Chapter 3: Pressure and Fluid Statics

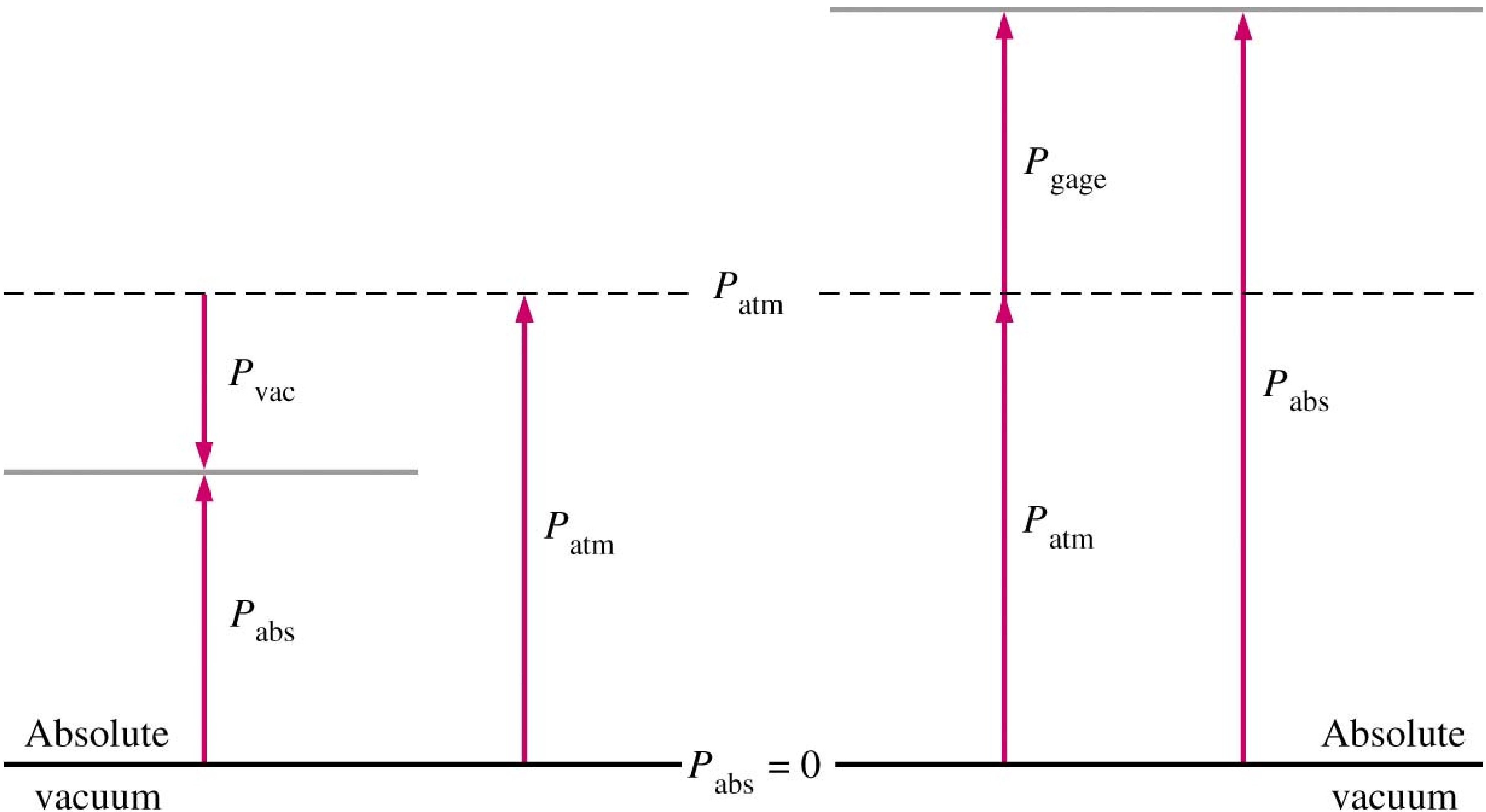
Pressure

- **Pressure** is defined as a *normal force exerted by a fluid per unit area*.
- Units of pressure are N/m^2, which is called a ***pascal*** (*Pa*).
- Since the unit *Pa* is too small for pressures encountered in practice, *kilopascal* (1 *kPa* = 10^3 *Pa*) and *megapascal* (1 *MPa* = 10^6 *Pa*) are commonly used.
- Other units include *bar*, *atm*, kgf/cm^2, $lbf/in^2 = psi$.

Absolute, gage, and vacuum pressures

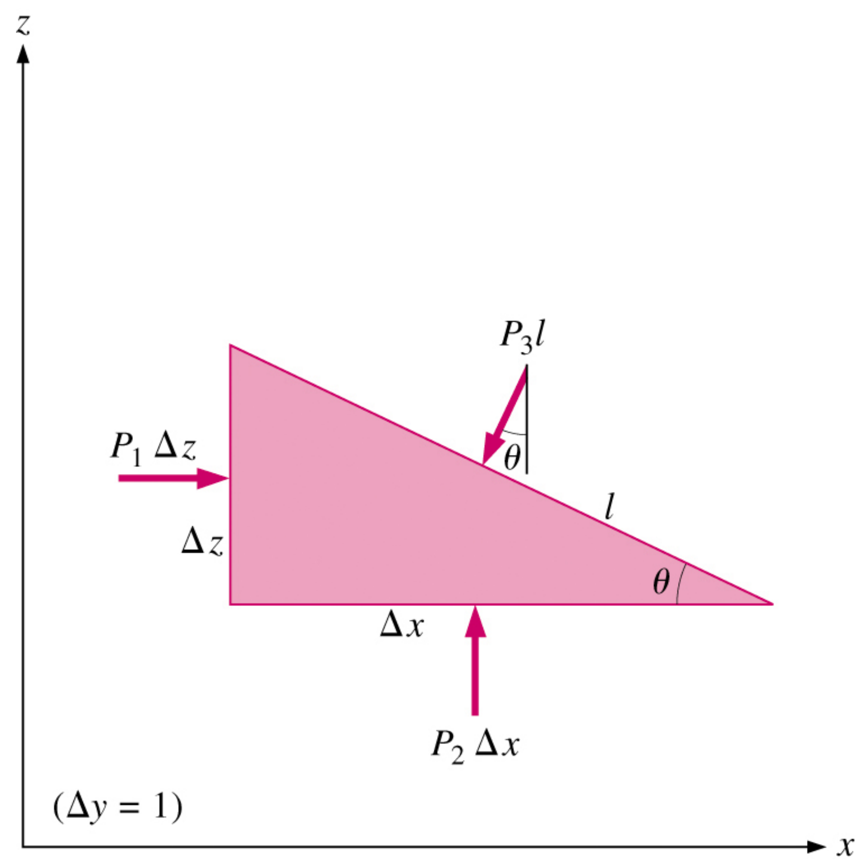
- Actual pressure at a give point is called the **absolute pressure**.
- Most pressure-measuring devices are calibrated to read zero in the atmosphere, and therefore indicate **gage pressure**, $P_{\text{gage}} = P_{\text{abs}} - P_{\text{atm}}$.
- Pressure below atmospheric pressure are called **vacuum pressure**, $P_{\text{vac}} = P_{\text{atm}} - P_{\text{abs}}$.

Absolute, gage, and vacuum pressures

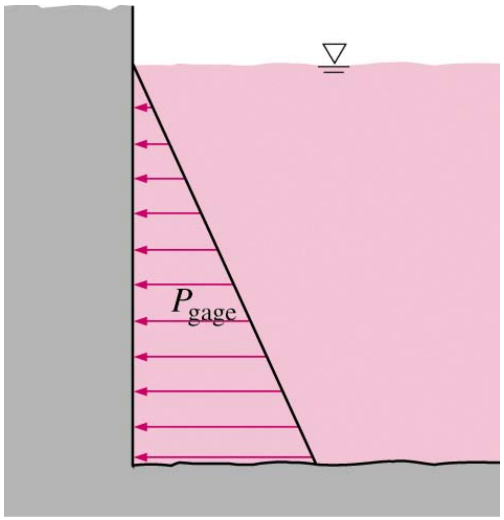


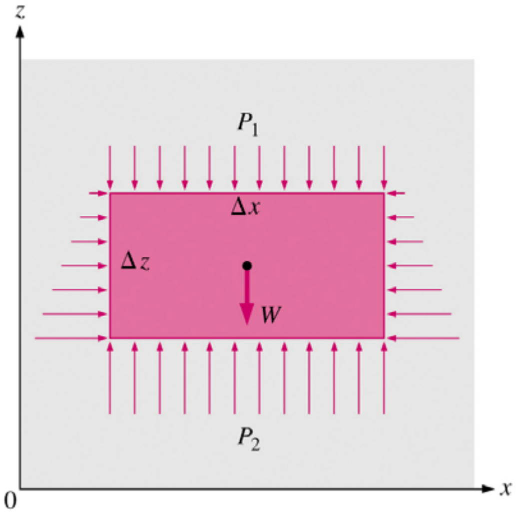
Pressure at a Point

- Pressure at any point in a fluid is the same in all directions.
- Pressure has a magnitude, but not a specific direction, and thus it is a scalar quantity.
- When $\Delta z \rightarrow 0$
  $P_1 = P_2 = P_3$



Variation of Pressure with Depth





- In the presence of a gravitational field, pressure increases with depth because more fluid rests on deeper layers.
- To obtain a relation for the variation of pressure with depth, consider rectangular element
  - Force balance in *z*-direction gives

$$\sum F_z = ma_z = 0$$

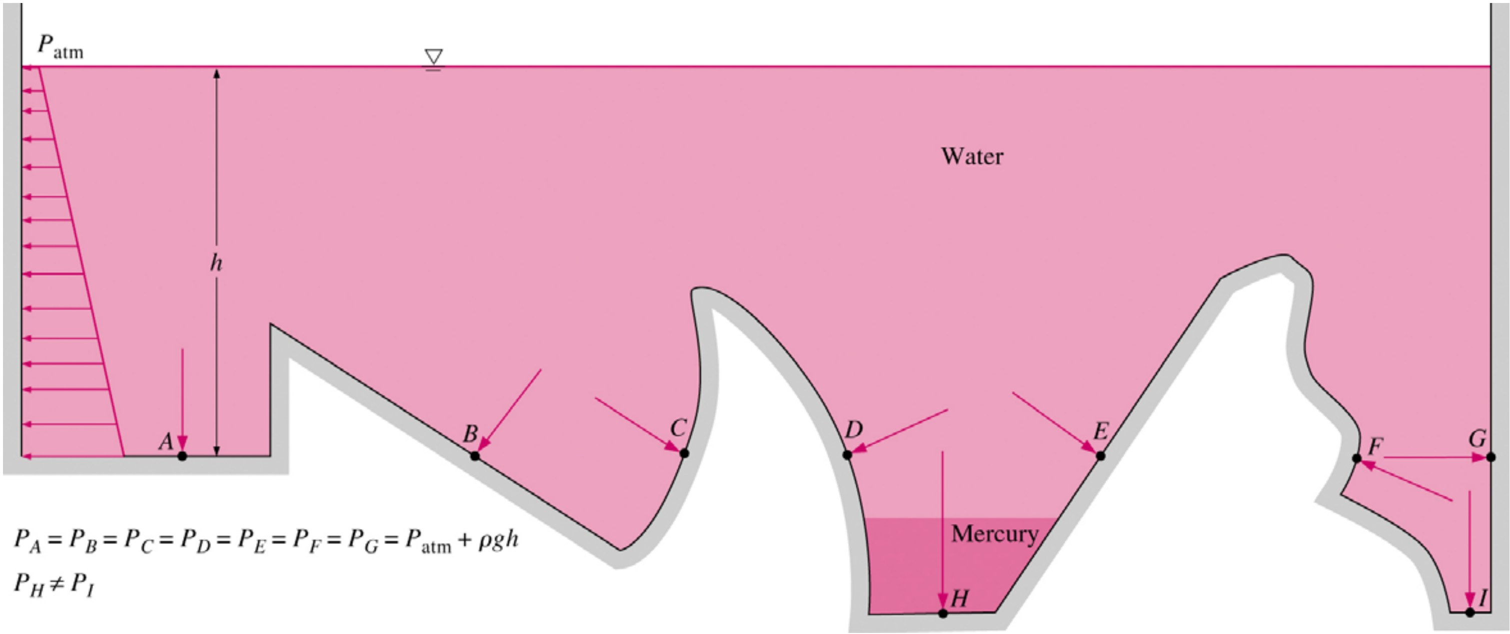
$$P_2 \Delta x - P_1 \Delta x - \rho g \Delta x \Delta z = 0$$

  - Rearranging gives

$$\boxed{\Delta P = P_2 - P_1 = \rho g \Delta z = \gamma_s \Delta z}$$

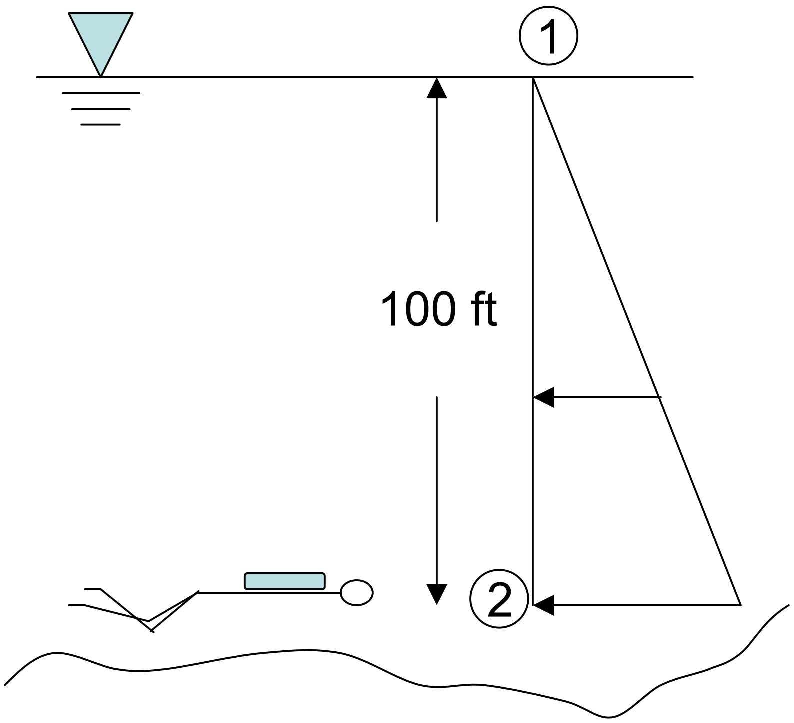
Variation of Pressure with Depth

- Pressure in a fluid at rest is independent of the shape of the container: $\boxed{dP/dz = -\rho g}$
- Pressure is the same at all points on a horizontal plane in a given fluid.



Scuba Diving and Hydrostatic Pressure

Scuba Diving and Hydrostatic Pressure



If you hold your breath on ascent, your lung volume would increase by a factor of 4, which would result in embolism and/or death.

- Pressure on diver at 100 ft?

$$P_{gage,2} = \rho g z = \left(998 \frac{kg}{m^3}\right)\left(9.81 \frac{m}{s^2}\right)(100\ ft)\left(\frac{1m}{3.28\ ft}\right)$$

$$= 298.5 kPa \left(\frac{1 atm}{101.325 kPa}\right) = 2.95 atm$$

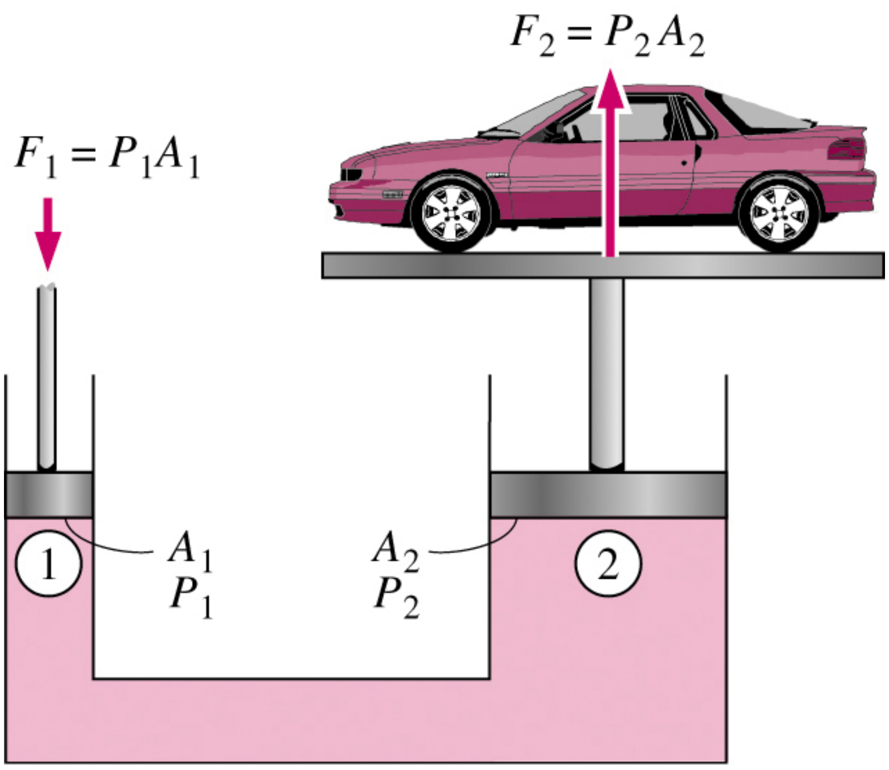
$$P_{abs,2} = P_{gage,2} + P_{atm} = 2.95 atm + 1 atm = \boxed{3.95 atm}$$

- Danger of emergency ascent?

$$P_1 V_1 = P_2 V_2 \qquad \text{Boyle's law}$$

$$\frac{V_1}{V_2} = \frac{P_2}{P_1} = \frac{3.95 atm}{1 atm} \approx 4$$

Pascal's Law

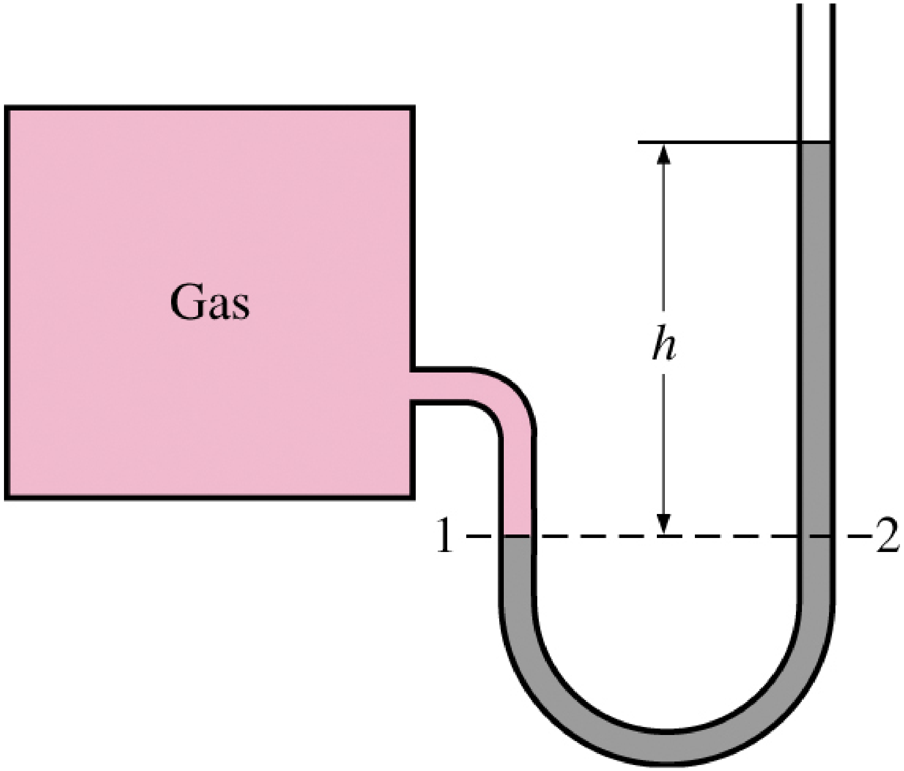


- Pressure applied to a confined fluid increases the pressure throughout by the same amount.
- In picture, pistons are at same height:

$$P_1 = P_2 \rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2} \rightarrow \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

- Ratio A_2/A_1 is called *ideal mechanical advantage*

The Manometer

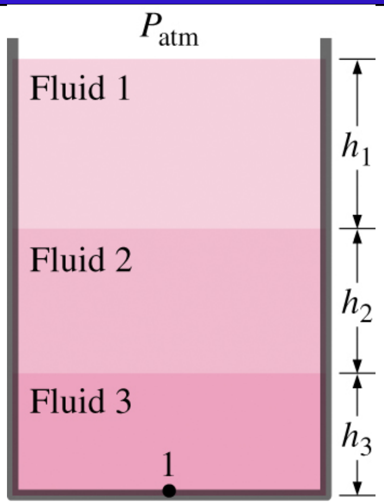


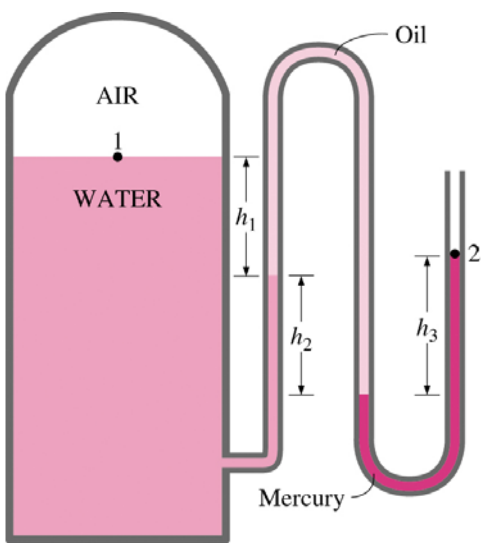
$$P_1 = P_2$$

$$P_2 = P_{atm} + \rho g h$$

- An elevation change of Δz in a fluid at rest corresponds to $\Delta P/\rho g$.
- A device based on this is called a **manometer.**
- A manometer consists of a U-tube containing one or more fluids such as mercury, water, alcohol, or oil.
- Heavy fluids such as mercury are used if large pressure differences are anticipated.

Multifluid Manometer

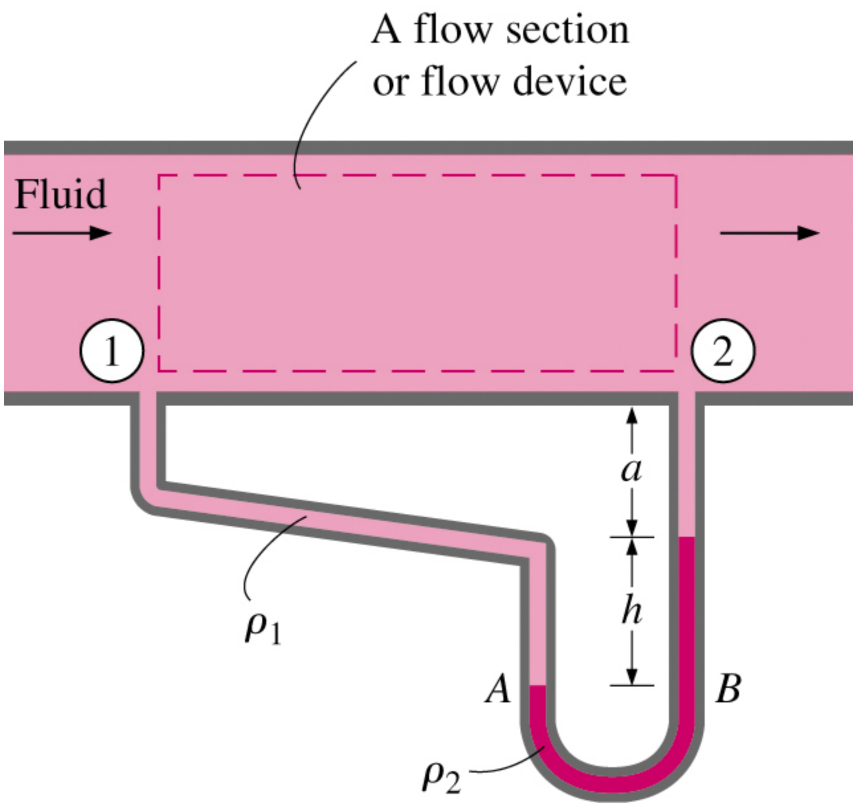




- For multi-fluid systems
  - Pressure change across a fluid column of height *h* is $\Delta P = \rho gh$.
  - Pressure increases downward, and decreases upward.
  - Two points at the same elevation in a continuous fluid are at the same pressure.
  - Pressure can be determined by adding and subtracting ρgh terms.

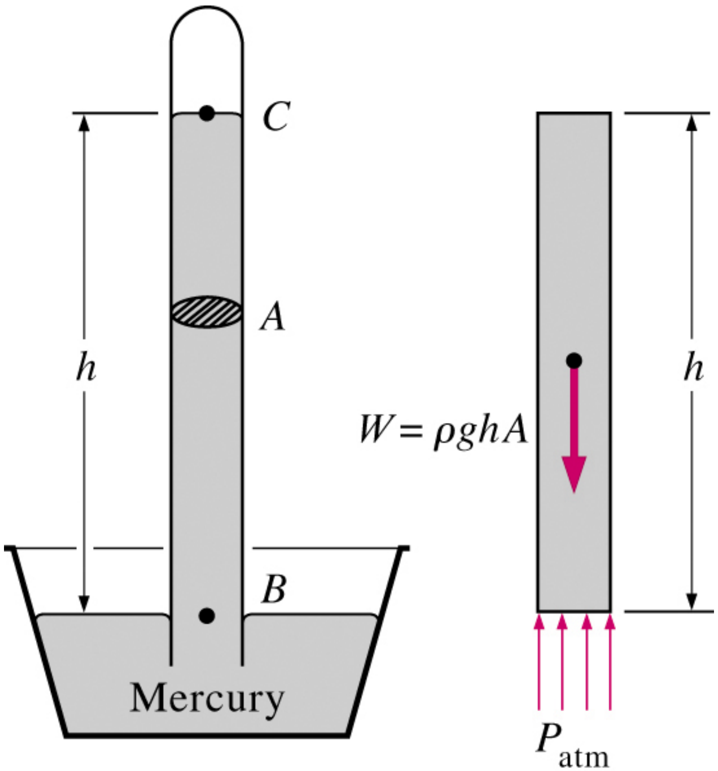
$$P_2 - \rho_1 g h_1 - \rho_2 g h_2 + \rho_3 g h_3 = P_1$$

Measuring Pressure Drops



- Manometers are well-suited to measure pressure drops across valves, pipes, heat exchangers, etc.
- Relation for pressure drop P_1-P_2 is obtained by starting at point 1 and adding or subtracting ρgh terms until we reach point 2.
- If fluid in pipe is a gas, $\rho_2 >> \rho_1$ and $P_1 - P_2 = \rho_2 gh$

The Barometer



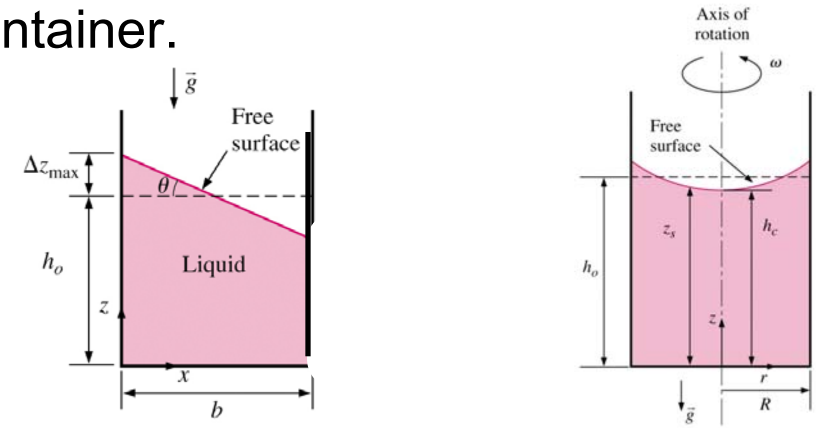
$$P_C + \rho g h = P_{atm}$$

$$P_{atm} = \rho g h$$

- Atmospheric pressure is measured by a device called a **barometer** (Evangelista Torricelli, circa 1635); thus, atmospheric pressure is often referred to as the *barometric pressure*.
- P_C can be taken to be zero since there is only Hg vapor above point C, with a very low pressure relative to P_{atm}.
- *Standard atmosphere* is the pressure produced by a column of Hg *760 mm* in height at *0°C*
- Atmospheric pressure decrease with elevation and this has many effects: cooking, nose bleeds, engine performance, aircraft performance.

Rigid-Body Motion (non-Galilean Frames)

- There are special cases where a body of fluid can undergo rigid-body motion: linear acceleration, and uniform rotation of a cylindrical container.



- In these cases, no shear is developed: the fluid is at rest is a non-Galilean reference system. A fictitious **inertial force** is introduced.
- Newton's 2nd law of motion can be used to derive an equation of motion for a fluid that acts as a rigid body

$$\boxed{\nabla P = \rho(\boldsymbol{g} - \boldsymbol{a}) = \boldsymbol{f}_{\text{effective}}}$$

- In Cartesian coordinates: $\frac{\partial P}{\partial x} = -\rho a_x, \frac{\partial P}{\partial y} = -\rho a_y, \frac{\partial P}{\partial z} = -\rho(g + a_z)$

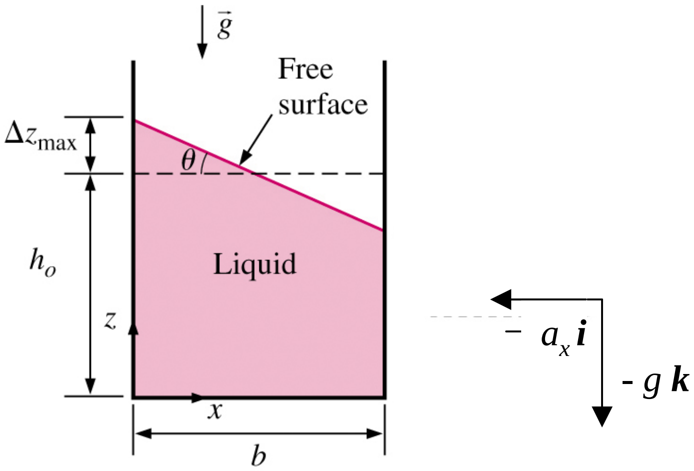
Linear Acceleration

Container is moving on a straight path

$$a_x \neq 0, a_y = a_z = 0$$

$$\frac{\partial P}{\partial x} = -\rho a_x, \frac{\partial P}{\partial y} = 0, \frac{\partial P}{\partial z} = -\rho g$$

$$\boxed{\nabla P = \boldsymbol{f}_{\text{effective}}}$$



Total differential of P: $dP = -\rho a_x dx - \rho g dz$

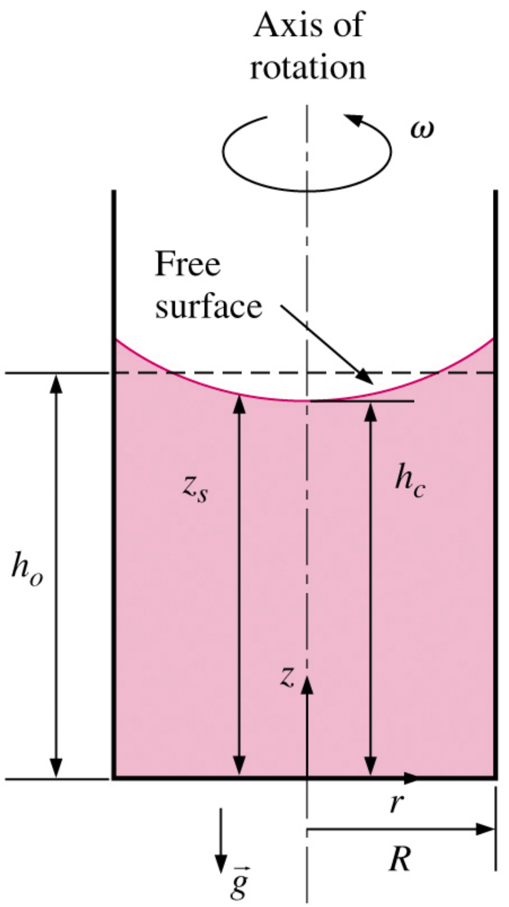
Equation of free surface: $z = -a_x x/g + k$

Pressure difference between any 2 points in the liquid:

$$P_2 - P_1 = -\rho a_x (x_2 - x_1) - \rho g (z_2 - z_1)$$

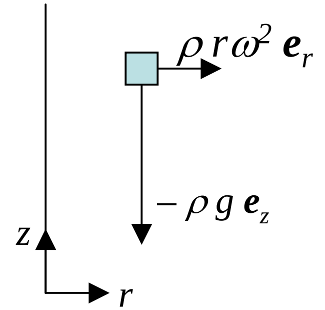
The free surface is orthogonal to $\boldsymbol{f}_{\text{effective}}$

Rotation in a Cylindrical Container

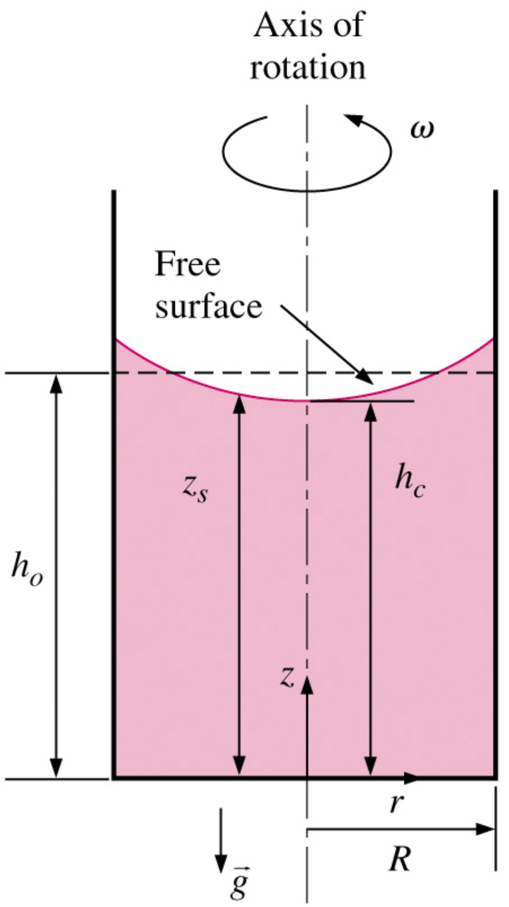


- The fluid rotates because the fluid is viscous and the no-slip condition applies
- In the rotating reference frame the fluid is at rest, hence the Coriolis force vanishes
- For an outside observer every fluid molecule is subject to its weight and to a centripetal acceleration equal to $\boldsymbol{a}_{\text{centripetal}} = -r\,\omega^2\,\boldsymbol{e}_r$
- The inertial force in the rotating frame of reference is the "centrifugal force" = - $\rho\,\boldsymbol{a}_{\text{centripetal}}$
- $\nabla P = \boldsymbol{f}_{\text{effettiva}} = \rho\,r\,\omega^2\,\boldsymbol{e}_r - \rho\,g\,\boldsymbol{e}_z$

  (the slope of $\boldsymbol{f}_{\text{effettiva}}$ in (r, z) is $-g/r\,\omega^2$)



Rotation in a Cylindrical Container



Container is rotating about the z-axis

$$\frac{\partial P}{\partial r} = \rho r \omega^2, \frac{\partial P}{\partial \theta} = 0, \frac{\partial P}{\partial z} = -\rho g$$

Total differential of *P*

$$dP = \rho r \omega^2 dr - \rho g dz$$

On an isobar, d*P* = 0

$$\frac{dz_{isobar}}{dr} = \frac{r\omega^2}{g} \rightarrow z_{isobar} = \frac{\omega^2}{2g} r^2 + C_1$$

$\longrightarrow$ any isobar is orthogonal to $\boldsymbol{f}_{\text{effective}}$

Equation of the free surface

$$z_s = h_0 - \frac{\omega^2}{4g}\left(R^2 - 2r^2\right)$$

Fluid Statics

- **Fluid Statics** deals with problems associated with fluids at rest.
- In fluid statics, there is no relative motion between adjacent fluid layers.
- Therefore, there is no shear stress in the fluid trying to deform it.
- The only stress in fluid statics is *normal stress*
  - Normal stress is due to pressure
  - Variation of pressure is due only to the weight of the fluid → fluid statics is only relevant in presence of gravity fields.
- Applications: Floating or submerged bodies, water dams and gates, liquid storage tanks, etc.

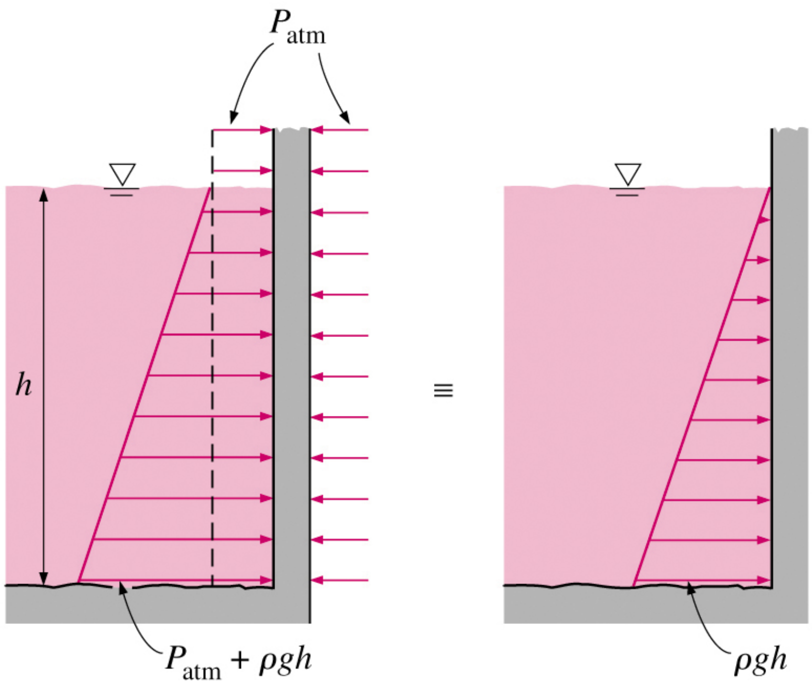
Hoover Dam

Hoover Dam

Hoover Dam

- Example of elevation head z converted to velocity head $V^2/2g$. We'll discuss this in more detail in Chapter 5 (Bernoulli equation).

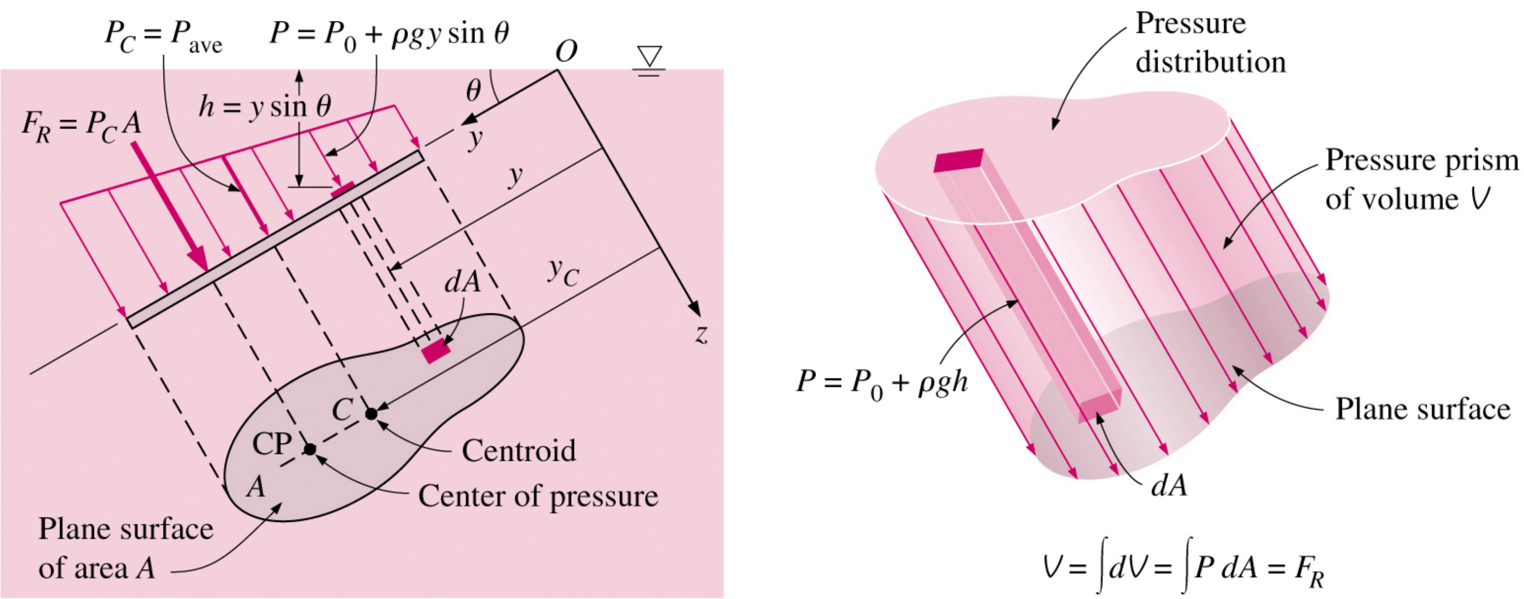
Hydrostatic Forces on Plane Surfaces



(*a*) P_{atm} considered (*b*) P_{atm} subtracted

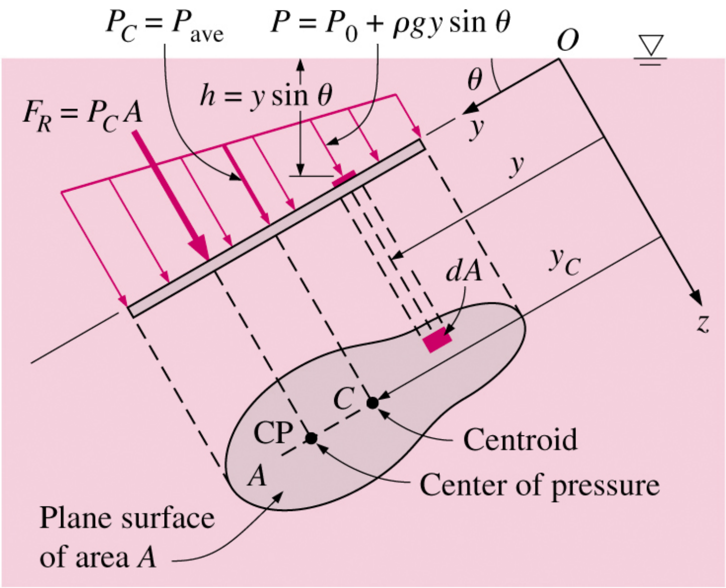
- On a submerged *plane* surface, the hydrostatic forces form a system of parallel forces
- For many applications, magnitude and location of application, which is called **center of pressure**, must be determined.
- Atmospheric pressure P_{atm} can be neglected when it acts on both sides of the surface.

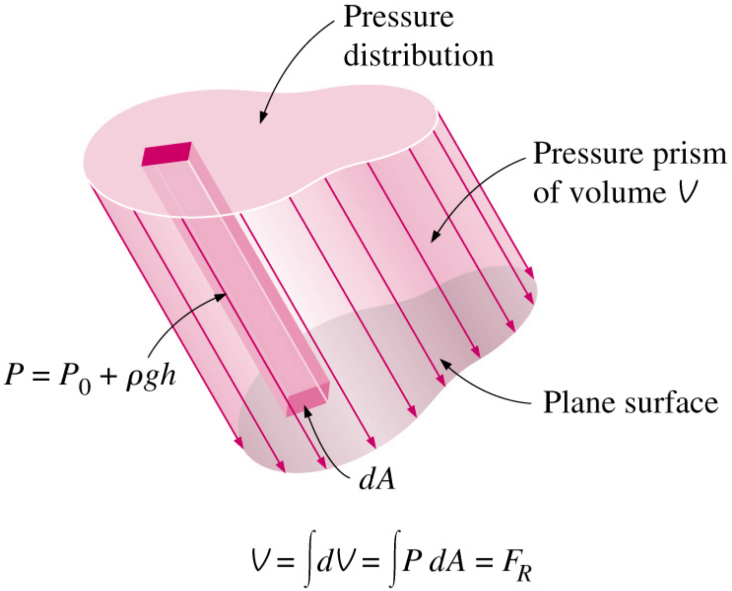
Resultant Force



The magnitude of the resultant force $\boldsymbol{F}_R$ acting on a plane surface of a completely submerged plate in a homogenous fluid is equal to the product of the pressure P_C at the centroid of the surface and the area A of the surface (which lies on the x,y plane).

Resultant Force

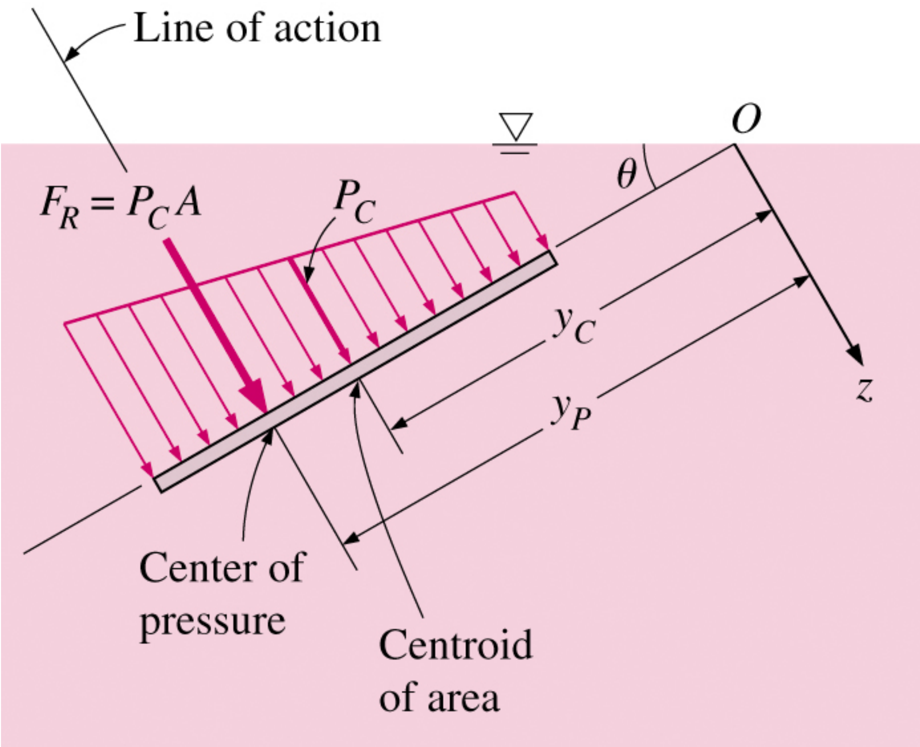




$$F_R = \int_A P\,dA = P_0 A + \rho g \sin\theta \int_A y\,dA =$$

$$= P_0 A + \rho g \sin\theta\ y_C A = (P_0 + \rho g h_C) A = P_C A$$

Center of Pressure

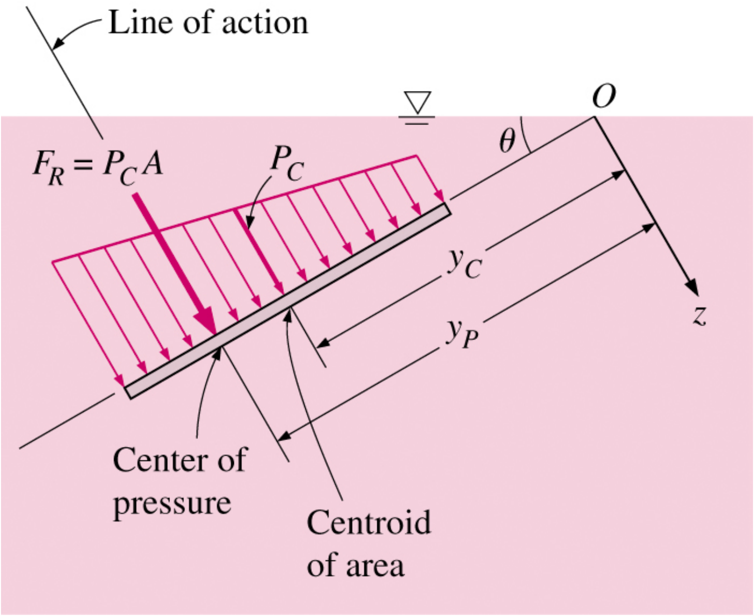


- Line of action of resultant force $F_R = P_C A$ does not pass through the centroid of the surface. It lies underneath, where pressure is higher.
- Vertical location of **Center of Pressure** is determined by equating the moment of the resultant force to the moment of the distributed pressure force. If $P_0 = 0$ the result is:

$$y_p = y_C + \frac{I_{xx,C}}{y_c A} \geq y_C$$

- $I_{xx,C}$ is tabulated for simple geometries.

Center of Pressure



$$y_P F_R = \int_A yP\, dA = \int_A y\,(P_0 + \rho\, g\, y \sin\theta)\, dA =$$

$$P_0 \int_A y\, dA + \rho\, g \sin\theta \int_A y^2\, dA =$$

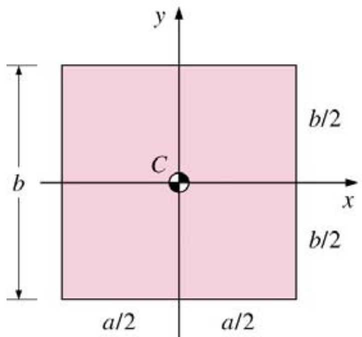
$$P_0\ y_C A + \rho\, g \sin\theta\, I_{xx,O}$$

$I_{xx,O}$ = area moment of inertia about x

Customarily an axis parallel to x but passing through the centroid of the area is considered, so that Steiner theorem can be applied:

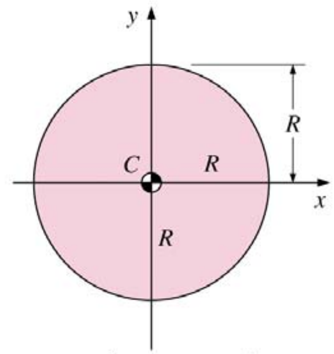
$$I_{xx,O} = I_{xx,C} + y_c^2 A$$

Centroid and Centroidal Moments of Inertia



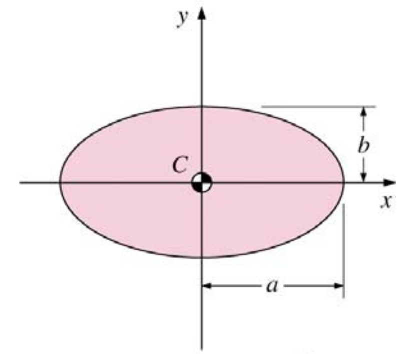
$A = ab,\ I_{xx,\,C} = ab^3/12$

(*a*) Rectangle



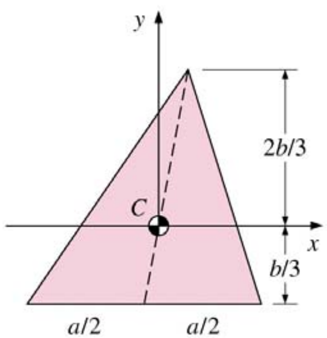
$A = \pi R^2,\ I_{xx,\,C} = \pi R^4/4$

(*b*) Circle



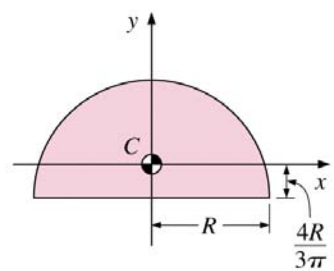
$A = \pi ab,\ I_{xx,\,C} = \pi ab^3/4$

(*c*) Ellipse



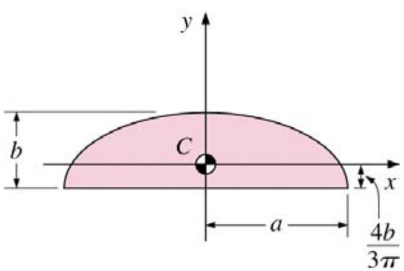
$A = ab/2,\ I_{xx,\,C} = ab^3/36$

(*d*) Triangle



$A = \pi R^2/2,\ I_{xx,\,C} = 0.109757R^4$

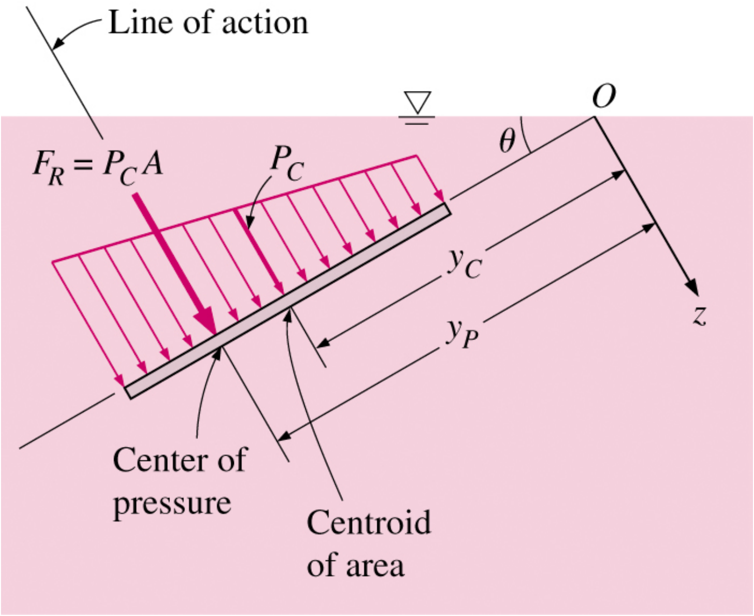
(*e*) Semicircle



$A = \pi ab/2,\ I_{xx,\,C} = 0.109757ab^3$

(*f*) Semiellipse

Center of Pressure



By taking the moment of the elementary forces about the *y*-axis it is easy to see that the

$$x_P F_R = \rho g \sin\theta \int_A xy\, dA \quad \text{(provided } P_0 \text{ is balanced)}$$

$I_{xy,O}$ = centrifugal moment of inertia about x-y

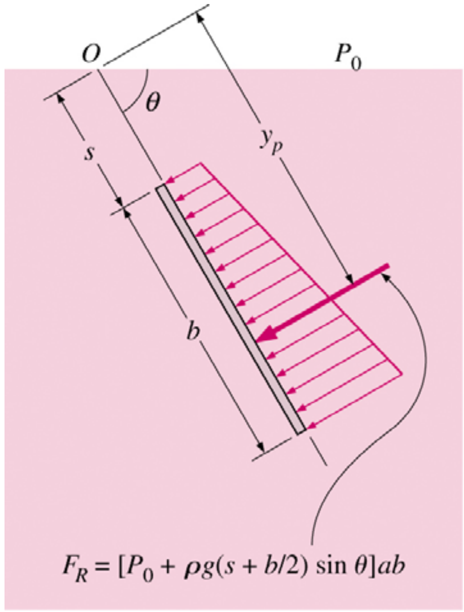
$$x_P = x_C + \frac{I_{xy,C}}{y_C A}$$

and if the plane surface is symmetric about the *y*-axis it is obvious that $x_P = x_C = 0$.

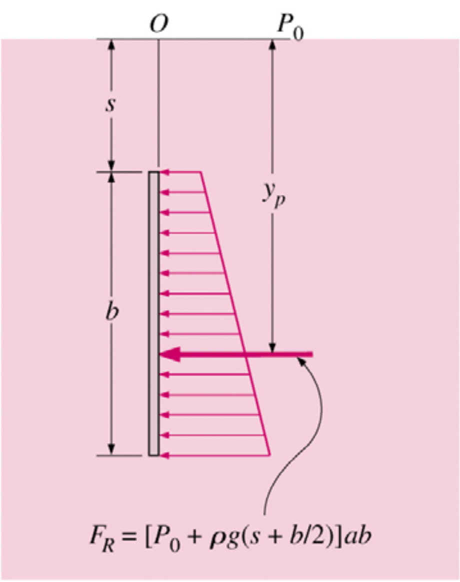
If the force is due to a <u>gas</u> at constant *P*, center of pressure and centroid coincide.

Submerged Rectangular Plates

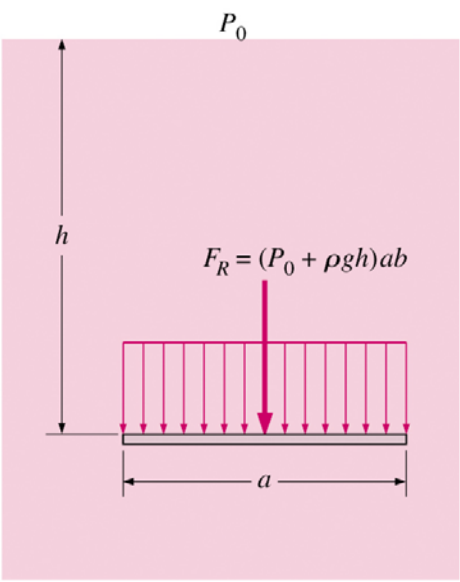
Submerged rectangular flat plate of height b and width a tilted at an angle θ from the horizontal and whose top edge is horizontal and at a distance s from the free surface …



(a) Tilted plate



(b) Vertical plate



(c) Horizontal plate

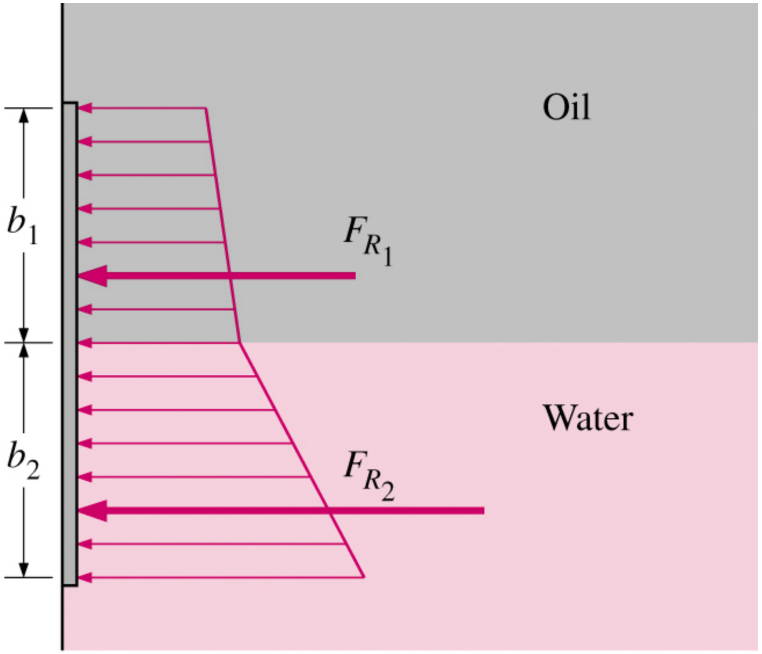
Multilayered Fluids

For a plane (or curved) surface submerged in a **multilayered fluid** of different densities, the hydrostatic force can be determined by considering different parts of surfaces in different fluids as different surfaces, finding the force on each part and then adding them using vector addition.

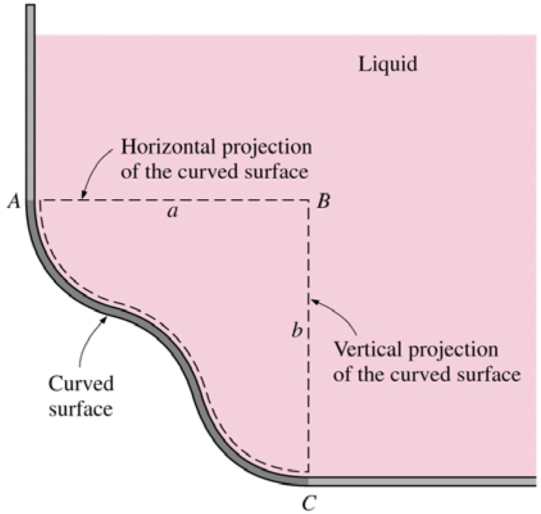
For a plane surface:

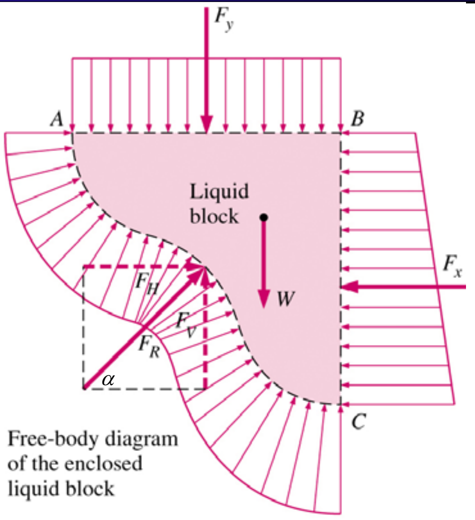
$$F_R = \sum F_{R,i} = \sum P_{C,i} \ A_i$$

where $P_{C,i} = P_0 + \rho_i g h_{C,i}$. The line of action of the equivalent force can be determined by equaling the moment of the equivalent force to the sum of the moments of the individual forces.



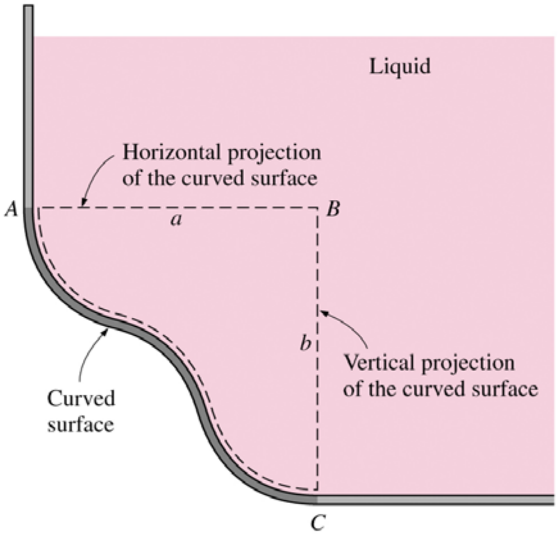
Hydrostatic Forces on Curved Surfaces

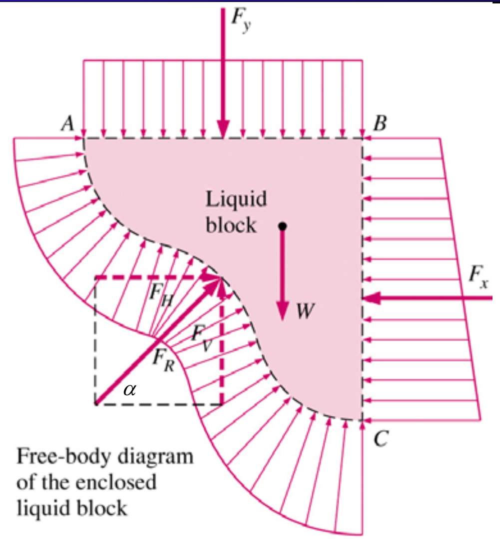


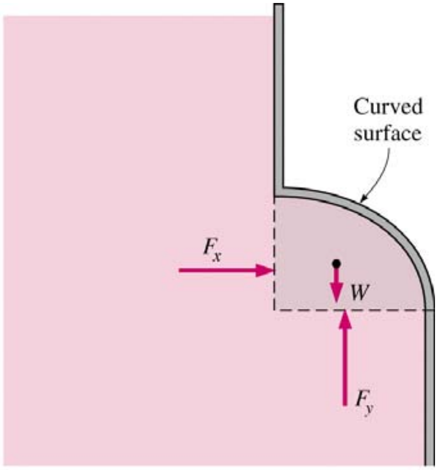


- $\boldsymbol{F_R}$ on a curved surface is more involved since it requires integration of the pressure forces that change direction along the curved surface.
- Easiest approach to determine the resultant force:
  - isolate a liquid block (no walls),
  - consider the *reaction* $\boldsymbol{F_R}$ of the wall to the fluid (Newton's third law),
  - the fluid block is in static equilibrium; through a balance of forces the horizontal and vertical components $\boldsymbol{F_H}$ and $\boldsymbol{F_V}$ can be found separately.

Hydrostatic Forces on Curved Surfaces







- Horizontal force component on curved surface:

$$F_H = F_x$$

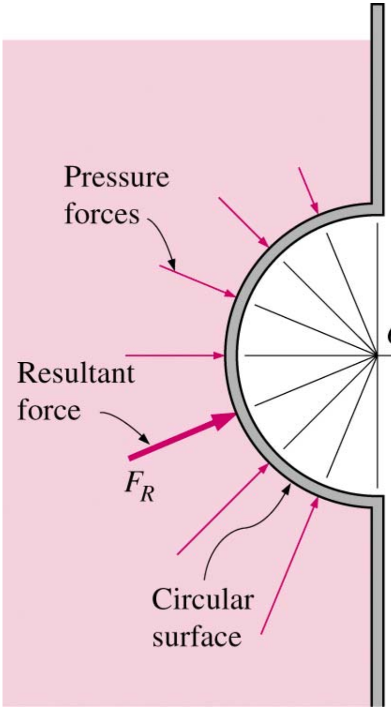
- Vertical force component on curved surface:

$$F_V = F_y + W = F_y + \rho g V$$

(if the curved surface is above the liquid, the weight of the liquid block should be subtracted, not added)

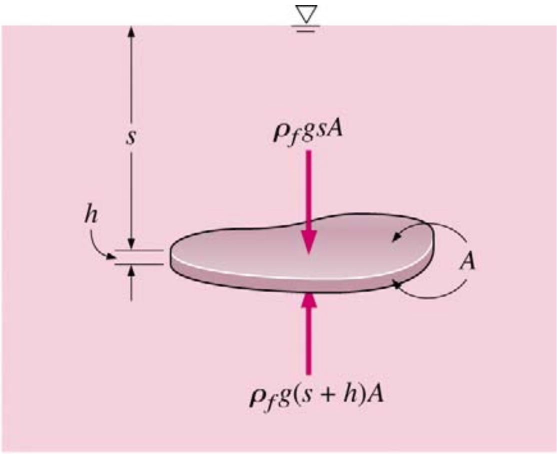
Hydrostatic Forces on Curved Surfaces

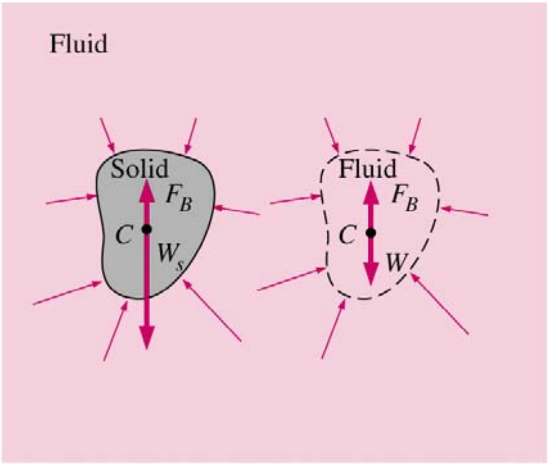
- Magnitude of force: $F_R=(F_H^2+F_V^2)^{1/2}$
- Angle of force: $\alpha = tan^{-1}(F_V/F_H)$
- Line of action of the resultant force can be determined by taking a moment about an appropriate point.
- In the simple case in which the curved surface is a *circular arc*, the resultant force always goes through the center of the circle, because the individual components are all normal to the surface and pass through O.



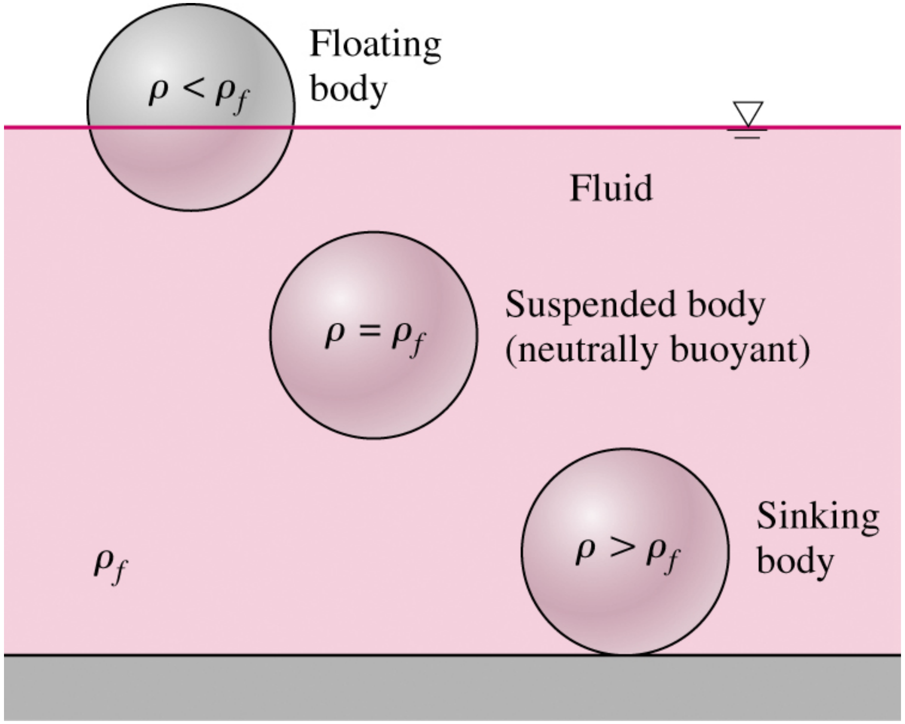
Buoyancy and Stability

- Buoyancy is due to the fluid displaced by a body. $F_B = \rho_f g V$.
- **Archimedes' principle**: The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.



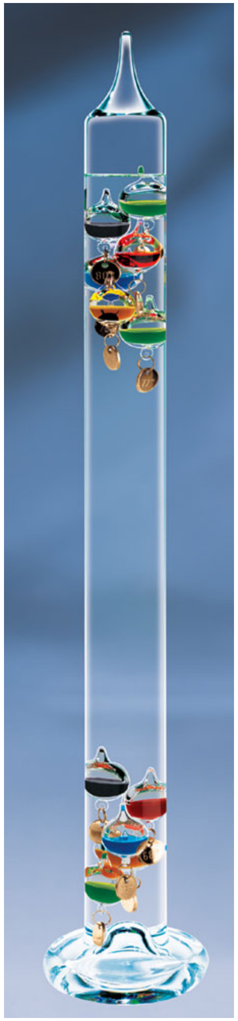


Buoyancy and Stability



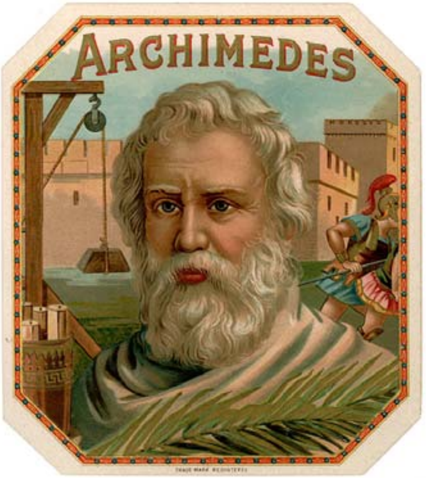
- Buoyancy force F_B is equal only to the displaced volume $\rho_f g V_{displaced}$.
- Three scenarios possible
  1. $\rho_{body} < \rho_{fluid}$: Floating body
  2. $\rho_{body} = \rho_{fluid}$: Neutrally buoyant
  3. $\rho_{body} > \rho_{fluid}$: Sinking body

Example 1: The Galilean Thermometer



- Galileo's thermometer is made of a sealed glass cylinder containing a clear liquid.
- Suspended in the liquid are a number of weights, which are sealed glass containers with colored liquid for an attractive effect.
- As the liquid changes temperature it changes density and the suspended weights rise and fall to stay at the position where their density is equal to that of the surrounding liquid.
- If the weights differ by a very small amount and ordered such that the least dense is at the top and most dense at the bottom they can form a temperature scale.

Example 2: The Golden Crown of Hiero II, King of Syracuse



- Archimedes, 287-212 B.C.
- Hiero, 306-215 B.C.
- Hiero learned of a rumor where the goldsmith replaced some of the gold in his crown with silver. He then asked Archimedes to determine whether the crown was pure gold.
- Archimedes had to develop a nondestructive testing method

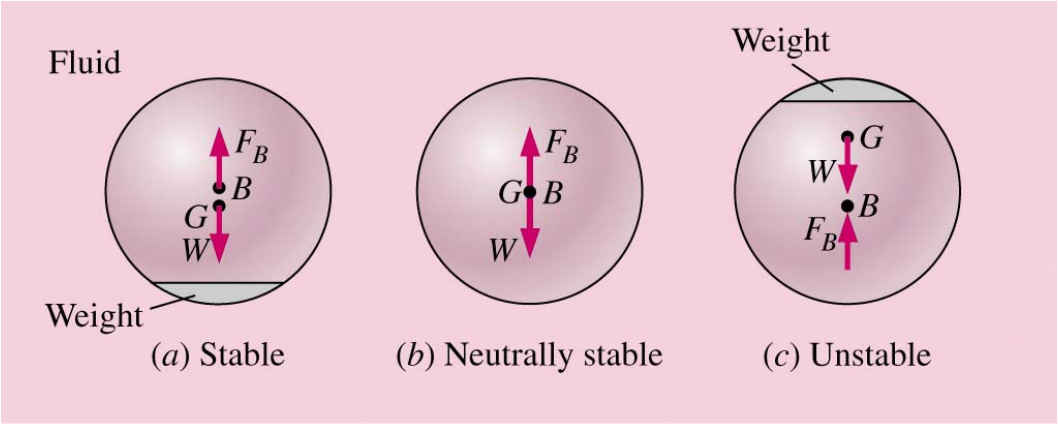
Example 2: The Golden Crown of Hiero II, King of Syracuse

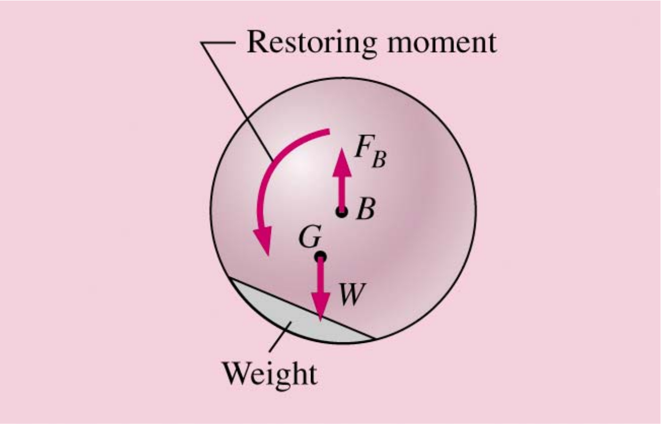
- The weight of the crown and nugget are the same in air: $W_c = \rho_c \mathcal{V}_c g = W_n = \rho_n \mathcal{V}_n g$.
- If the crown is pure gold, $\rho_c = \rho_n$ which means that the volumes must be the same, $\mathcal{V}_c = \mathcal{V}_n$.
- In water, the buoyancy force is $B = \rho_{H2O} \mathcal{V} g$.
- If the scale becomes unbalanced once immersed, then $\mathcal{V}_c \neq \mathcal{V}_n$, which means that $\rho_c \neq \rho_n$
- Goldsmith was shown to be a fraud!

Example 3: Hydrostatic Bodyfat Testing

- What is the best way to measure body fat?
- Hydrostatic Bodyfat Testing using Archimedes Principle!
- Process
  - Measure body weight $W=\rho_{body}\mathcal{V}\ g$
  - Get in tank, expel all air, and measure apparent weight W_a
  - Buoyancy force $B = W-W_a = \rho_{H2O}\mathcal{V}\ g$. This allows to retrieve body volume $\mathcal{V}$.
  - Body density can be computed ρ_{body}=W/$\mathcal{V}g$.
  - Body fat can be computed from formulas.

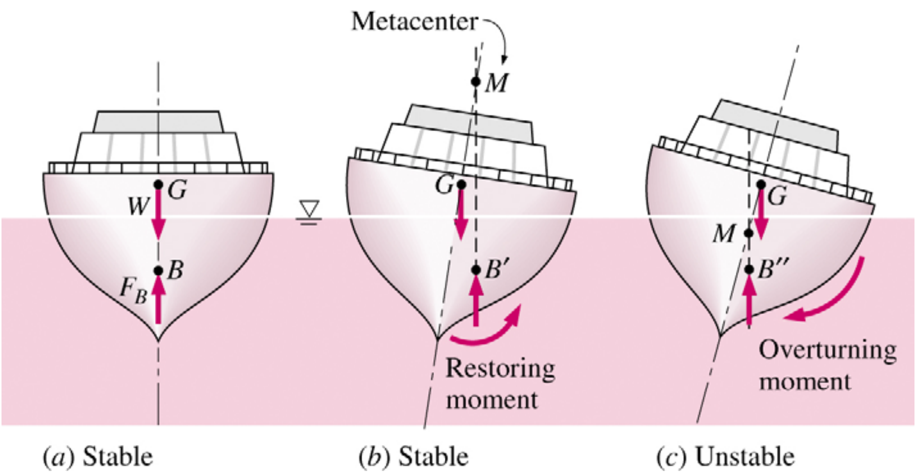
Stability of Immersed Bodies





- Rotational stability of immersed bodies depends upon relative location of *center of gravity G* and *center of buoyancy B* (which is the centroid of the displaced volume).
  - *G* below *B*: **stable** (small disturbances generate a restoring force that returns the body to its original, equilibrium position)
  - *G* above *B*: **unstable** (any disturbance causes the body to diverge from its original position)
  - *G* coincides with *B*: **neutrally stable** (when displaced the body stays at its new location)

Stability of Floating Bodies



- If body is bottom heavy (*G* lower than *B*), it is always stable.
- Unlike immersed bodies, floating bodies can be stable when *G* is higher than *B* due to shift in location of center of buoyancy and creation of restoring moment.
- Measure of stability is the metacentric height *GM*. If *GM*>0 (*i.e*. *M* is above *G*), ship is stable.