Chapter 5: Mass, Bernoulli, and Energy Equations

Introduction

- This chapter deals with 3 equations commonly used in fluid mechanics
  - The mass equation is an expression of the conservation of mass principle.
  - The Bernoulli equation is concerned with the conservation of kinetic, potential, and flow energies of a fluid stream and their conversion to each other.
  - The energy equation is a statement of the conservation of energy principle.

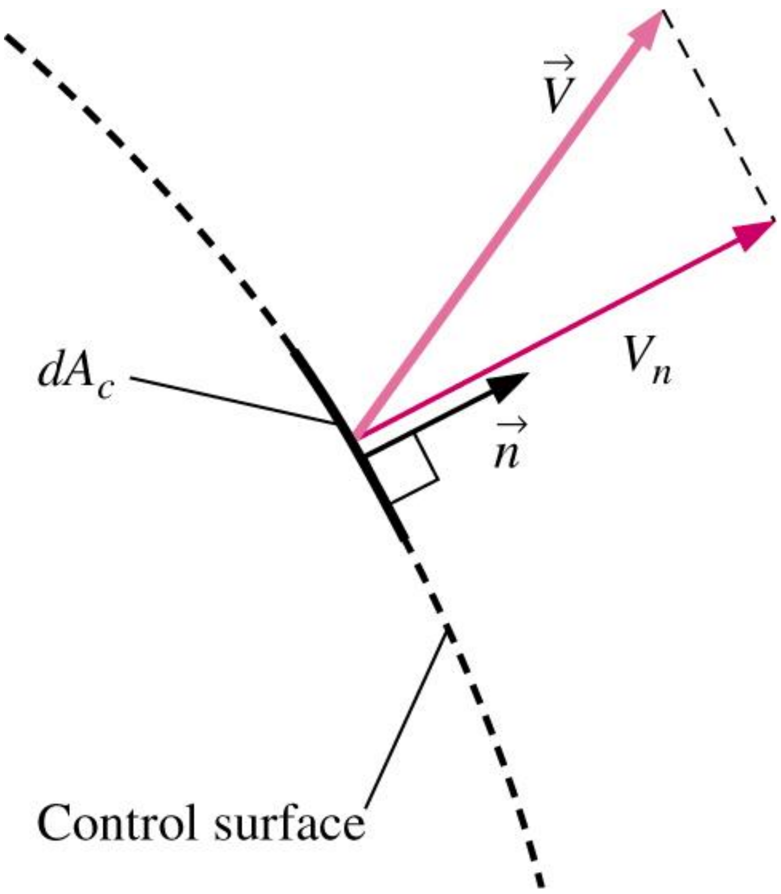
Objectives

- After completing this chapter, you should be able to
  - Apply the mass equation to balance the incoming and outgoing flow rates in a flow system.
  - Recognize various forms of mechanical energy, and work with energy conversion efficiencies.
  - Understand the use and limitations of the Bernoulli equation, and apply it to solve a variety of fluid flow problems.
  - Work with the energy equation expressed in terms of heads, and use it to determine turbine power output and pumping power requirements.

Conservation of Mass

- Conservation of mass principle is one of the most fundamental principles in nature.
- Mass, like energy, is a conserved property, and it cannot be created or destroyed during a process.
- For *closed systems* mass conservation is implicit since the mass of the system remains constant during a process.
- For *control volumes*, mass can cross the boundaries which means that we must keep track of the amount of mass entering and leaving the control volume.

Mass and Volume Flow Rates

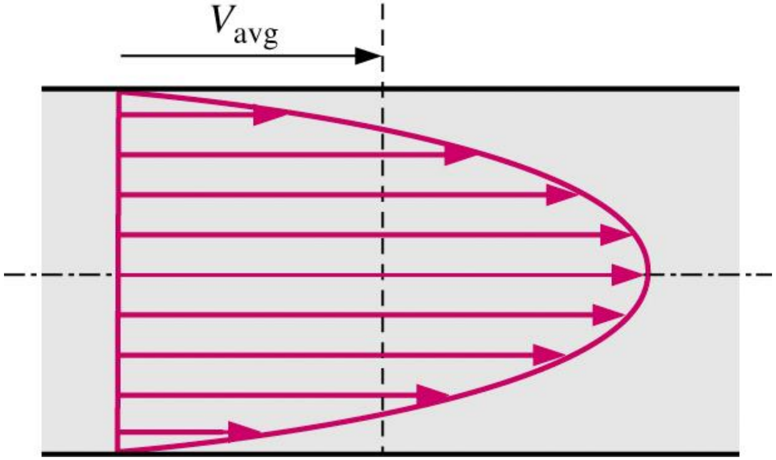


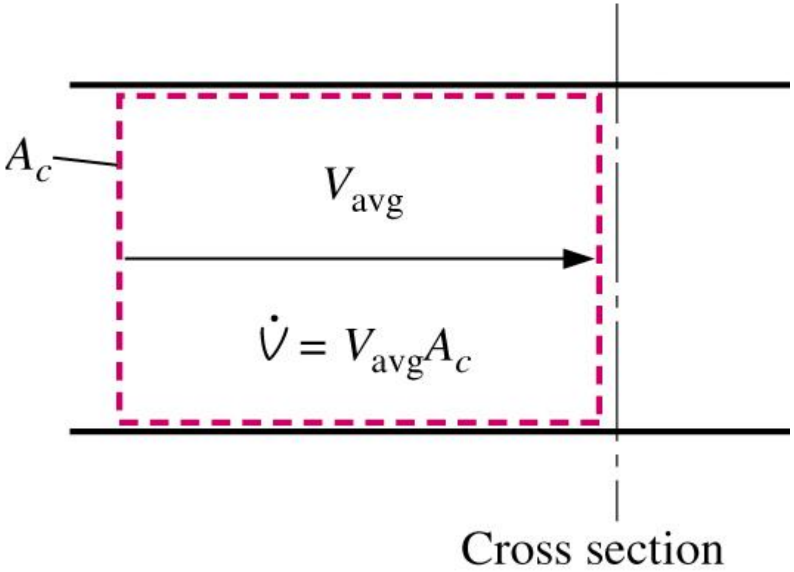
- The amount of mass flowing through a control surface per unit time is called the **mass flow rate** and is denoted $\dot{m}$
- The dot over a symbol is used to indicate *time rate of change*.
- Flow rate across the entire cross-sectional area of a pipe or duct is obtained by integration

$$\dot{m} = \int_{A_c} \delta m = \int_{A_c} \rho V_n dA_c$$

- While this expression for $\dot{m}$ is exact, it is not always convenient for engineering analyses.

Average Velocity and Volume Flow Rate





- Integral in $\dot{m}$ can be replaced with average values of ρ and V_n

$$V_{avg} = \frac{1}{A_c}\int_{A_c} V_n dA_c$$

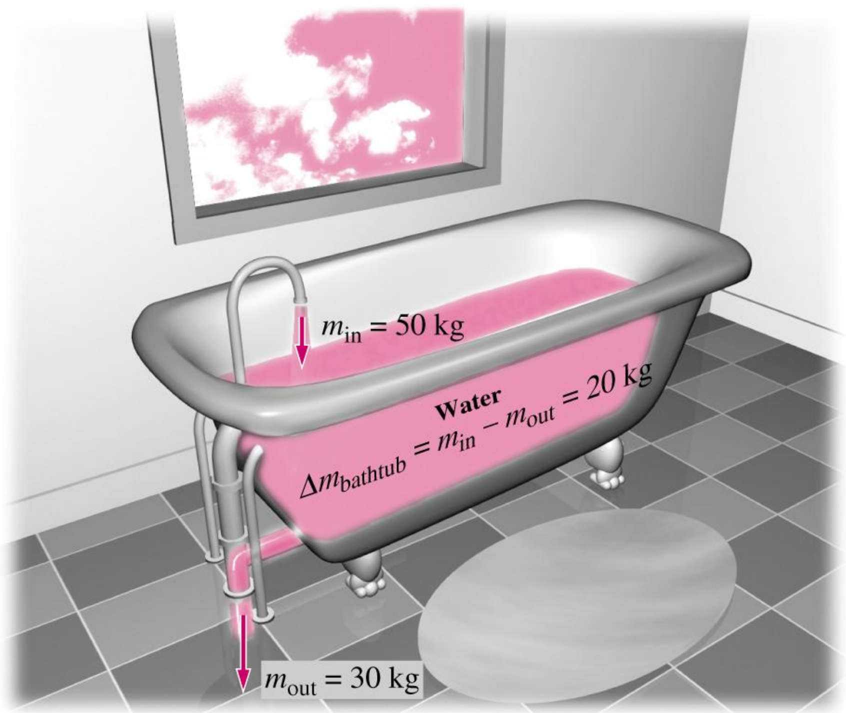
- For many flows, variation of ρ is very small: $\dot{m} = \rho V_{avg} A_c$
- Volume flow rate $\dot{V}$ is given by

$$\dot{V} = \int_{A_c} V_n dA_c = V_{avg} A_c = VA_c$$

(use V to denote avg velocity)

- Note: many textbooks use Q to denote volume flow rate.
- Mass and volume flow rates are related by $\dot{m} = \rho\,\dot{V}$

Conservation of Mass Principle

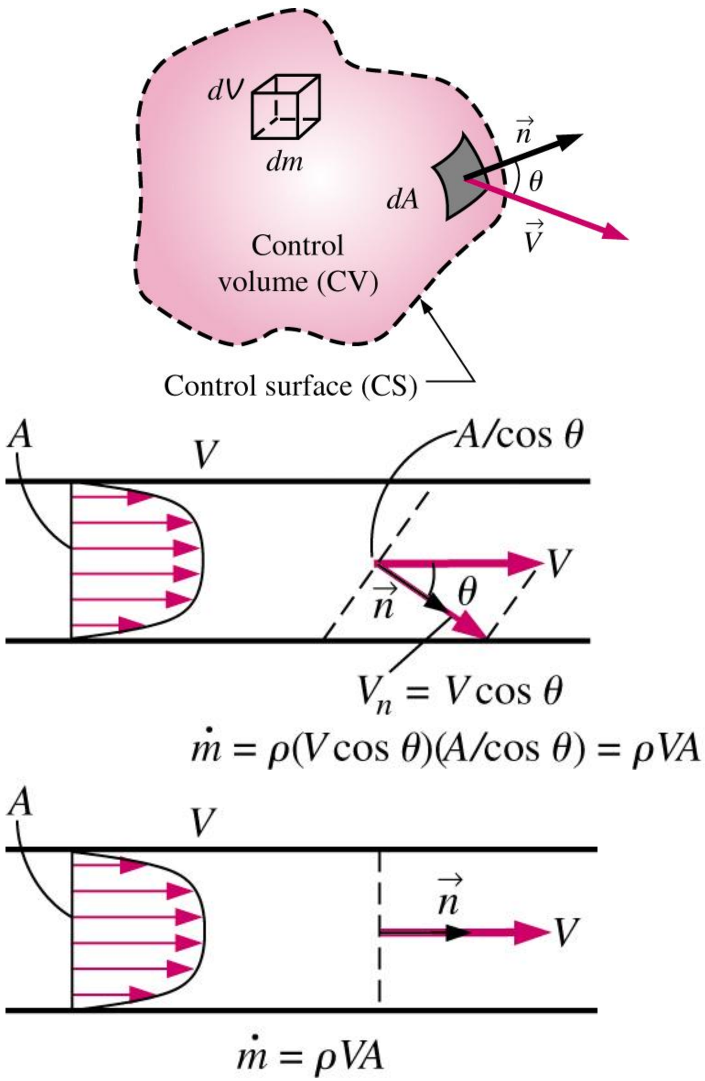


- The **conservation of mass principle** can be expressed as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt} \qquad \text{(kg/s)}$$

- Where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the CV, and dm_{CV}/dt is the rate of change of mass within the CV.

Conservation of Mass Principle



- For CV of arbitrary shape
  - rate of change of mass within the CV

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho\, dV$$

  - net mass flow rate

$$\dot{m}_{net} = \int_{CS} \delta\dot{m} = \int_{CS} \rho V_n dA = \int_{CS} \rho\left(\vec{V}\cdot\vec{n}\right) dA$$

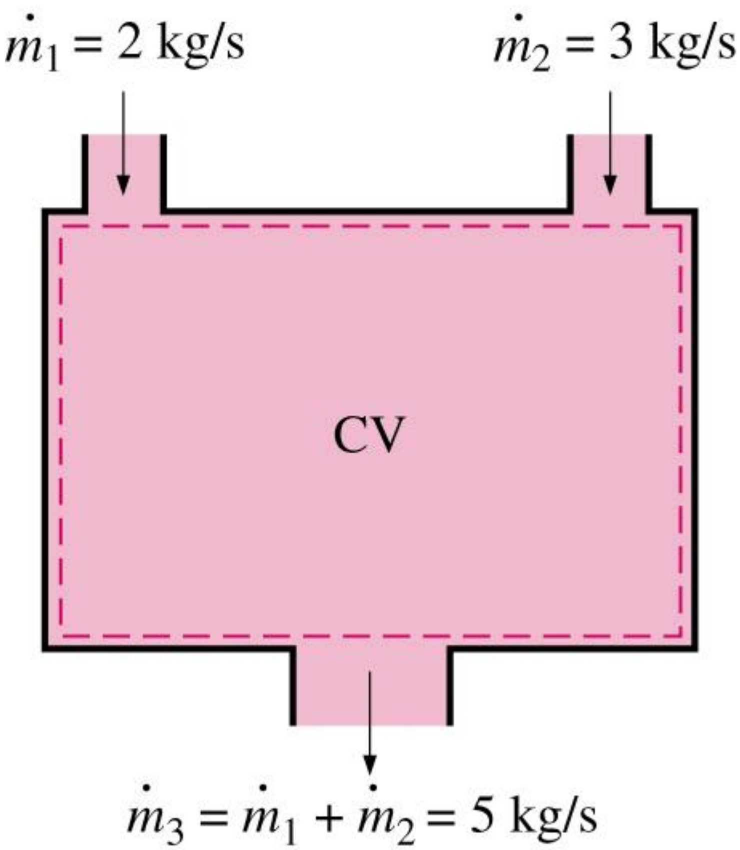
$$\dot{m}_{net} > 0 \longrightarrow \text{net outflow}$$

- Therefore, general conservation of mass for a fixed CV is:

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho\left(\vec{V}\cdot\vec{n}\right) dA = 0$$

(see also RTT with $B = m$ and $b = 1$)

Steady—Flow Processes



- For steady flow, the total amount of mass contained in *CV* is constant: m_{CV} = constant.
- Total amount of mass entering must be equal to total amount of mass leaving

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

- For incompressible flows,

$$\sum_{in} V_n A_n = \sum_{out} V_n A_n$$

Mechanical Energy

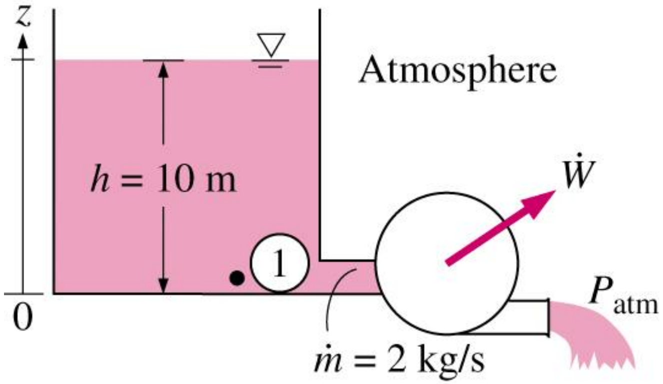
- **Mechanical energy** can be defined as *the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine.*
- Flow P/ρ, kinetic $V^2/2$, and potential gz energy are forms of mech. energy (per unit mass): $e_{mech} = P/\rho + V^2/2 + gz$
- Mechanical energy *per unit mass* change of a fluid during incompressible flow becomes

$$\Delta e_{mech} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$

- In the absence of losses, Δe_{mech} represents the work supplied to the fluid ($\Delta e_{mech} > 0$) or extracted from the fluid ($\Delta e_{mech} < 0$).

Mechanical Energy

$$\Delta e_{mech} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)$$

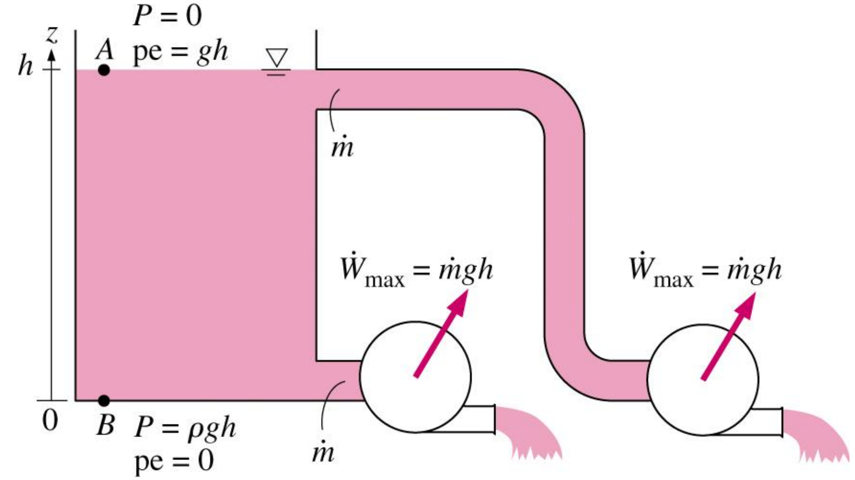


$$\dot{W}_{max} = \dot{m}\frac{P_1 - P_{atm}}{\rho} = \dot{m}\frac{\rho g h}{\rho} = \dot{m}gh$$

$$= (2\text{ kg/s})(9.81\text{ m/s}^2)(10\text{ m})$$

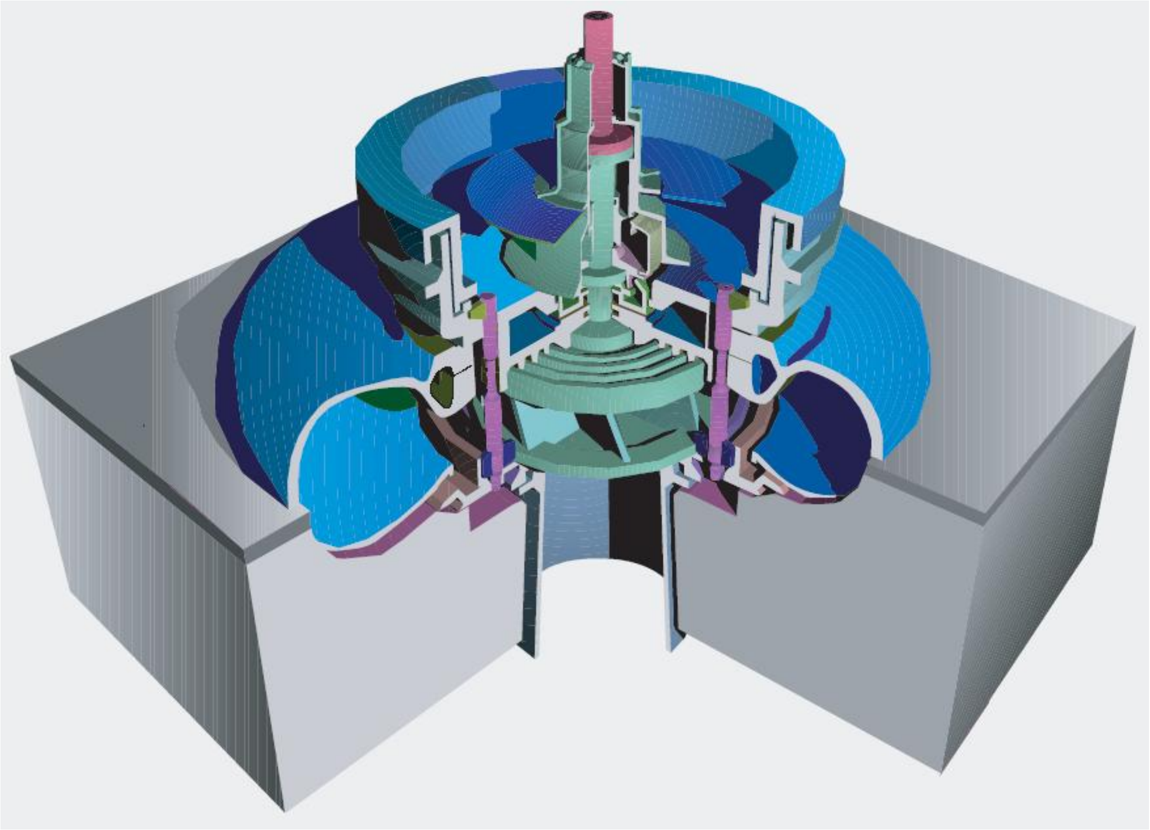
$$= 196\text{ W}$$

If no changes in flow velocity or elevation, the power produced by an ideal hydraulic turbine is proportional to the pressure drop of water across the turbine.



An ideal hydraulic turbine produces the same work per unit mass $w_{turbine} = gh$ whether it receives water from the top or from the bottom of the container (in the absence of irreversible losses).

Francis Hydraulic Turbine



Distributor, Francis runner and shaft

The choice of Francis, Kaplan or Pelton runners is based on the water mass flow rate and on the available ‘’head’’.

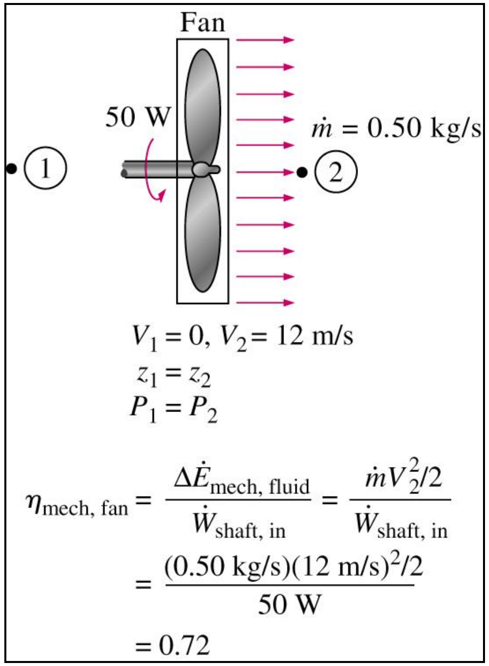
Efficiency

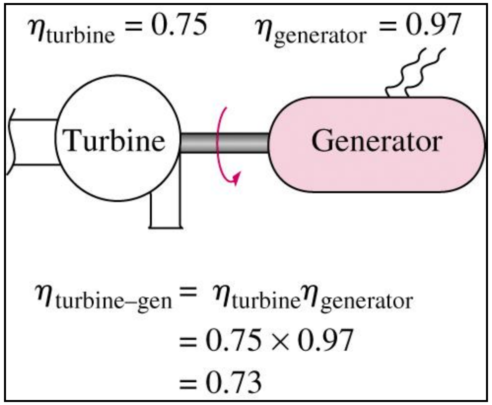
- Transfer of e_{mech} is usually accomplished by a rotating shaft: *shaft work*
- Pump, fan, compressor: receives shaft work (*e.g.*, from an electric motor) and transfers it to the fluid as mechanical energy (less frictional losses) ⟹ fluid pressure is raised
- Turbine: converts e_{mech} of a fluid to shaft work.
- In the absence of irreversibilities (*e.g*. friction), the **mechanical efficiency** of a device or process can be defined as

$$\eta_{mech} = \frac{E_{mech,out}}{E_{mech,in}} = 1 - \frac{E_{mech,loss}}{E_{mech,in}}$$

- If η_{mech} < 100% losses have occurred during conversion.

Pump and Turbine Efficiencies





- In fluid systems, we are usually interested in increasing/decreasing the pressure, velocity, and/or elevation of a fluid: pumps or turbines.
- In these cases, efficiency is better defined as the ratio of supplied or extracted work vs. rate of increase in mechanical energy

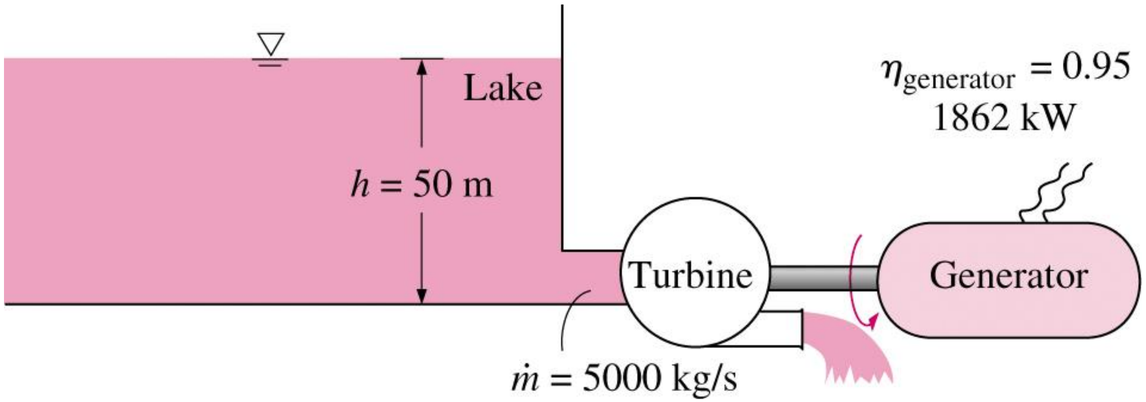
$$\eta_{pump} = \frac{\Delta \dot{E}_{mech,\, fluid}}{\dot{W}_{shaft,in}} = \frac{\text{rate of increase in the mechanical energy of the fluid}}{\text{mechanical energy input}}$$

$$\eta_{turbine} = \frac{\dot{W}_{shaft,out}}{\left| \Delta \dot{E}_{mech,\, fluid} \right|}$$

- *Overall efficiency* must include motor or generator efficiency, ratio of power output to input (shaft power and electric power).

Turbine and Generator Efficiencies

Example:



Determine:

(a) overall efficiency of the turbine-generator,

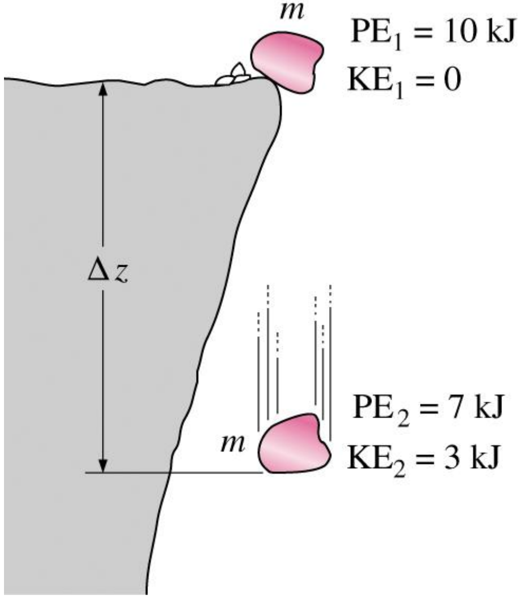
(b) mechanical efficiency of the turbine,

(c) shaft power supplied by the turbine to the generator, and

(d) the irreversible losses through each component.

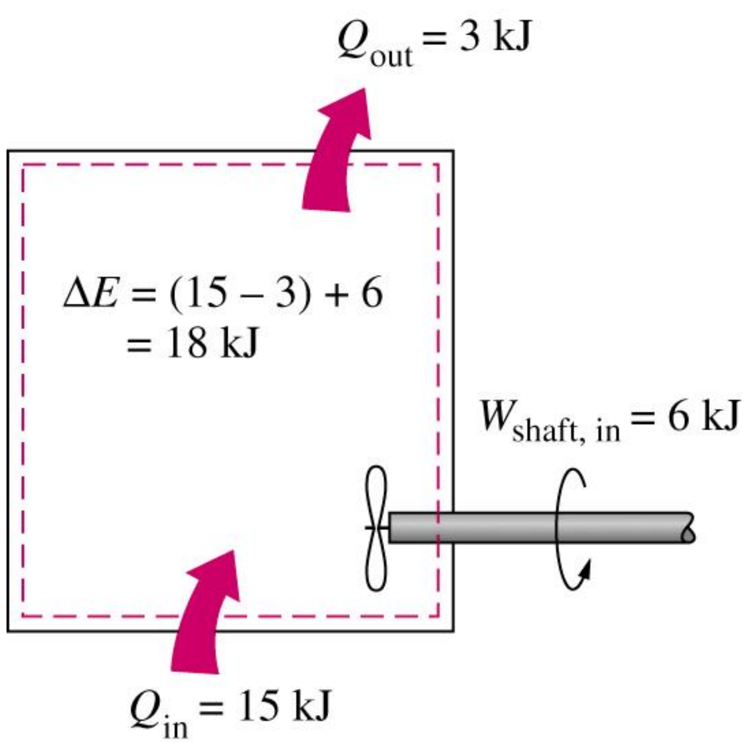
General Energy Equation

- One of the most fundamental laws in nature is the **1st law of thermodynamics**, which is also known as the **conservation of energy principle:** *energy can be neither created nor destroyed during a process; it can only change forms*



- Falling rock, picks up speed as PE is converted to KE.
- If air resistance is neglected, PE + KE = constant

General Energy Equation



- The energy content of a fixed quantity of mass (closed system) can be changed by two mechanisms: *heat transfer* Q and *work transfer* W.
- Conservation of energy for a closed system can be expressed in rate form as

$$\dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{dE_{sys}}{dt}$$

- Net rate of heat transfer to the system (> 0 if the system is heated):

$$\dot{Q}_{net,in} = \dot{Q}_{in} - \dot{Q}_{out}$$

- Net power input to the system (> 0 if work is done on the system):

$$\dot{W}_{net,in} = \dot{W}_{in} - \dot{W}_{out}$$

General Energy Equation

- Recall general RTT

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho\, b\, dV + \int_{CS} \rho b \left(\vec{V}_r \cdot \vec{n}\right) dA$$

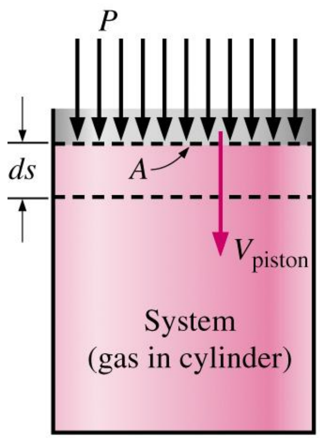
- "Derive" energy equation using $B=E$ and $b=e$

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{d}{dt}\int_{CV} \rho e\, dV + \int_{CS} \rho e \left(\vec{V}_r \cdot \vec{n}\right) dA$$

- Break power into rate of shaft work and pressure work (neglect *viscous work*, and that done by other forces such as electric, magnetic and surface tension)

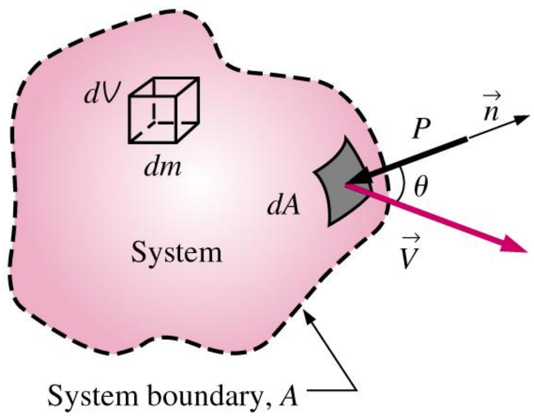
$$\dot{W}_{net,in} = \dot{W}_{shaft,net,in} + \dot{W}_{pressure,net,in} = \dot{W}_{shaft,net,in} - \int_A P\left(\vec{V} \cdot \vec{n}\right) dA$$

General Energy Equation



- Where does expression for pressure work come from?
- When piston moves down by ds under the influence of $F=PA$, the work done *on* the system is $\delta W_{boundary}=PAds$.
- If we divide both sides by dt, we have

$$\delta \dot{W}_{pressure} = \delta \dot{W}_{boundary} = PA\frac{ds}{dt} = PAV_{piston}$$



- For generalized *system* deforming under the influence of pressure:

$$\boxed{\delta \dot{W}_{pressure} = -PdAV_n = -PdA\left(\vec{V}\cdot\vec{n}\right)}$$

- Note sign conventions:
  - $\vec{n}$ is outward pointing normal
  - negative sign ensures that work done is positive when is done *on* the system.

General Energy Equation

- Moving integral for rate of pressure work to RHS of energy equation results in:

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e \, dV + \int_{CS} \left(\frac{P}{\rho} + e \right) \rho \left(\vec{V}_r \cdot \vec{n} \right) dA$$

- Recall that P/ρ is the **flow work,** which is the work associated with pushing a fluid into or out of a CV per unit mass.
- For fixed CV pressure work can exist only where the fluid enters and leaves the CV.

General Energy Equation

- As with the mass equation, practical analysis is often facilitated when $P/\rho + e$ is averaged across inlets and outlets of a fixed CV:

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e \, dV + \sum_{out} \dot{m}\left(\frac{P}{\rho} + e\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + e\right)$$

$$\dot{m} = \int_{A_C} \rho\left(\vec{V} \cdot \vec{n}\right) dA_c$$

- Since $e = u + ke + pe = u + V^2/2 + gz$

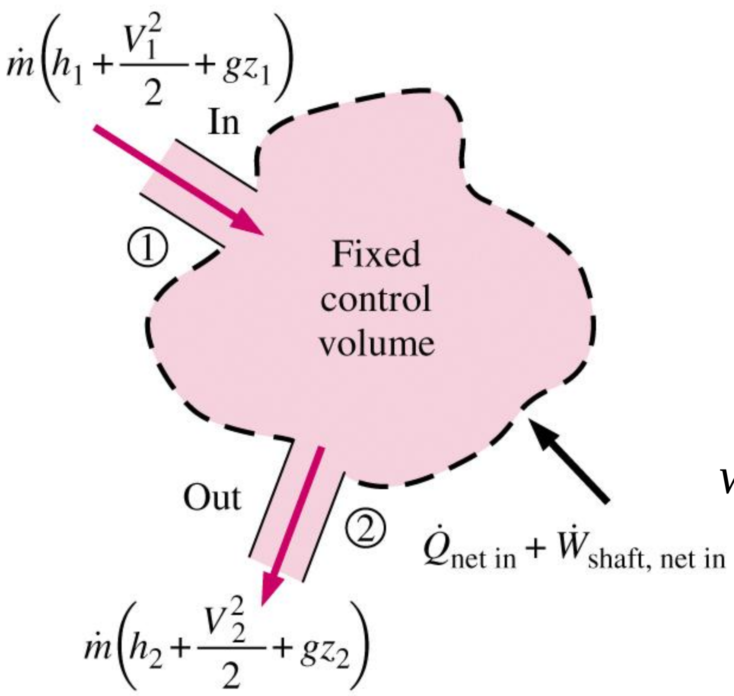
$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho \, e \, dV + \sum_{out} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right)$$

Energy Analysis of Steady Flows

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \sum_{out} \dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(h + \frac{V^2}{2} + gz\right)$$

- For steady flow, time rate of change of the energy content of the CV is zero.
- This equation states: *the net rate of energy transfer to a CV by heat and work during steady flow is equal to the difference between the rates of outgoing and incoming energy flows with mass.*

Energy Analysis of Steady Flows



- For **single-stream devices**, mass flow rate is constant. The steady flow energy equation per unit mass reads:

$$q_{net,in}+w_{shaft,net,in}=h_2-h_1+\frac{V_2^2-V_1^2}{2}+g\left(z_2-z_1\right)$$

$$w_{shaft,net,in}+\frac{P_1}{\rho_1}+\frac{V_1^2}{2}+gz_1=\frac{P_2}{\rho_2}+\frac{V_2^2}{2}+gz_2+\left(u_2-u_1-q_{net,in}\right)$$

$$\frac{P_1}{\rho_1}+\frac{V_1^2}{2}+gz_1+w_{pump}=\frac{P_2}{\rho_2}+\frac{V_2^2}{2}+gz_2+w_{turbine}+e_{mech,loss}$$

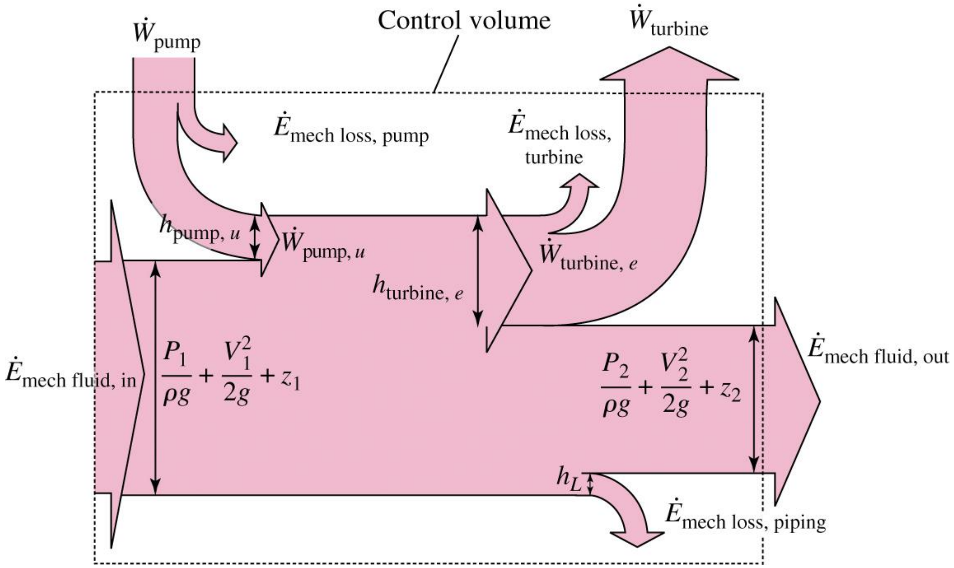
since $e_{mech,loss}=u_2-u_1-q_{net\ in}$, and
$w_{shaft,net,in}=w_{shaft,in}-w_{shaft,out}=w_{pump}-w_{turbine}$

Energy Analysis of Steady Flows

- Divide by *g* to get each term in units of length

$$\frac{P_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1+h_{pump}=\frac{P_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2+h_{turbine}+h_L$$

Magnitude of each term is now expressed as an equivalent column height of fluid, i.e., *Head*



The Bernoulli Equation

- If we neglect piping losses, and have a system without pumps or turbines

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 \quad = C$$

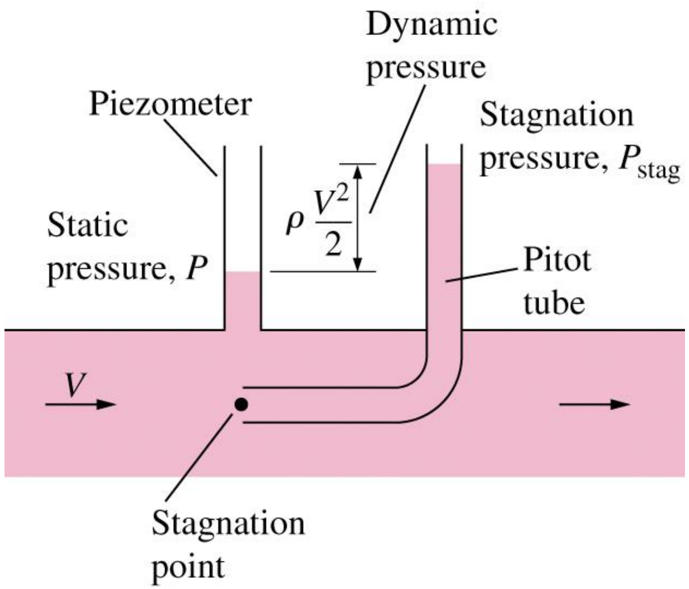
- This is **Bernoulli's equation**
- It can be derived in a formal way by using Newton's second law of motion (see text, p. 187). The formal derivation holds that *in steady flow the sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline, when compressibility and frictional effects can be neglected.* Also: *total pressure along a streamline is constant.*
- 3 terms above correspond to static, dynamic, and hydrostatic head (pressure).

The Pitot Tube

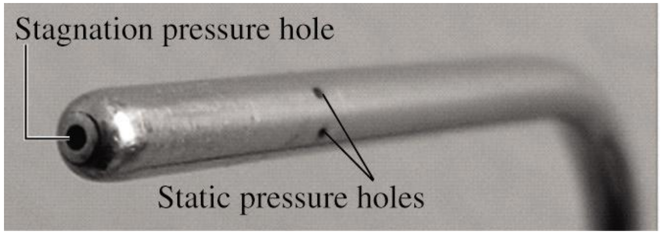
The sum of static and dynamic pressure is called the *stagnation pressure*, *i.e.*

$$P_{stag} = P + \rho V^2/2.$$

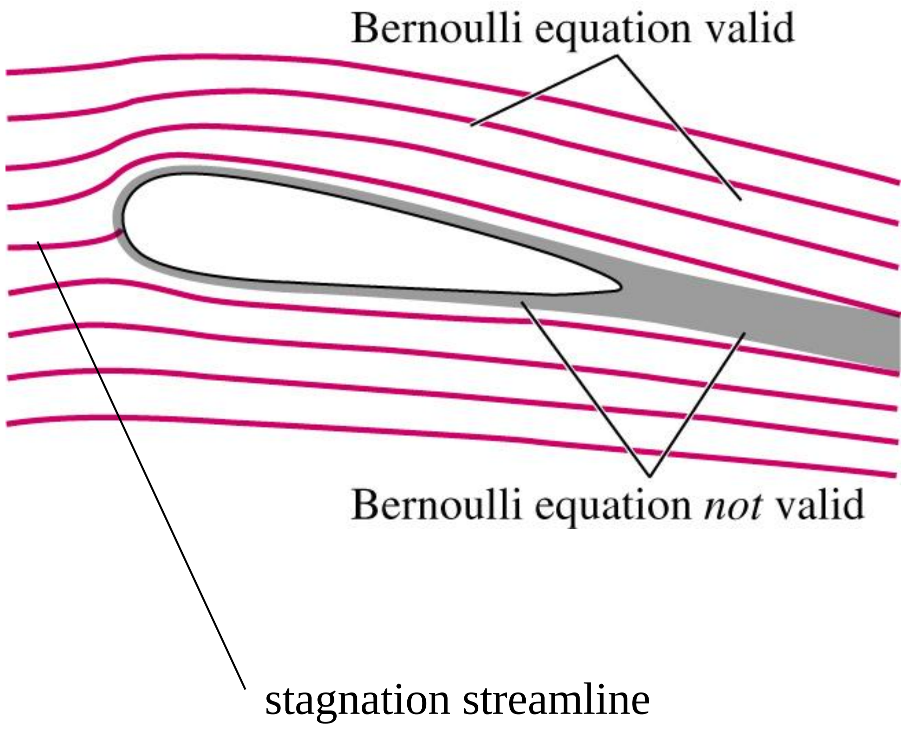
It represents the pressure at a point where the fluid is brought to a complete stop isentropically.



$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$



The Bernoulli Equation



- The **Bernoulli equation** is an *approximate relation between pressure, velocity*, and *elevation and is valid in regions of steady, incompressible flow where net frictional forces are negligible.*
- Equation is useful in flow regions outside of boundary layers and wakes.

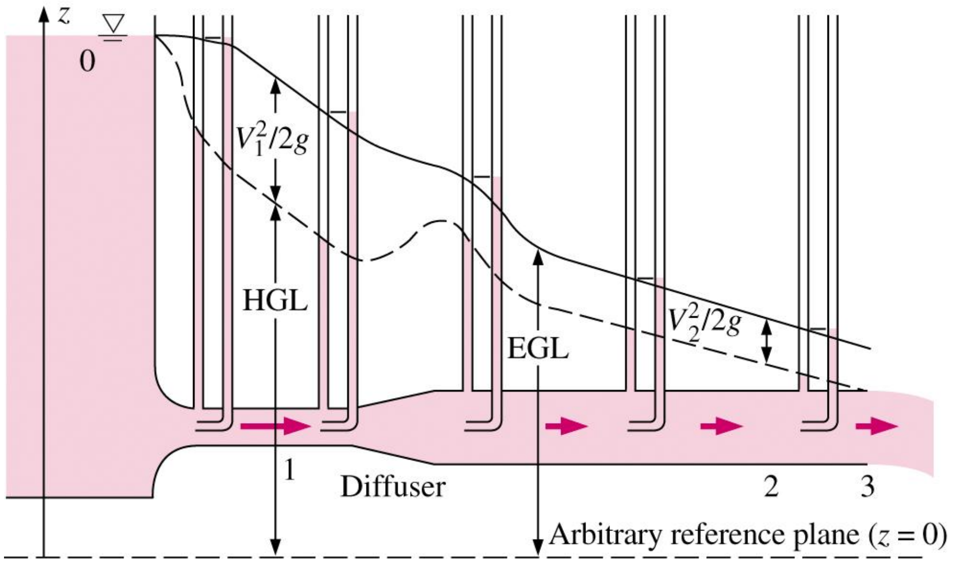
The Bernoulli Equation

- Limitations on the use of the Bernoulli equation:

$$P + \rho V^2/2 + \rho g z = \text{constant (along a streamline)}$$

  - Steady flow: $d/dt = 0$
  - Frictionless flow
  - No shaft work: $w_{pump} = w_{turbine} = 0$
  - Incompressible flow: ρ = constant
  - No heat transfer: $q_{net,in} = 0$
  - Applied along a streamline (except when the flow is irrotational, in which case the constant C is the same throughout the flow field, Chapt. 10)

HGL and EGL



Frictional losses are present.

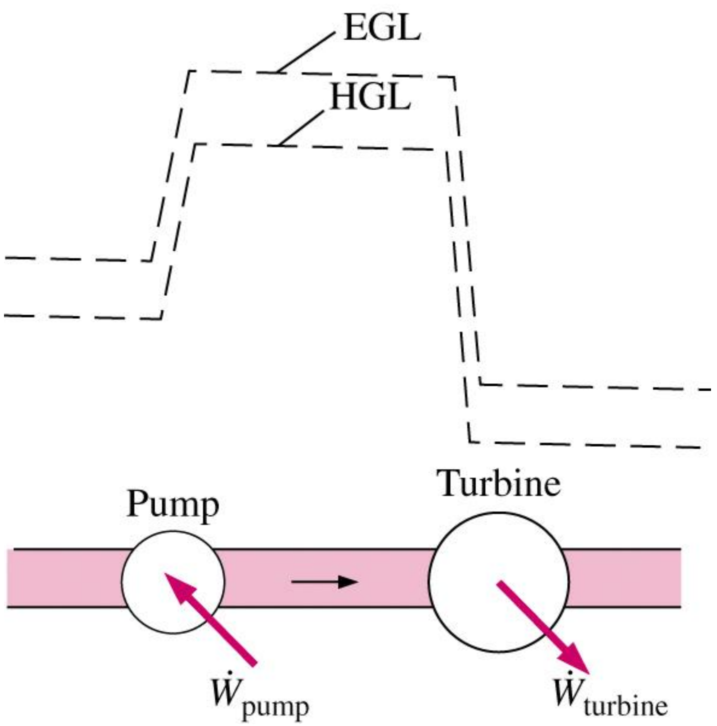
- It is often convenient to plot mechanical energy graphically using heights.
- Hydraulic Grade Line

$$HGL = \frac{P}{\rho g} + z$$

- Energy Grade Line (or total energy)

$$EGL = \frac{P}{\rho g} + \frac{V^2}{2g} + z$$

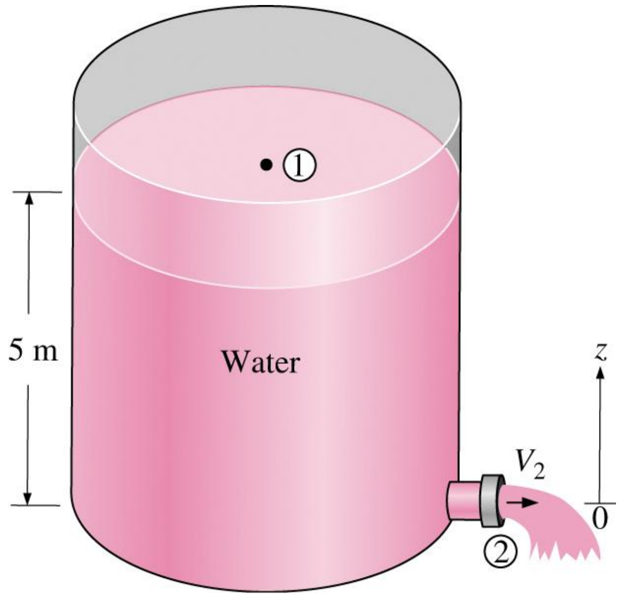
HGL and EGL



Steep jump (or drop) in EGL and HGL appear whenever mechanical energy is added to (or removed from) the fluid by a pump (or a turbine).

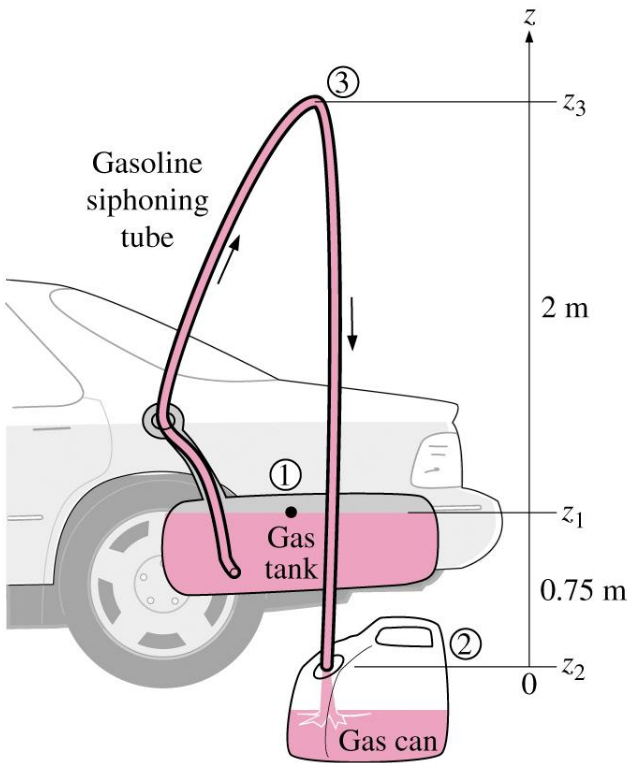
Examples and Applications

Bernoulli equation



Flow irrotational and incompressible; flow quasi-steady (water drains slowly); no frictional losses nor flow disturbances.
Exit velocity is given by $V_2 = (2\,g\,z_1)^{1/2}$ (Torricelli equation)

Bernoulli equation



Siphon diameter = 4 mm; $\rho_{gasoline}$ = 750 kg/m^3; no frictional losses. Determine the time to siphon 4L of gasoline, and the pressure at point 3 (careful of cavitation at point 3!).

Examples and Applications

Bernoulli equation

When compressibility effects are important, the Bernoulli equation takes the form:

$$\int dP/\rho + V^2/2 + gz = \text{constant (along a streamline)}$$

Simplify the relation above for

(*a*) the isothermal expansion or compression of an ideal gas, and

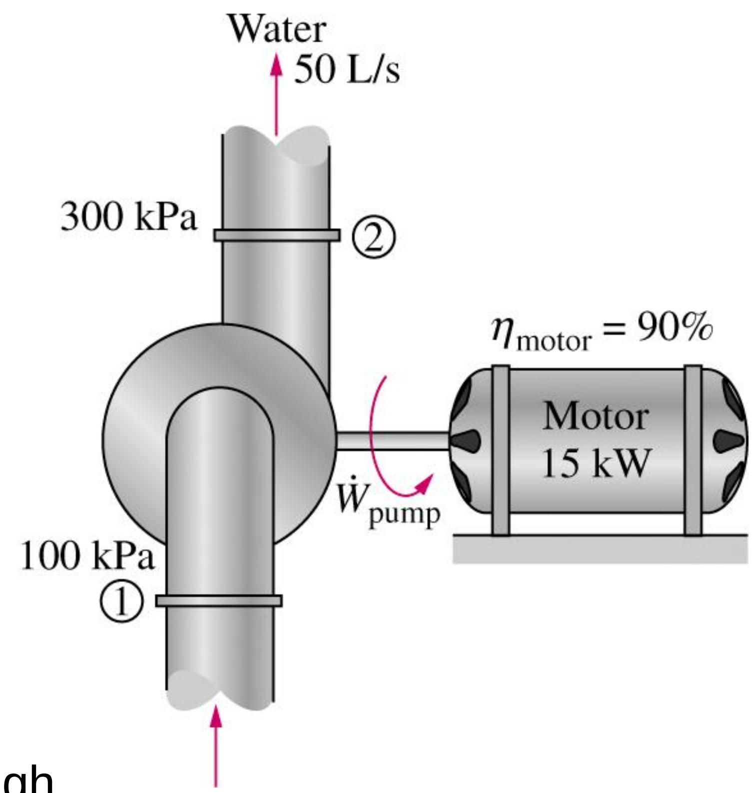
(*b*) the isentropic flow of an ideal gas.

Examples and Applications

Energy analysis of steady flows

$$\Delta e_{mech} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$

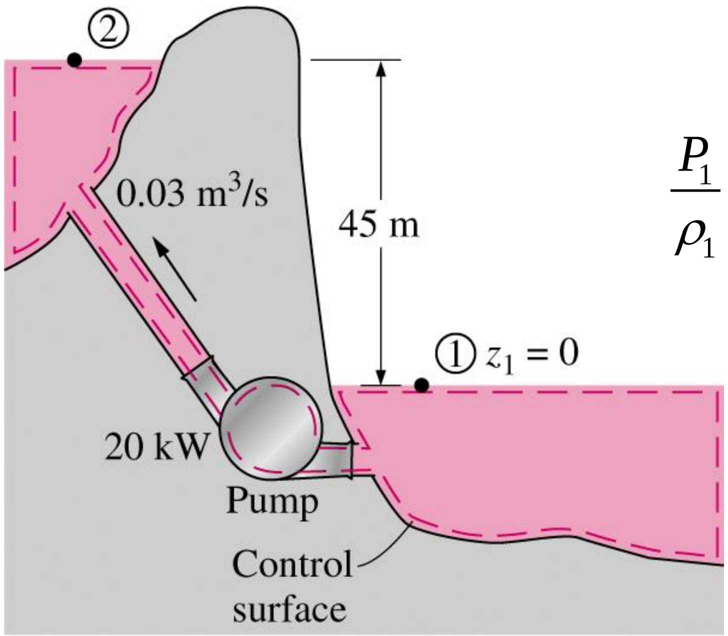
$$\Delta \dot{E}_{mech,fluid} = \dot{m}\left[\frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)\right]$$



Determine:
(a) the mechanical efficiency of the pump, and
(b) the temperature rise of water as it flows through the pump, because of mechanical inefficiency. The specific heat of water is 4.18 kJ/(kg °C)

Examples and Applications

Energy analysis of steady flows



$$\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 + w_{pump} = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + w_{turbine} + e_{mech,loss}$$

Determine the mechanical power lost (and the head loss h_L) while pumping water to the upper reservoir.