Chapter 6: Momentum Analysis of Flow Systems

Introduction

- Fluid flow problems can be analyzed using one of three basic approaches: differential, experimental, and integral (or control volume).
- In Chap. 5, control volume forms of the mass and energy equation were developed and used.
- In this chapter, we complete control volume analysis by presenting the integral momentum equation.
  - Review Newton's laws and conservation relations for momentum.
  - Use RTT to develop linear and angular momentum equations for control volumes.
  - Use these equations to determine forces and torques acting on the CV.

Objectives

- After completing this chapter, you should be able to
  - Identify the various kinds of forces and moments acting on a control volume.
  - Use control volume analysis to determine the forces associated with fluid flow.
  - Use control volume analysis to determine the moments caused by fluid flow and the torque transmitted.

Newton's Laws

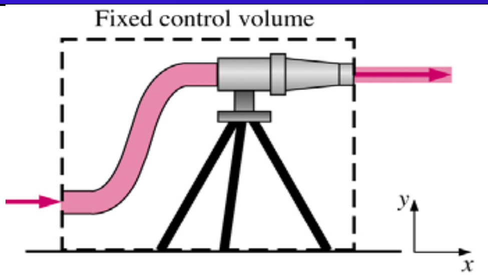
- Newton's laws are relations between motions of bodies and the forces acting on them.
  - **First law**: a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.
  - **Second law**: the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

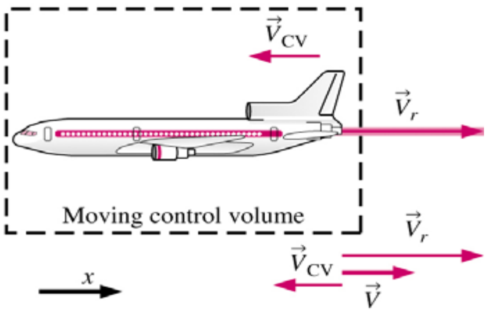
$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\left(m\vec{V}\right)}{dt}$$

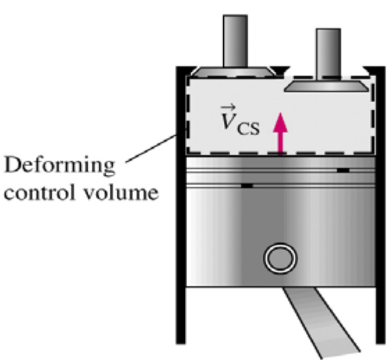
(analogous expression for the angular momentum ***H***)

  - **Third law**: when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

Choosing a Control Volume







- CV is arbitrarily chosen by fluid dynamicist, however, selection of CV can either simplify or complicate analysis.
  - Clearly define all boundaries. Analysis is often simplified if CS is normal to flow direction.
  - Clearly identify all fluxes crossing the CS.
  - Clearly identify forces and torques *of interest* acting on the CV and CS.
- Fixed, moving, and deforming control volumes.
  - For moving CV, use relative velocity,

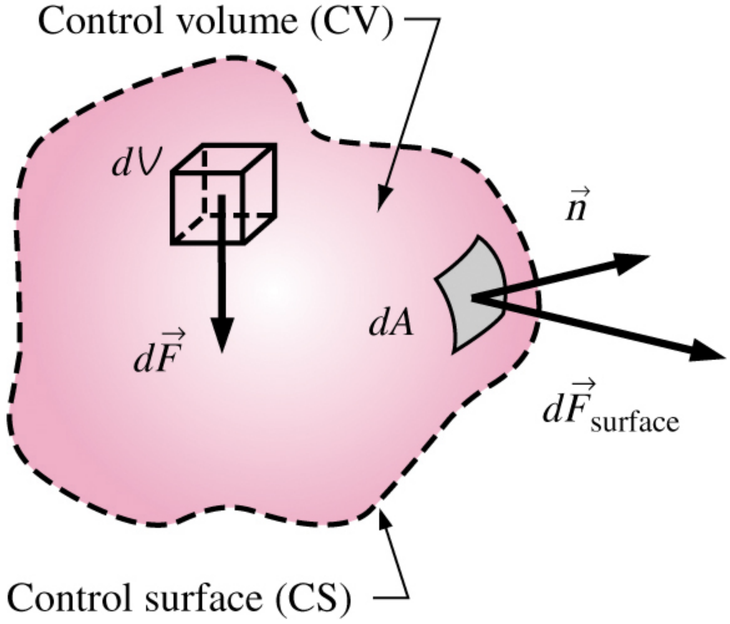
$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

  - For deforming CV, use relative velocity for all deforming control surfaces,

$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

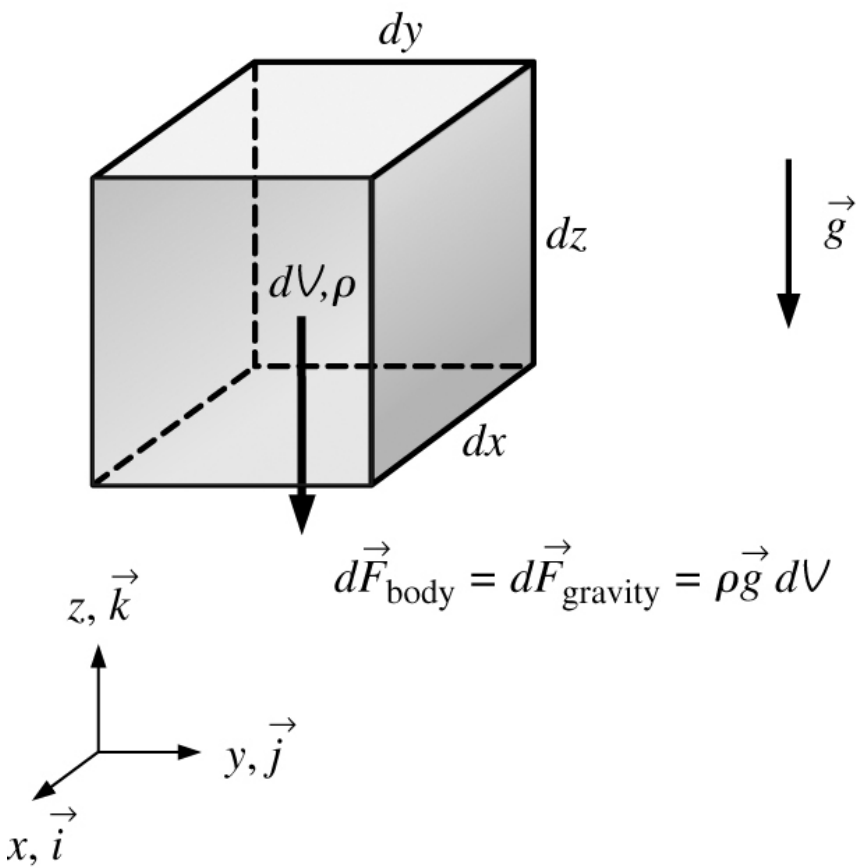
Forces Acting on a CV

- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).



- Body forces act on each volumetric portion dV of the CV.
- Surface forces act on each portion dA of the CS.

Body Forces



- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The different body force

$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}d\mathcal{V}$$

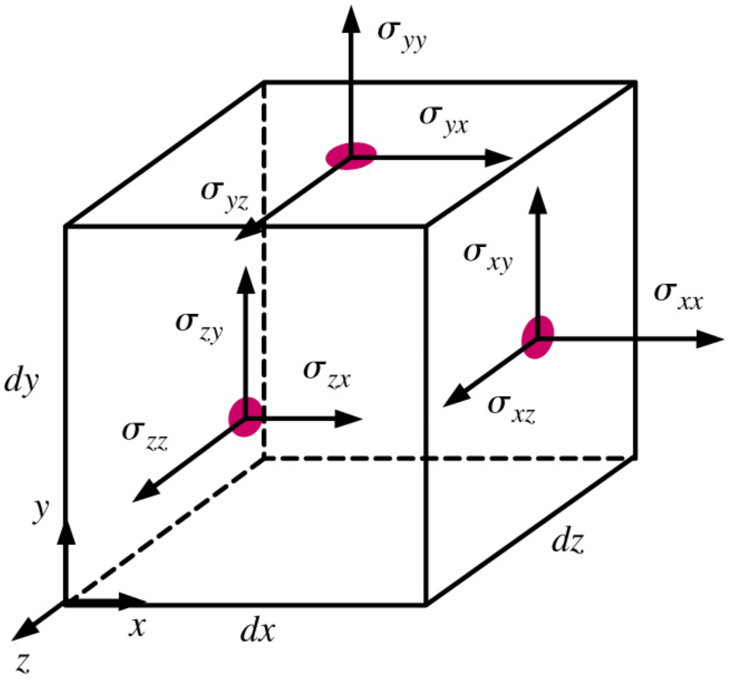
- Typical convention is that $\vec{g}$ acts in the negative *z*-direction,

$$\vec{g} = -g\vec{k}$$

- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho\vec{g}d\mathcal{V} = m_{CV}\vec{g}$$

Surface Forces



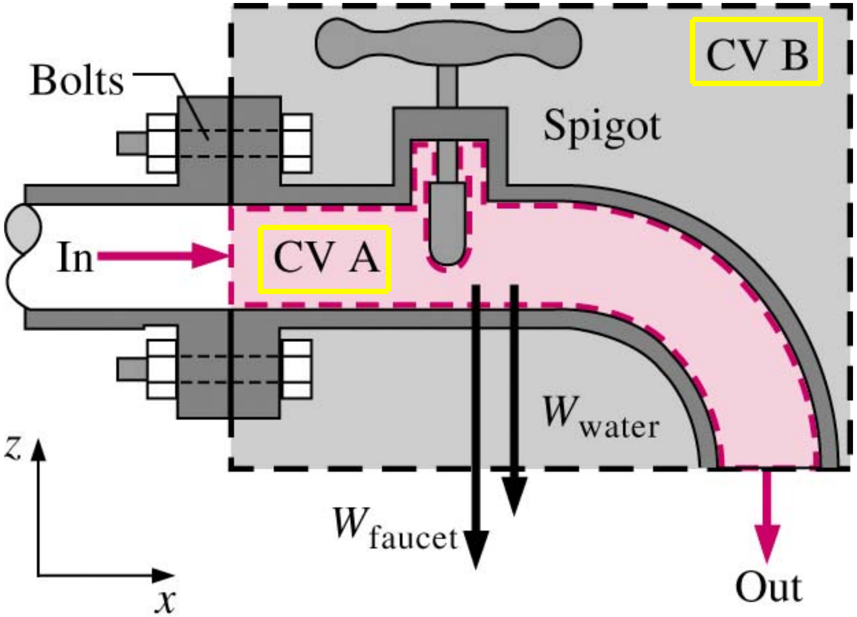
- Surface forces include both normal and tangential components
- Diagonal components σ_{xx}, σ_{yy}, σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components σ_{xy}, σ_{xz}, etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

Second order stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Body and Surface Forces



- Surface integrals are cumbersome.
- Careful selection of CV allows expression of total force in terms of more readily available quantities like weight, *gage* pressure, and reaction forces.
- Goal is to choose CV to expose only the forces to be determined and a minimum number of other forces: in the example to the left CV B is a wiser choice than CV A.

$$\sum \vec{F} = \underbrace{\sum \vec{F}_{gravity}}_{\text{body force}} + \underbrace{\sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other}}_{\text{surface forces}}$$

Linear Momentum Equation

- Newton's second law for a system of mass *m* subjected to a force $\vec{F}$ is expressed in a Galilean coordinate system as

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)$$

- Use RTT with $b = \vec{V}$ and $B = m\vec{V}$ to shift from system formulation of the control volume formulation

$$\frac{d\left(m\vec{V}\right)_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$$

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$$

Linear Momentum Equation

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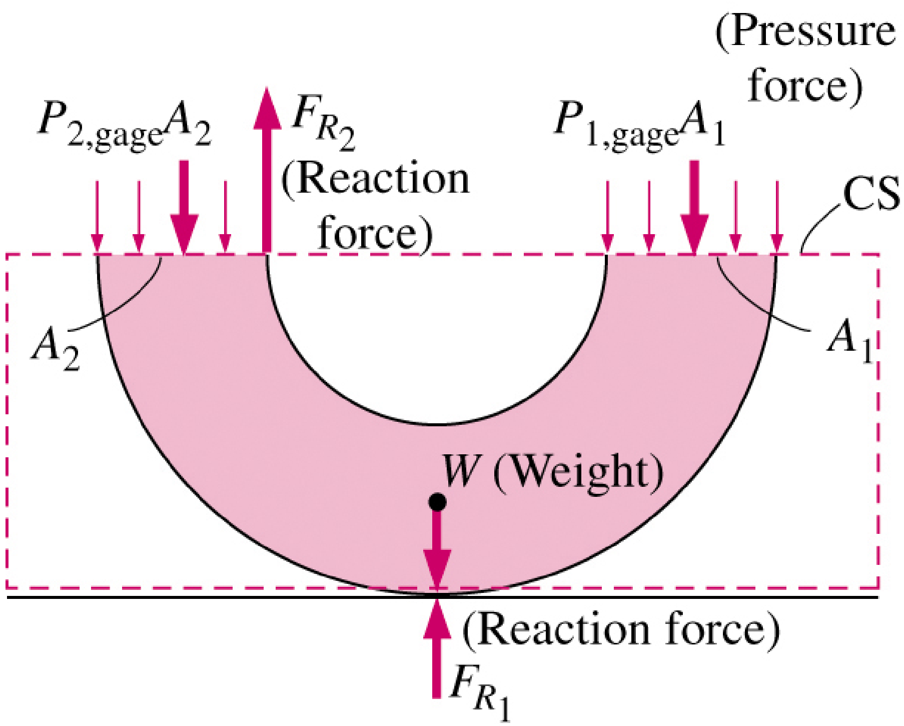


An 180° elbow supported by the ground

The linear momentum equation is commonly used to calculate the forces (usually on support systems, connectors, flanges, etc.) induced by the flow. In most flow systems the force $\sum \vec{F}$ consists of weights, (gage) pressure forces, and reaction forces.

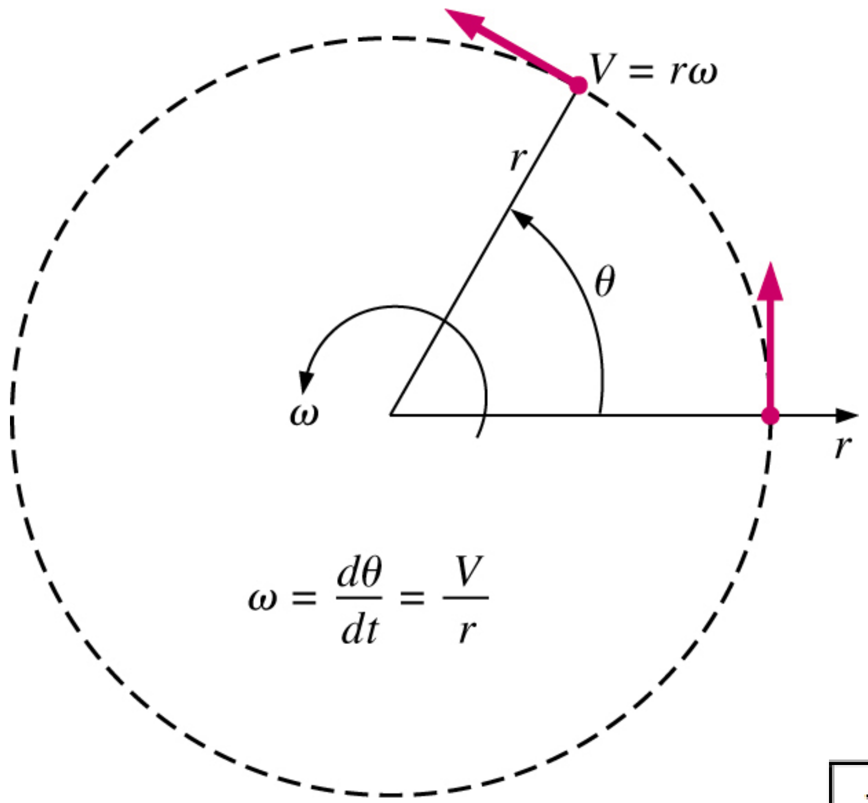
Special Cases

- Steady Flow $$\sum \vec{F} = \int_{CS} \rho \vec{V} \left(\vec{V}_r \cdot \vec{n} \right) dA$$
- Average velocities $$\dot{m} = \int_{A_C} \rho \left(\vec{V} \cdot \vec{n} \right) dA_c = \rho V_{avg} A_c$$
- Approximate momentum flow rate when speed across inlets and outlets is almost uniform
$$\int_{CS} \rho \vec{V} \left(\vec{V} \cdot \vec{n} \right) dA \approx \rho V_{avg} A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$
- To account for error, use *momentum-flux correction factor* β
$$\beta = \frac{1}{A_c} \int_{A_C} \left(\frac{V}{V_{avg}} \right)^2 dA_c$$
$$\boxed{\sum \vec{F} = \frac{d}{dt} \int \rho \vec{V} \, d\mathcal{V} + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}}$$

Angular Momentum

- Motion of a rigid body can be considered to be the combination of
  - the translational motion of its center of mass (U_x, U_y, U_z)
  - the rotational motion about its center of mass (ω_x, ω_y, ω_z)
- Translational motion can be analyzed with linear momentum equation.
- Rotational motion is analyzed with angular momentum equation.
- Together, a general body motion can be described as a 6–degree–of–freedom (6DOF) system.

Review of Rotational Motion



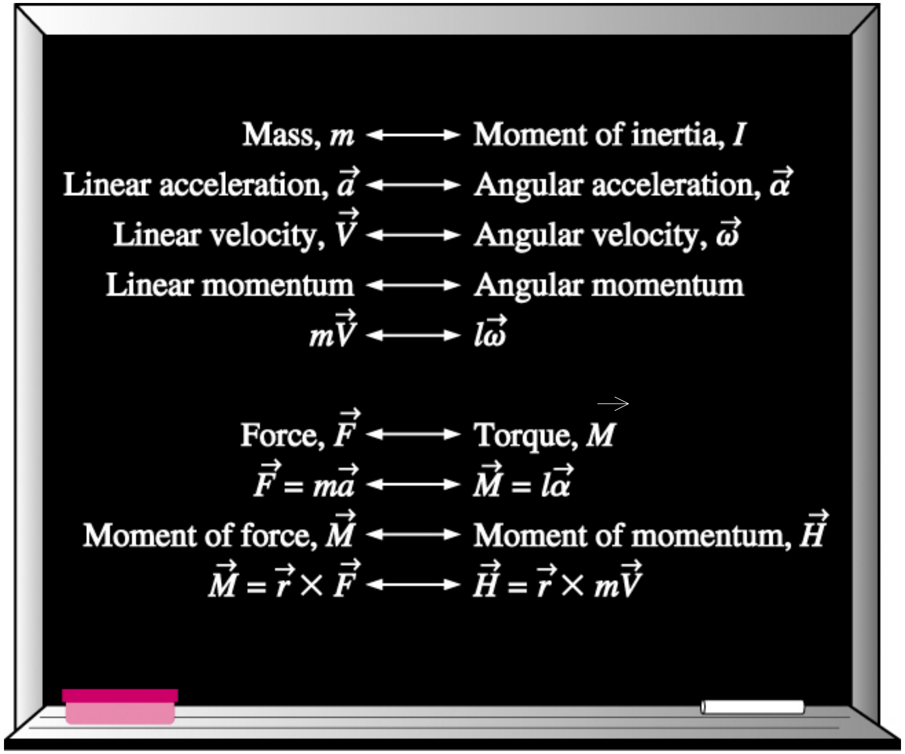
Angular velocity ω is the angular distance θ traveled per unit time, and angular acceleration α is the rate of change of angular velocity.

$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

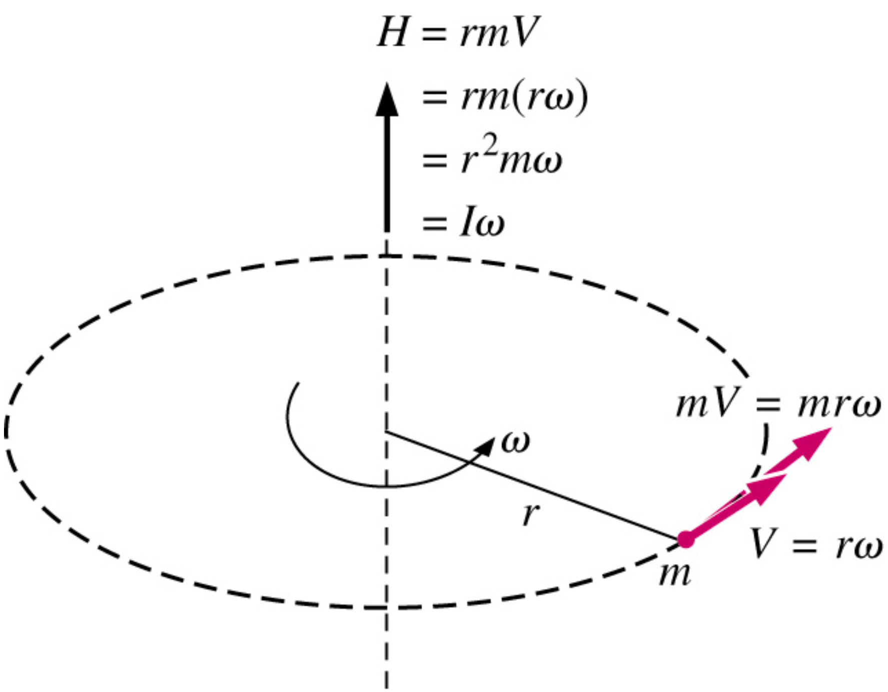
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$$\boxed{V = r\omega \text{ and } a_t = r\alpha}$$

Review of Angular Momentum



Analogy between corresponding linear and angular quantities.



Angular momentum of a point mass m rotating at angular velocity ω at distance r from the axis of rotation.

Review of Angular Momentum

- Moment of a force: $\vec{M} = \vec{r} \times \vec{F}$
- **Angular** momentum: $\vec{H} = \vec{r} \times m\vec{V}$
- For a system: $\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$

$$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$$

- Therefore, the angular momentum equation for a Galilean system can be written as: $\sum \vec{M} = \frac{d\vec{H}_{sys}}{dt}$
- To derive angular momentum for a CV, use RTT with $B = \vec{H}$ and $b = \vec{r} \times \vec{V}$

Angular Momentum Equation for a CV

- General form

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, d\mathcal{V} + \int_{CS} (\vec{r} \times \vec{V})\rho \left(\vec{V}_r \cdot \vec{n}\right) \, dA$$

- Approximate form using average properties at inlets and outlets

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, d\mathcal{V} + \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$

- Steady flow

$$\sum \vec{M} = +\sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$