundary conditions

$$\text{wall:} \quad u_{r_2} = u_{z_2} = 0,$$

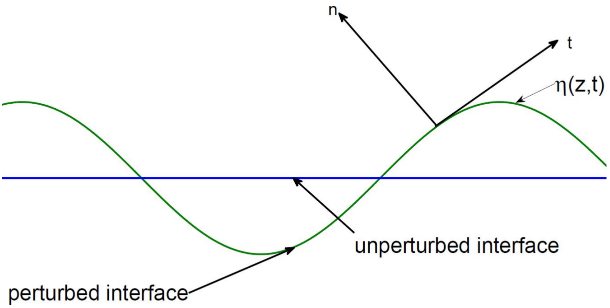
$$\text{axis:} \quad u_r = 0, \frac{\partial u_z}{\partial r} = 0, \frac{\partial p}{\partial r} = 0.$$

Linearize the continuity of velocities condition at the interface

$$U_{z_1}(R_0+\varepsilon\eta)+\varepsilon u_{z_1}(R_0+\varepsilon\eta)=U_{z_2}(R_0+\varepsilon\eta)+\varepsilon u_{z_2}(R_0+\varepsilon\eta)$$

flattening hypothesis, Taylor expansion around R_0

$$u_{z_1}(R_0)+\left.\frac{\partial U_{z_1}}{\partial r}\right|_{R_0}\eta=u_{z_2}(R_0)+\left.\frac{\partial U_{z_2}}{\partial r}\right|_{R_0}\eta$$



Modal expansion $u = \hat{u}(r)e^{i(kz-\omega t)}, p = \hat{p}(r)e^{i(kz-\omega t)}, \eta = \hat{\eta}e^{i(kz-\omega t)}$.

$$A\varphi = 0$$

where

$$A = \begin{pmatrix} [domain1] & [0] \\ [0] & [domain2] \\ interface & conditions \end{pmatrix}$$

$$\varphi = \begin{pmatrix} u_{r_1} & u_{z_1} & p_1 & u_{r_2} & u_{z_2} & p_2 & \eta \end{pmatrix}^T$$

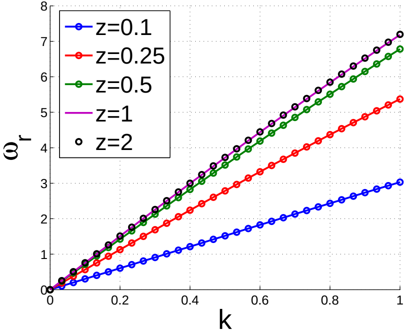
Non trivial solution if $det(A) = 0$, this leads to an eigenvalue problem for ω.

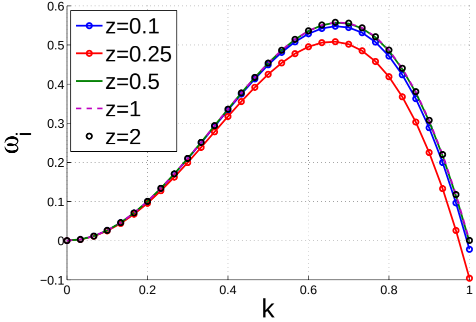
$$\omega = \omega_r(k) + i\omega_i(k)$$

or in a non dimensional form

$$\tilde{\omega} = \frac{16\mu_2 R_2}{\gamma}, \quad \tilde{k} = kR_{out}.$$

Local stability analysis performed in different sections z

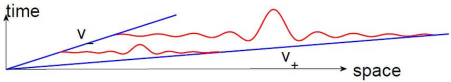


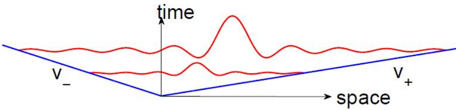


Real and imaginary part of $\tilde{\omega}$ for $Q = 0.7$, $\lambda = 0.5$, $Ka \approx 1$ and $R_1 = 0.5$
The dispersion relation can be written as in Guillot study

$$\tilde{\omega} = \alpha(z)\tilde{k} + iA(z)((\tilde{k}/b(z))^2 - (\tilde{k}/b(z))^4).$$

Absolute instability criterion

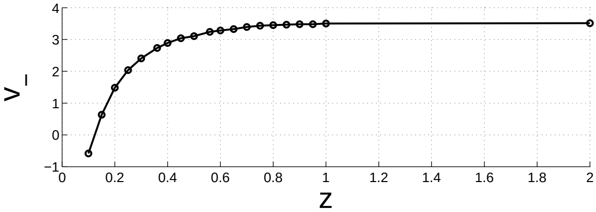




Velocity of the back front of the perturbation

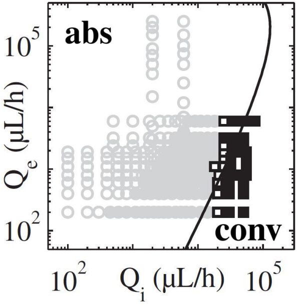
$$v_-(z) = \alpha(z) - A(z)\left(\frac{\sqrt{7}+5}{12b(z)^2} - \frac{\sqrt{7}+5}{36b(z)^4}\right)\sqrt{\frac{24b(z)^2}{\sqrt{7}-1}}$$

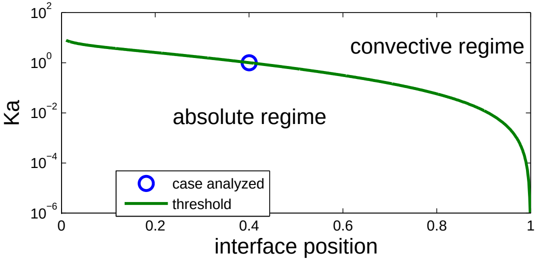
$$L_{abs} > \lambda_{min}, \qquad \text{where} \qquad \lambda_{min} = 2\pi/k_{max}$$



$$L_{abs} \approx 0.125 < \lambda_{min}$$

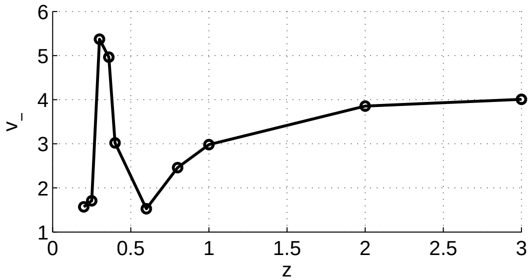
Region of convective and absolute instability



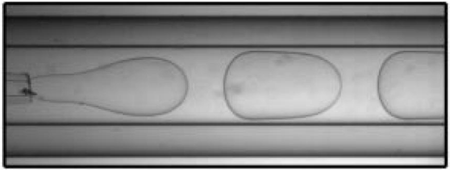


Analytical solution and experimental data presented by Guillot

Results



Numerical results for $Q = 0.5636$, $\lambda = 0.2$, $Ka = 1$, $R_1 = 0.2$



Experimental study by Guillot

Conclusions

- Better agreement with the experimental results respect to previous studies
- Change in the behavior of the perturbations from the developing region respect to the fully developed flow

Future developments

- Global stability analysis

Thank you,
Questions?