Chapter 4: Fluid Kinematics

Overview

- Fluid kinematics deals with the motion of fluids without considering the forces and moments which create the motion.
- Items discussed in this Chapter.
  - Material derivative and its relationship to Lagrangian and Eulerian descriptions of fluid flow.
  - Fundamental kinematic properties of fluid motion and deformation.
  - Reynolds Transport Theorem.

Lagrangian Description

- Lagrangian description of fluid flow tracks the position and velocity of individual particles.
- Based upon Newton's laws of motion.
- Difficult to use for practical flow analysis.
  - Fluids are composed of *billions* of molecules.
  - Interaction between molecules hard to describe/model.
- However, useful for specialized applications
  - Sprays, particles, bubble dynamics, rarefied gases.
  - Coupled Eulerian-Lagrangian methods.
- Named after Italian mathematician Joseph Louis Lagrange (1736-1813).

Eulerian Description

- Eulerian description of fluid flow: a **flow domain** or **control volume** is defined by which fluid flows in and out.
- We define **field variables** which are functions of space and time.
  - Pressure field, $P = P(x,y,z,t)$
  - Velocity field, $\vec{V} = \vec{V}(x,y,z,t)$

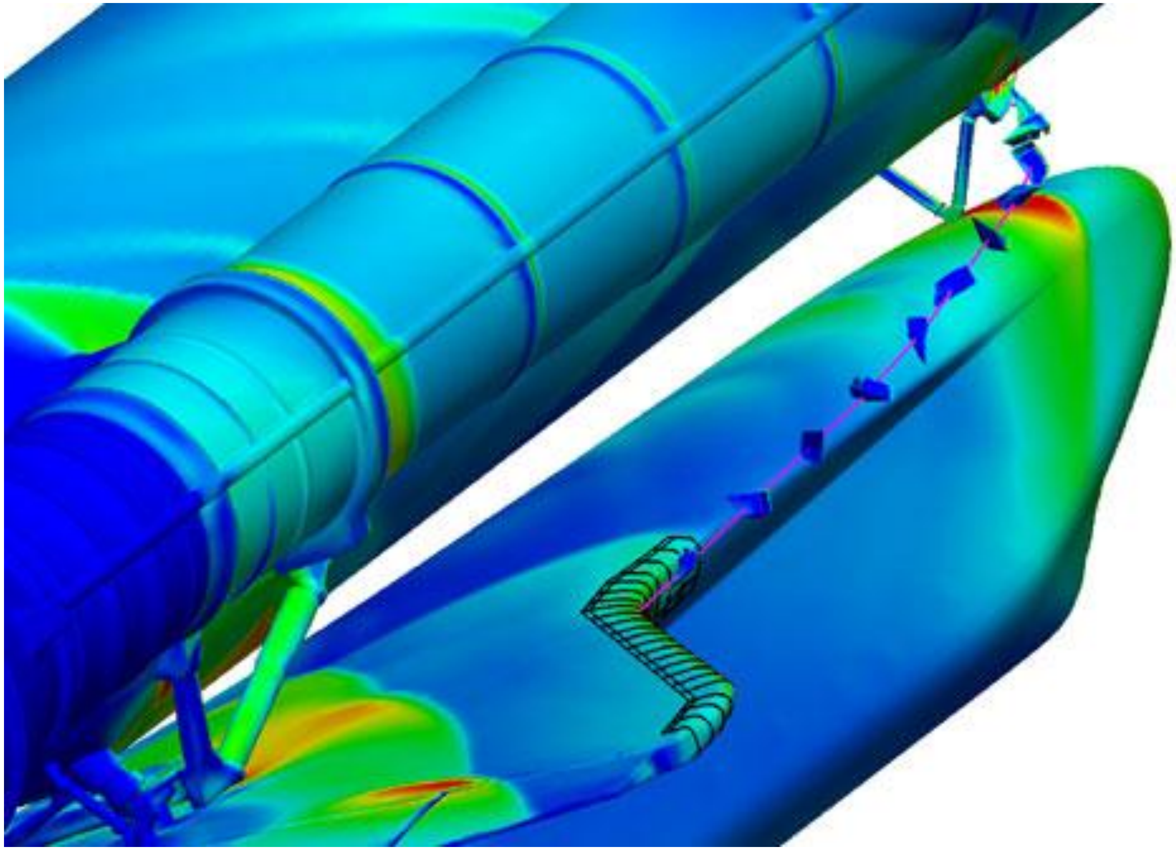
$$\vec{V} = u(x,y,z,t)\vec{i} + v(x,y,z,t)\vec{j} + w(x,y,z,t)\vec{k}$$

  - Acceleration field, $\vec{a} = \vec{a}(x,y,z,t)$

$$\vec{a} = a_x(x,y,z,t)\vec{i} + a_y(x,y,z,t)\vec{j} + a_z(x,y,z,t)\vec{k}$$

  - These (and other) field variables define the **flow field**.
- Well suited for formulation of initial boundary-value problems (PDE's).
- Named after Swiss mathematician Leonhard Euler (1707-1783).

Example: Coupled Eulerian-Lagrangian Method



Forensic analysis of Columbia accident: simulation of shuttle debris trajectory using Eulerian CFD for flow field and Lagrangian method for the debris.

Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity: $$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point is the same as the fluid velocity, $$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t), t\right)$$

- To take the time derivative, chain rule must be used.

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

Acceleration Field

- Since $$\frac{dx_{particle}}{dt} = u, \frac{dy_{particle}}{dt} = v, \frac{dz_{particle}}{dt} = w$$

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

- In vector form, the acceleration can be written as

$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

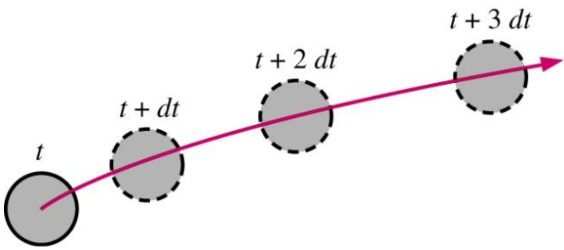
- First term is called the **local acceleration** and is nonzero only for unsteady flows.
- Second term is called the **advective** (or **convective**) **acceleration** and accounts for the effect of the fluid particle moving to a new location in the flow, where the velocity is different (it can thus be nonzero even for steady flows).

Material Derivative

- The total derivative operator *d/dt* is call the **material derivative** and is often given special notation, *D/Dt*.

$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

- Advective acceleration is nonlinear: source of many phenomena and primary challenge in solving fluid flow problems.
- Provides “transformation”' between Lagrangian and Eulerian frames.

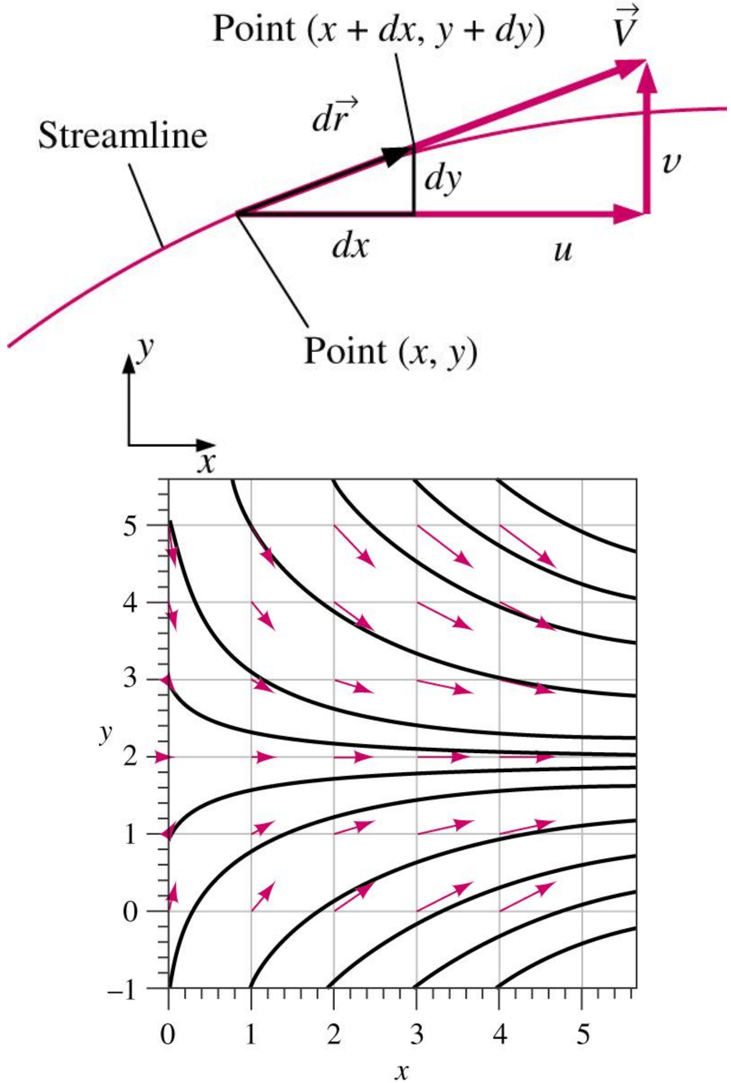


- Other names for the material derivative include: **total, particle, Lagrangian, Eulerian,** and **substantial** derivative.

Flow Visualization

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines
  - Refractive flow visualization techniques
  - Surface flow visualization techniques

Streamlines and streamtubes



- A **streamline** is a curve that is everywhere tangent to the *instantaneous* local velocity vector.
- Consider an infinitesimal arc length along a streamline:

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- By definition $d\vec{r}$ must be parallel to the local velocity vector

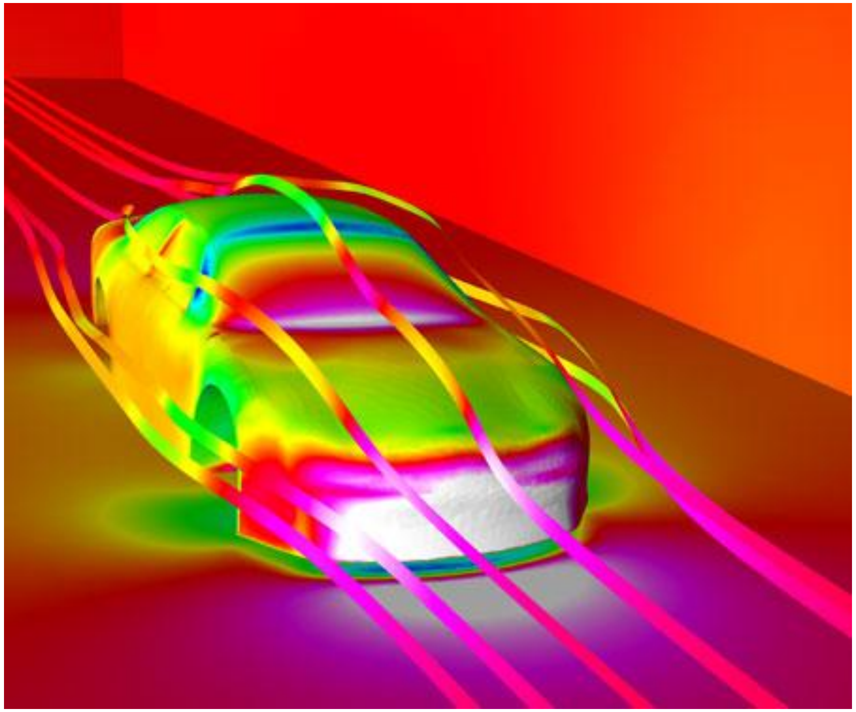
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- Geometric arguments result in the equation for a streamline

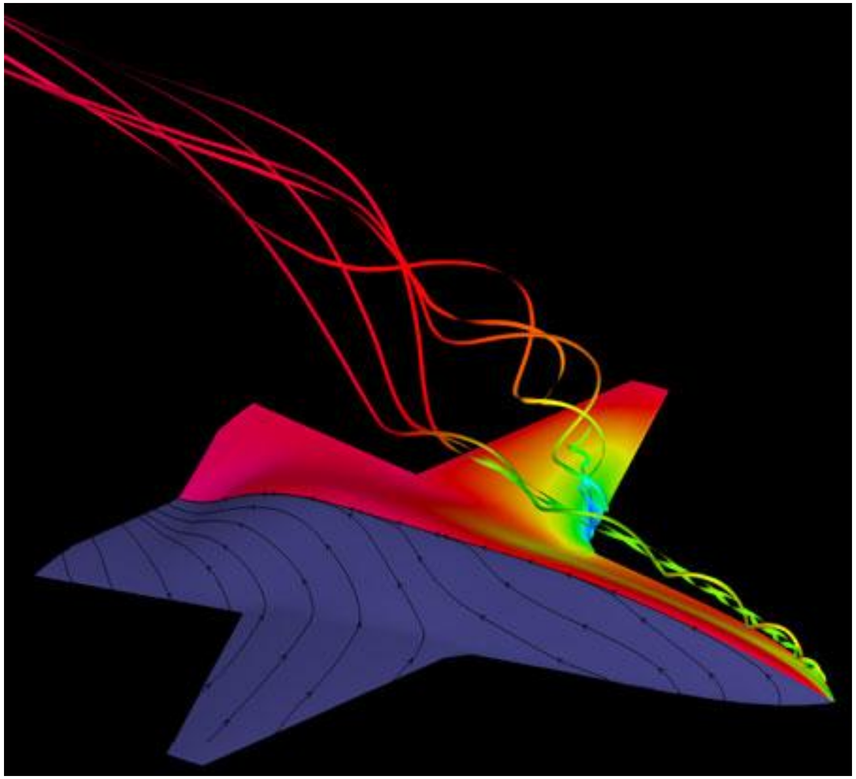
$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Streamlines and streamtubes

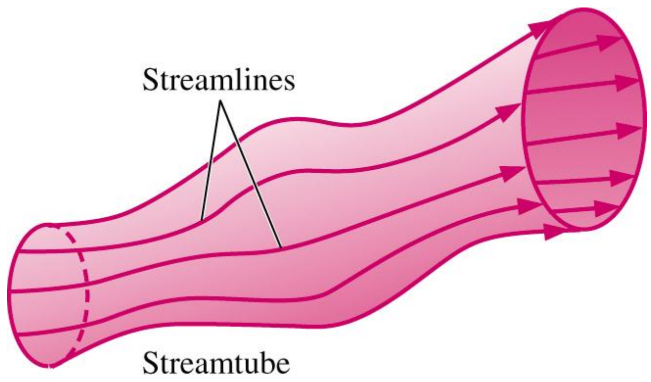
NASCAR surface pressure contours and streamlines



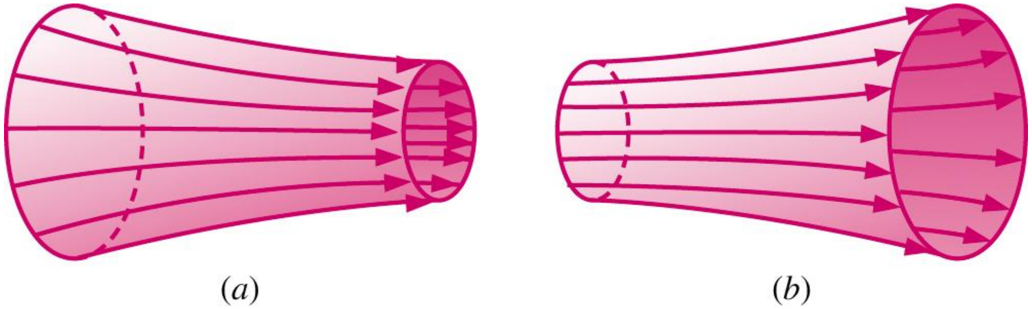
Airplane surface pressure contours, volume streamlines, and surface streamlines



Streamlines and streamtubes

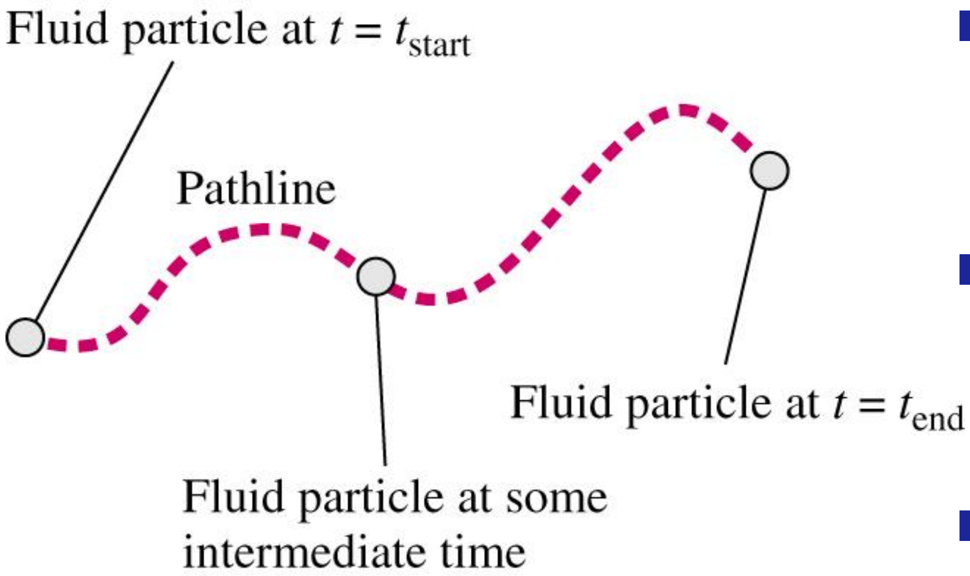


A **streamtube** consists of a bundle of individual streamlines. Since fluid cannot cross a streamline (by definition), fluid within a streamtube must remain there. Streamtubes are, obviously, instantaneous quantities and they may change significantly with time.



In the converging portion of an incompressible flow field, the diameter of the streamtube must decrease as the velocity increases, so as to conserve mass.

Pathlines



- A **pathline** is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

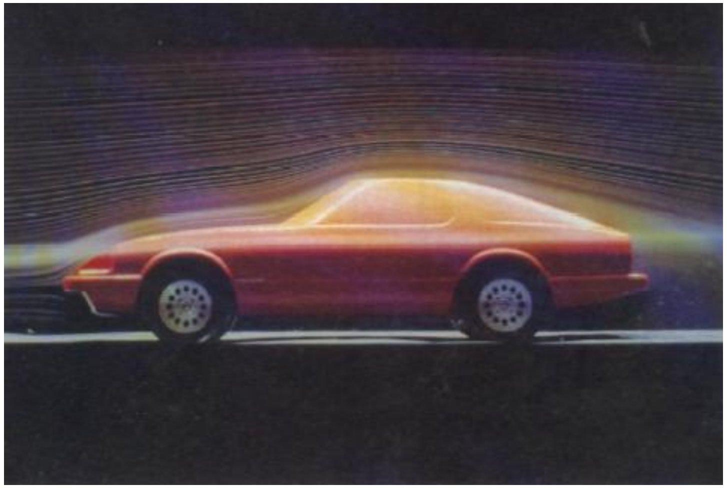
$$\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

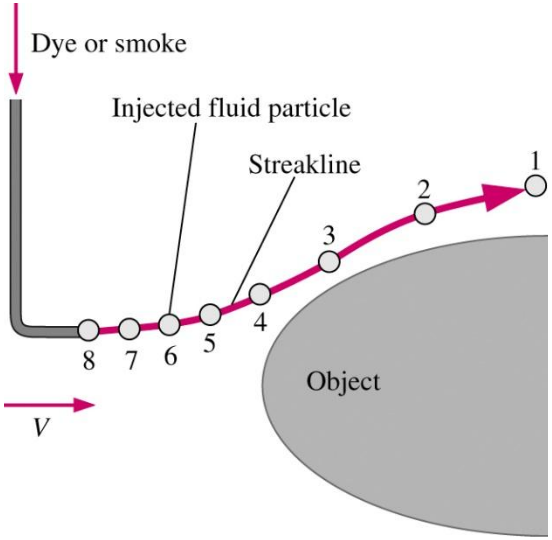
- Particle location at time t:

$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V} dt$$

- Particle Image Velocimetry (PIV) is a modern experimental technique to measure velocity field over a plane in the flow field.

Streaklines



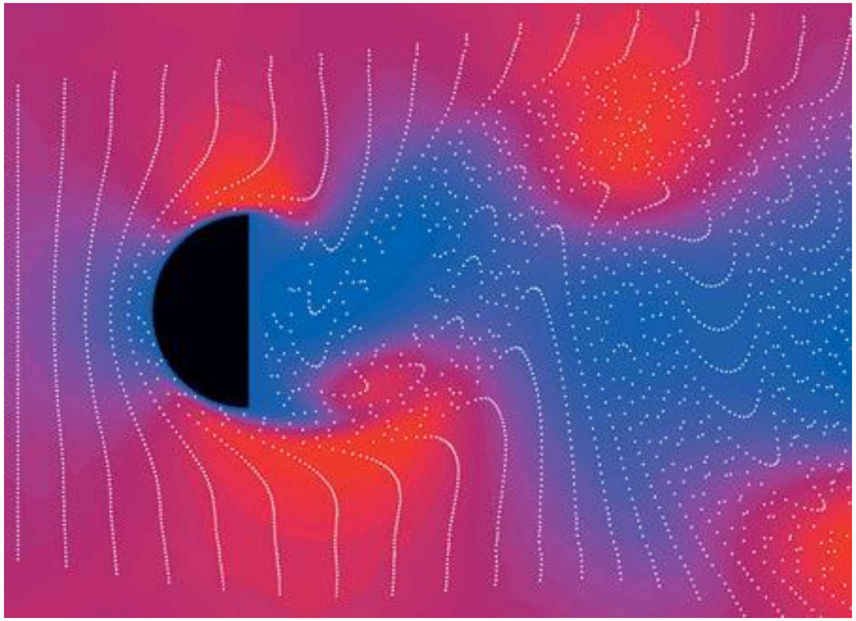


- A **streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: continuous introduction of dye (in a water flow) or smoke (in an airflow) from a point.

Comparisons

- If the flow is *steady*, streamlines, pathlines and streaklines are identical.
- For unsteady flows, they can be very different.
  - Streamlines provide an instantaneous picture of the flow field
  - Pathlines and streaklines are flow patterns that have a time history associated with them.

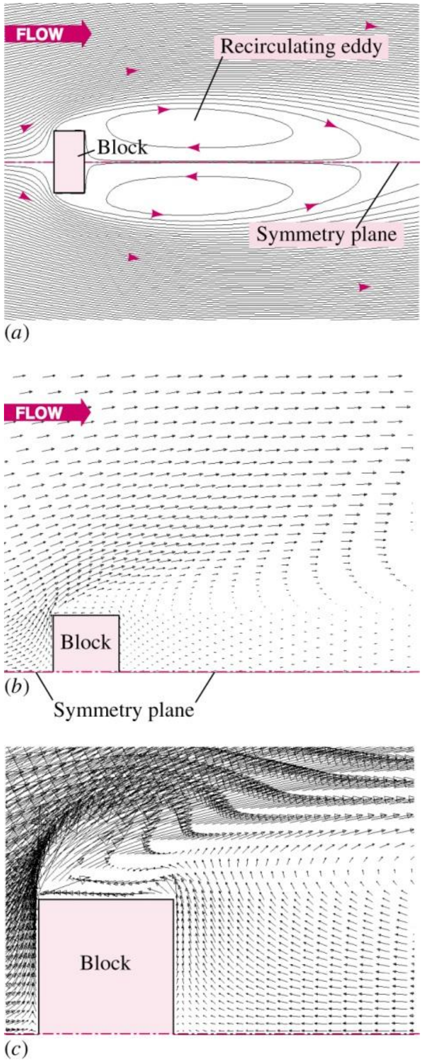
Timelines



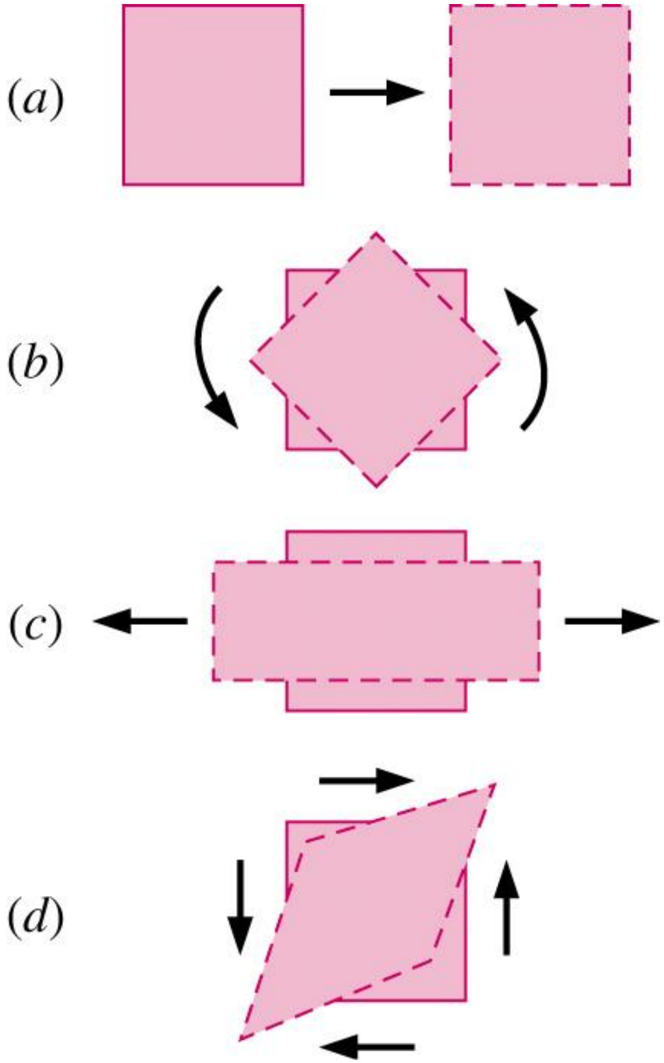
- A **timeline** is a set of adjacent fluid particles that were marked at the same (earlier) instant in time.
- Experimentally, timelines can be generated using a hydrogen bubble wire: a line is marked and its movement/deformation is followed in time.

Plots of Data

- A **Profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.
- A **Vector plot** is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.
- A **Contour plot** shows curves of constant values of a scalar property (or magnitude for a vector property) at an instant in time.



Kinematic Description



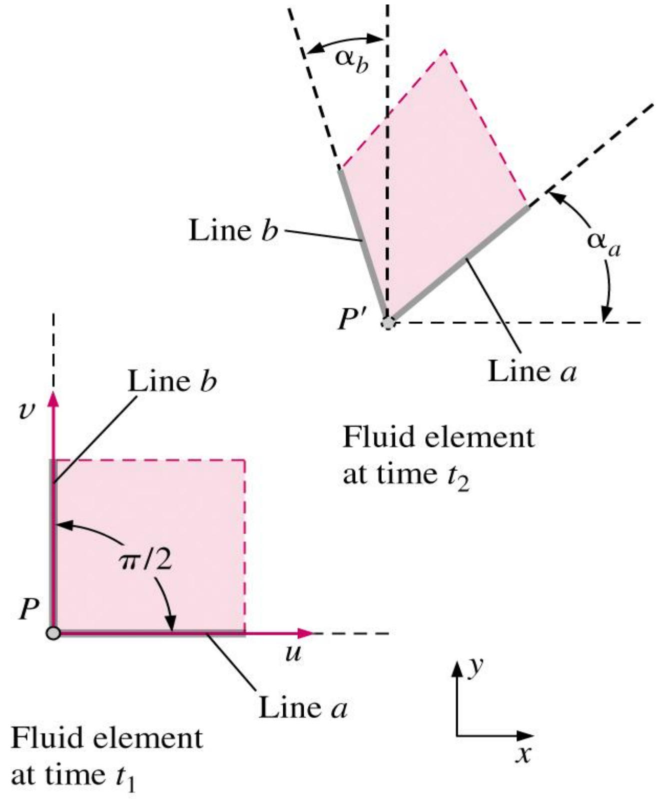
- In fluid mechanics (as in solid mechanics), an element may undergo four fundamental types of motion.
  - a) **Translation**
  - b) **Rotation**
  - c) **Linear strain**
  - d) **Shear strain**
- Because fluids are in constant motion, motion and deformation is best described in terms of rates
  - a) velocity: rate of translation
  - b) angular velocity: rate of rotation
  - c) linear strain rate: rate of linear strain
  - d) shear strain rate: rate of shear strain

Rate of Translation and Rotation

- To be useful, these *deformation rates* must be expressed in terms of velocity and derivatives of velocity
- The **rate of translation vector** is described mathematically as the *velocity vector*.
  In Cartesian coordinates:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- **Rate of rotation** (angular velocity) at a point is defined as the *average rotation rate* of two lines which are initially perpendicular and that intersect at that point.



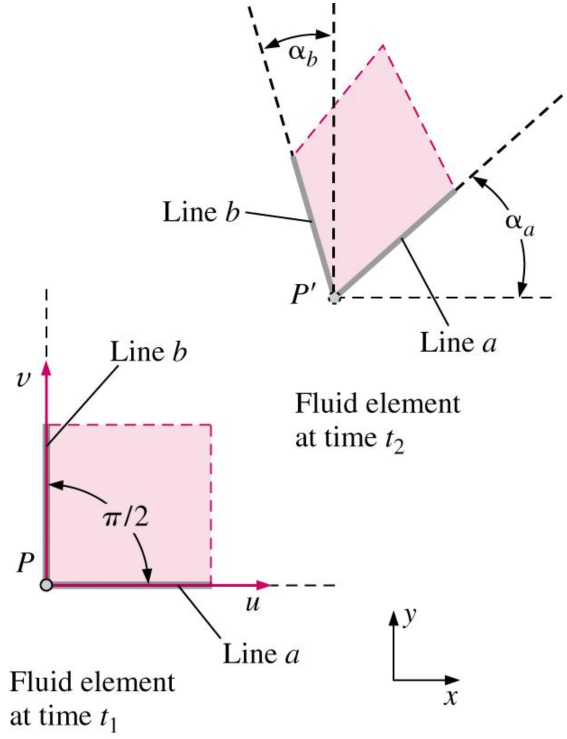
Rate of Rotation

In 2D the average rotation angle of the fluid element about the point P is

$$\bar{\alpha} = (\alpha_a + \alpha_b)/2$$

The rate of rotation of the fluid element about P is

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



In 3D the angular velocity **vector** is:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Linear Strain Rate

- **Linear Strain Rate** is defined as the *rate of increase in length per unit length.*
- In Cartesian coordinates

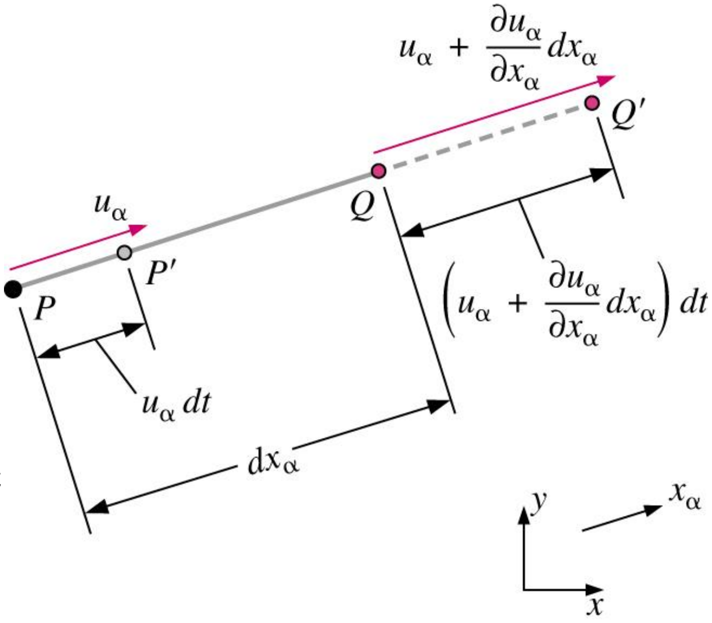
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

- The rate of increase of volume of a fluid element per unit volume is the **volumetric strain rate**, in Cartesian coordinates:

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

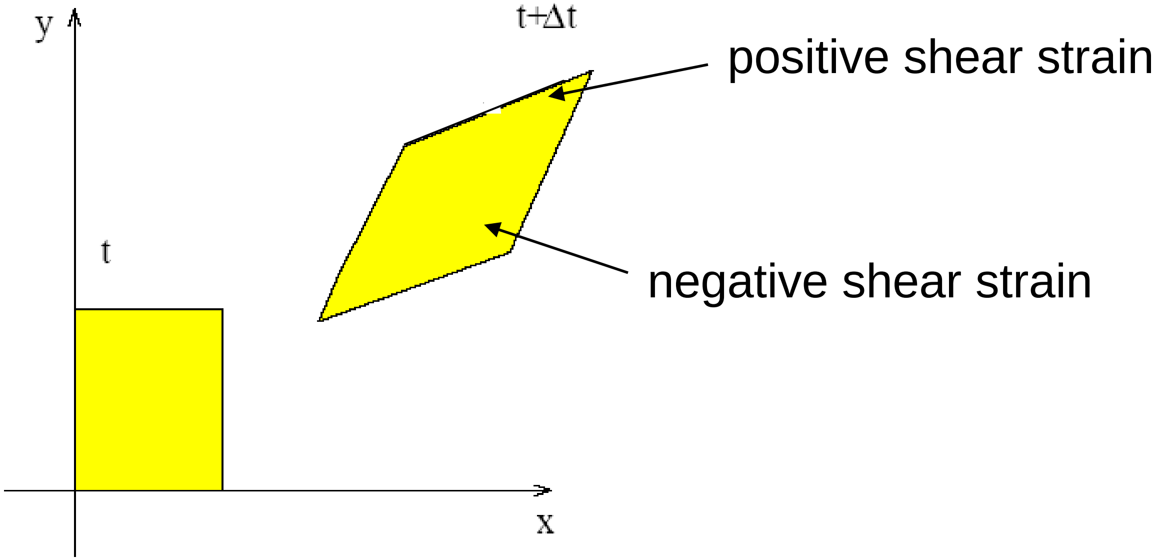
(we are talking about a *material volume*, hence the D)

- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.



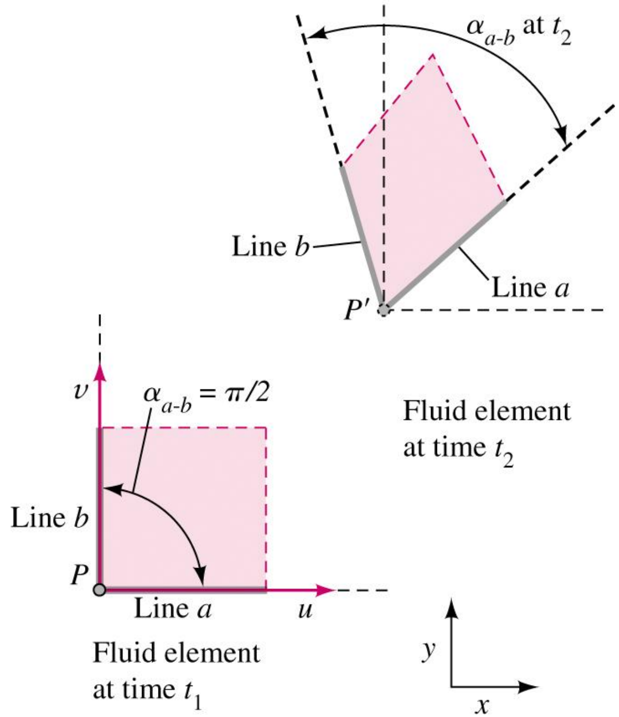
Shear Strain Rate

- **Shear Strain Rate** at a point is defined as *half the rate of decrease of the angle between two initially perpendicular lines that intersect at a point*.



Shear Strain Rate

- The shear strain at point P is $\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{a\text{-}b}$



- Shear strain rate can be expressed in Cartesian coordinates as:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

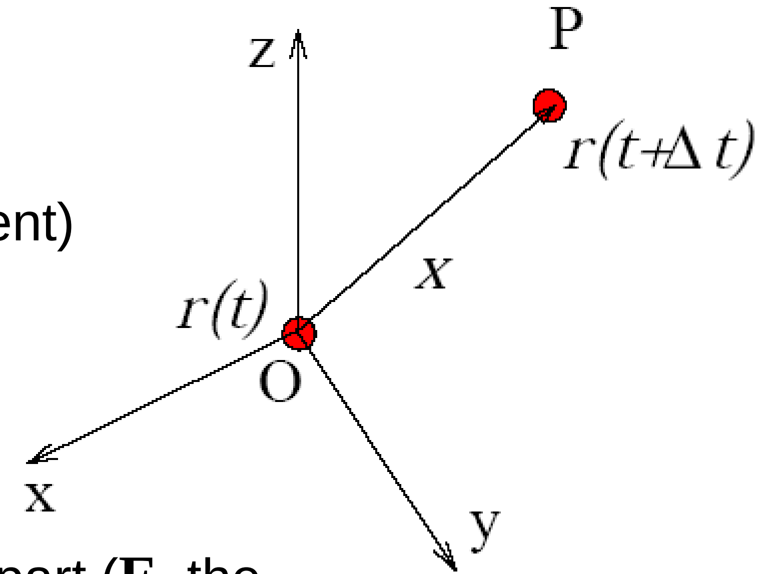
Shear Strain Rate

We can combine *linear* strain rate and *shear* strain rate into one <u>symmetric</u> second-order tensor called E: **strain-rate tensor**. In Cartesian coordinates:

$$\mathrm{E} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

State of Motion

Particle moves from O to P in time Δt



Taylor series around O (for a small displacement) yields:

$$\mathbf{u}_P = \mathbf{u}_O + \nabla \mathbf{u}\,|_O \cdot \mathbf{x} + \mathcal{O}(\mathbf{x}^2)$$

The tensor $\nabla \mathbf{u}$ can be split into a symmetric part ($\mathbf{E}$, the strain tensor) and an antisymmetric part ($\boldsymbol{\Omega}$, the rotation tensor) part as

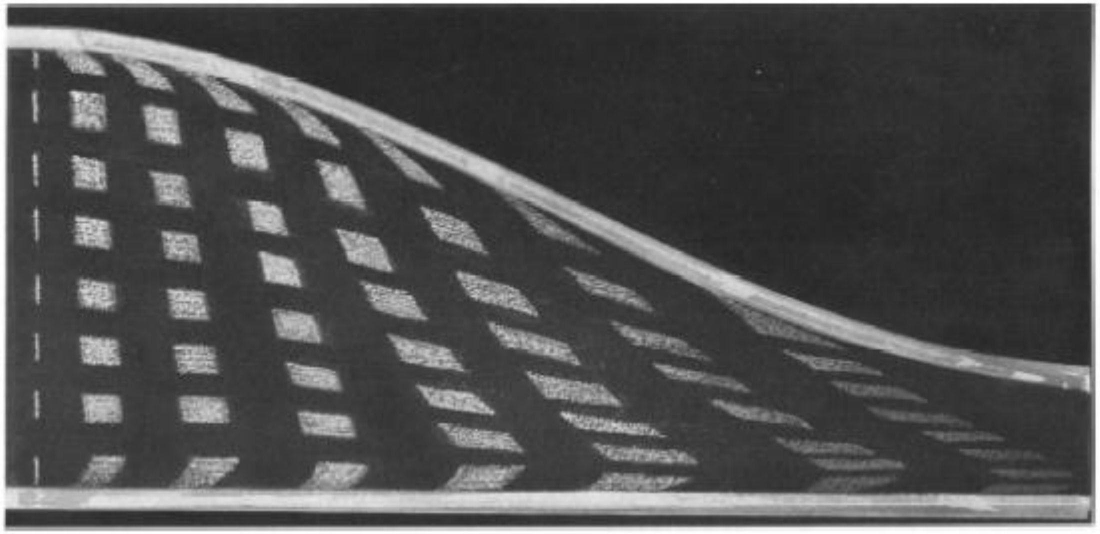
$$\nabla \mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) + \frac{1}{2}\left(\nabla \mathbf{u} - \nabla \mathbf{u}^T\right) = \mathbf{E} + \boldsymbol{\Omega}$$

so that $\underline{\mathbf{u}_P \cong \mathbf{u}_O + \mathbf{E} \cdot \mathbf{x} + \boldsymbol{\Omega} \cdot \mathbf{x}}$

Shear Strain Rate

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion
- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.
  - Feature extraction and flow visualization in CFD simulations.

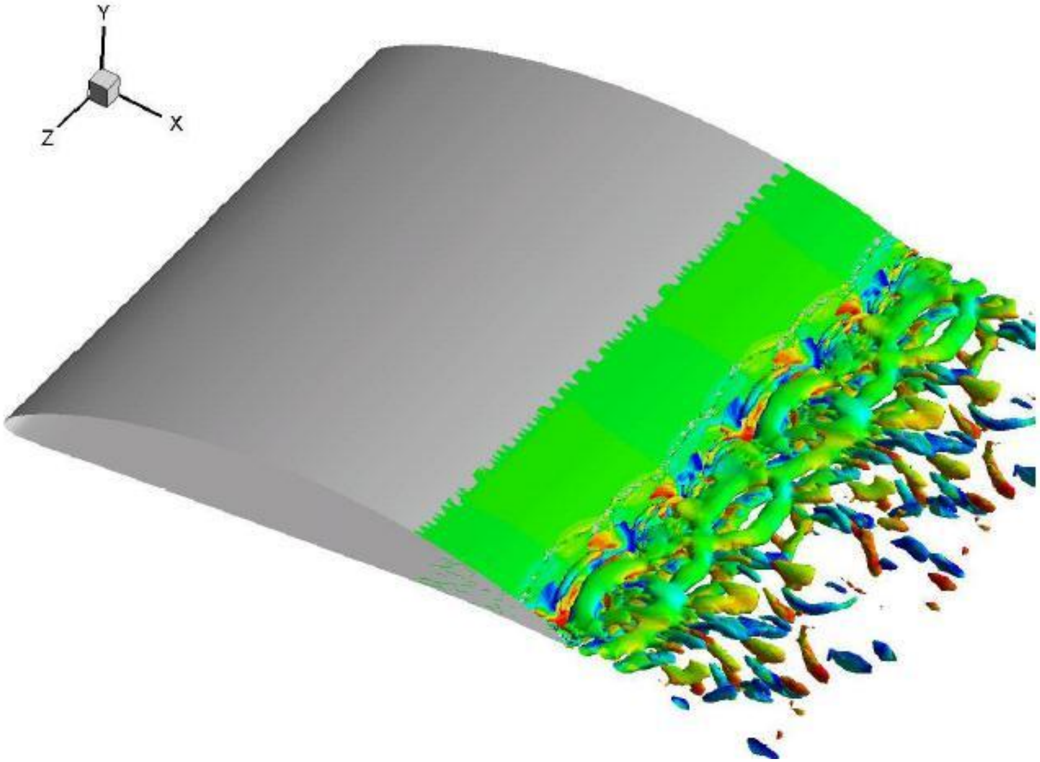
Translation, Rotation, Linear Strain, Shear Strain, and Volumetric Strain



Deformation of fluid elements (made visible with a tracer) during their compressible motion through a convergent channel; shear strain is more evident near the walls because of larger velocity gradients (a *boundary layer* is present there).

Strain Rate Tensor

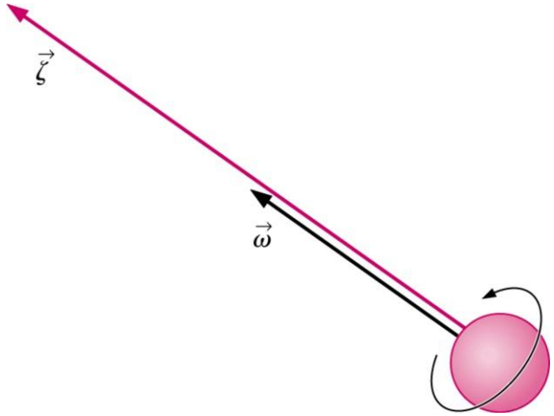
Example: Visualization of trailing-edge turbulent eddies for a hydrofoil with a beveled trailing edge



Feature extraction method is based upon eigen-analysis of the strain-rate tensor.

Vorticity and Rotationality

- The **vorticity vector** is defined as the *curl* of the velocity vector $\vec{\zeta} = \vec{\nabla} \times \vec{V}$
- Vorticity is equal to twice the angular velocity of a fluid particle: $\vec{\zeta} = 2\vec{\omega}$



Cartesian coordinates

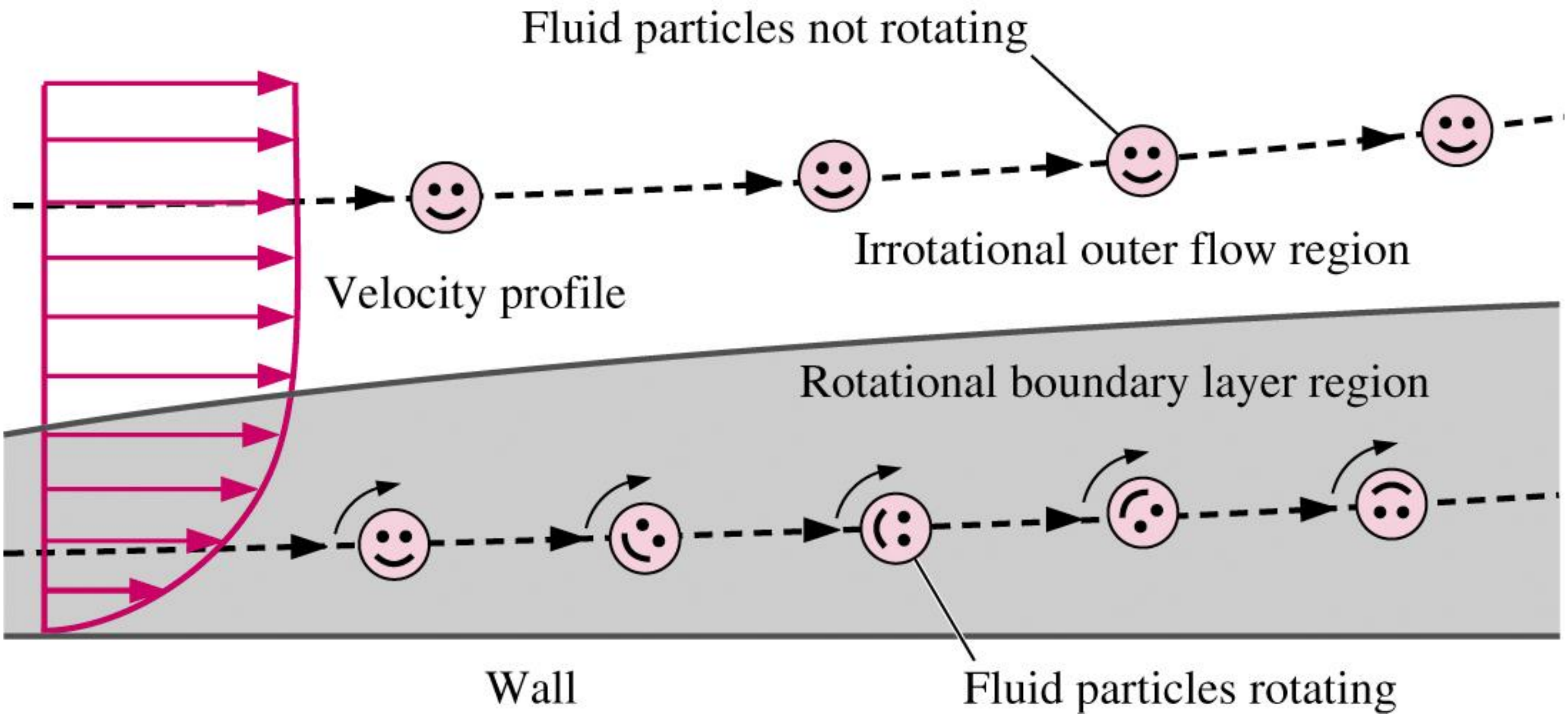
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$

Cylindrical coordinates

$$\vec{\zeta} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \vec{e}_\theta + \left(\frac{\partial \left(r u_\theta \right)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \vec{e}_z$$

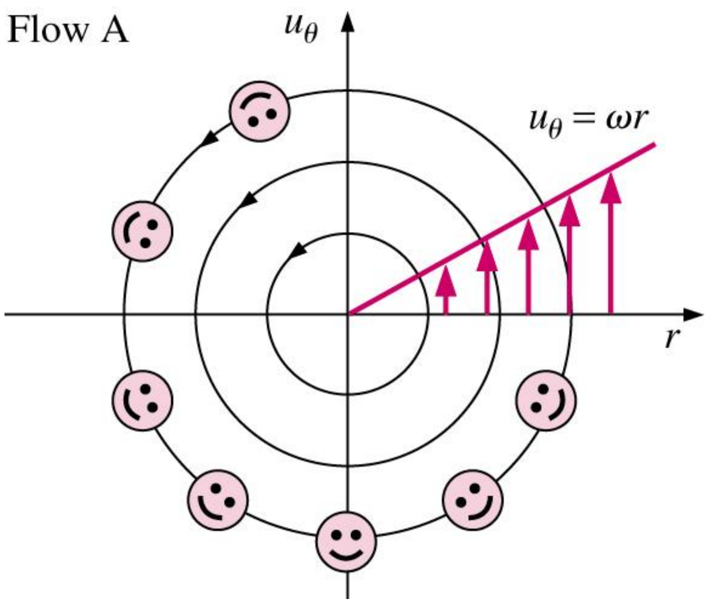
- In regions where $\zeta = 0$, the flow is called **irrotational.**
- Elsewhere, the flow is called **rotational.**

Vorticity and Rotationality



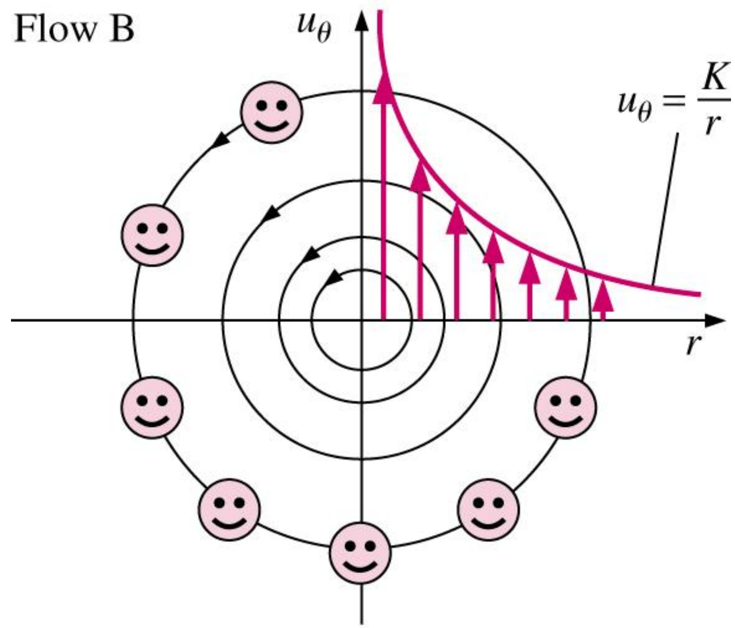
Comparison of Two Circular Flows

Special case: consider two flows with circular streamlines



$u_r = 0, u_\theta = \omega r$ *solid-body rotation*

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$



(b)

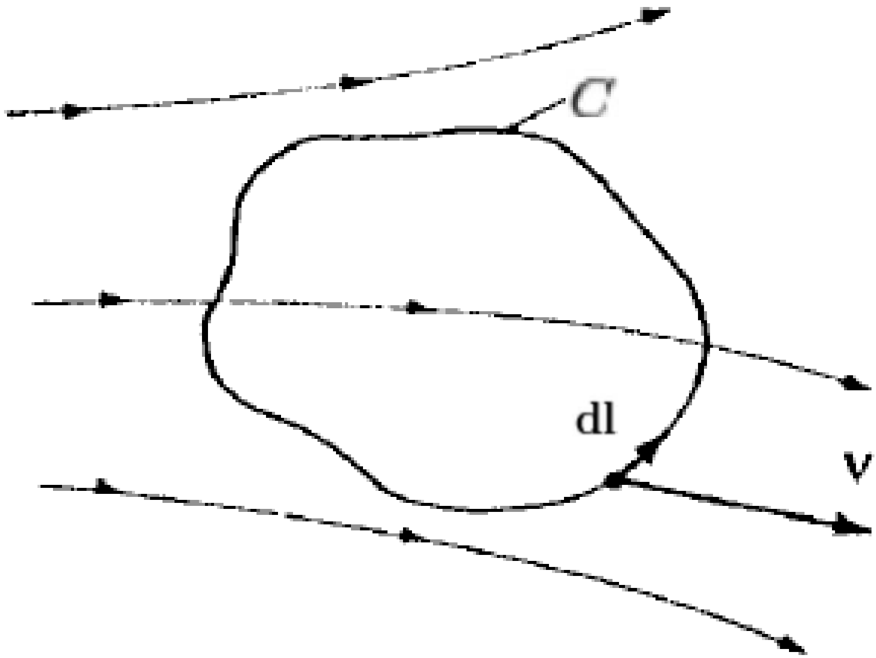
$u_r = 0, u_\theta = \frac{K}{r}$ *line vortex*

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

Circulation and vorticity

Circulation:

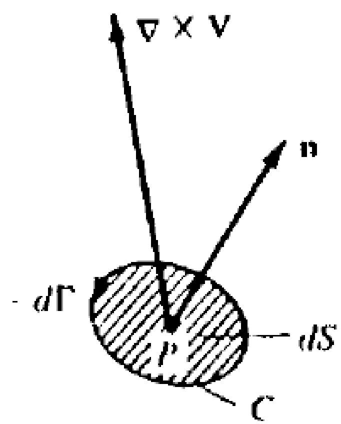
$$\Gamma = \oint_C \mathbf{V} \cdot \mathbf{dl}$$



Stokes theorem:

$$\Gamma = \oint_C \mathbf{V} \cdot \mathbf{dl} = \iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{dS}$$

If the flow is irrotational everywhere within the contour of integration C then $\Gamma = 0$.

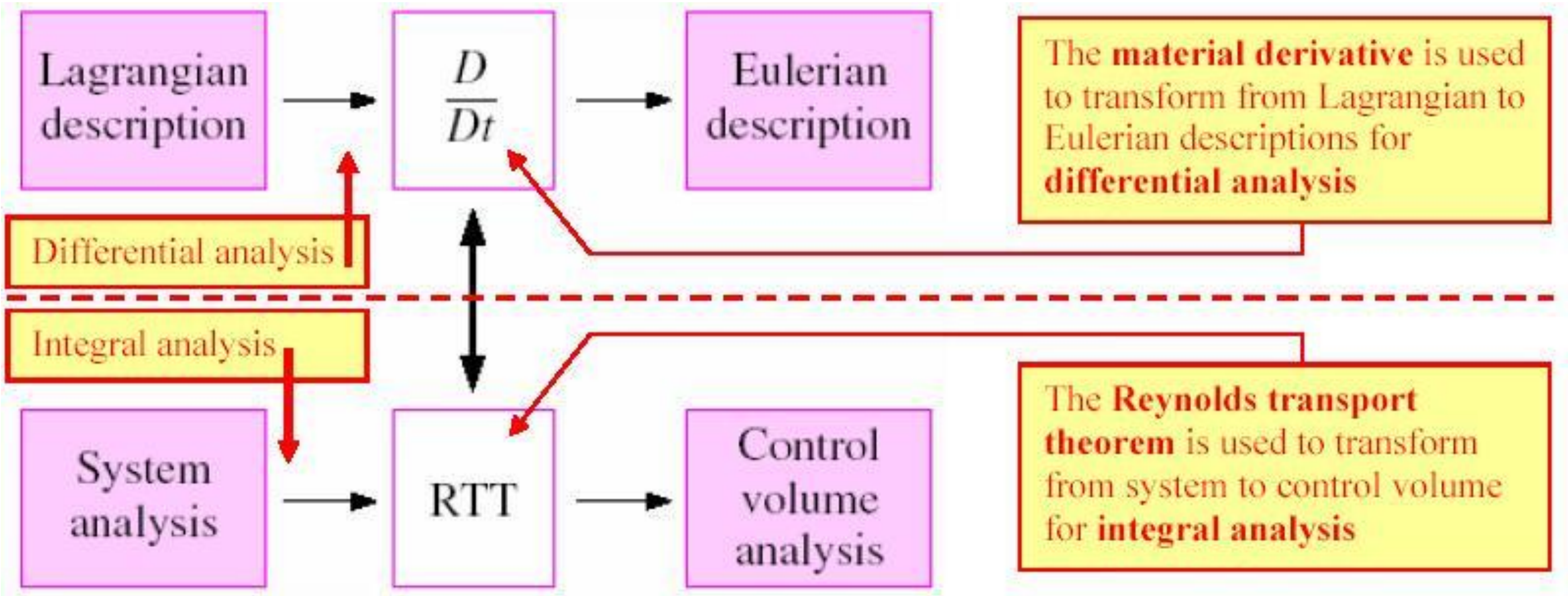


$$(\nabla \times \mathbf{V}) \cdot \mathbf{n} = \frac{d\Gamma}{dS}$$

Reynolds Transport Theorem (RTT)

- A **system** is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A **control volume** is a region in space chosen for study. Mass can cross a control surface.
- The fundamental conservation laws (conservation of mass, energy, and momentum) apply directly to systems.
- However, in most fluid mechanics problems, control volume analysis is preferred over system analysis (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to transform the conservation laws from a system to a control volume. This is accomplished with the Reynolds transport theorem (RTT).

Reynolds Transport Theorem (RTT)



There is a direct analogy between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields).

Reynolds Transport Theorem (RTT)

- Material derivative (differential analysis):

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right) b$$

- RTT, moving or deformable CV (integral analysis):

$$\vec{V}_r = \vec{V} - \vec{V}_{cs}$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \left(\vec{V}_r \cdot \vec{n}\right) dA$$

	Mass	Momentum	Energy	Angular momentum
B, Extensive properties	m	$m\vec{V}$	E	$\vec{H}$
b, Intensive properties	1	$\vec{V}$	e	$\left(\vec{r} \times \vec{V}\right)$

- In Chaps 5 and 6, we will apply RTT to conservation of mass, energy, linear momentum, and angular momentum.

Reynolds Transport Theorem (RTT)

- RTT, fixed *CV*:

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

  - Time rate of change of the property *B* of the closed system is equal to (Term 1) + (Term 2)
  - Term 1: time rate of change of *B* of the control volume
  - Term 2: net flux of *B* out of the control volume by mass crossing the control surface