Chapter 7: Dimensional Analysis and Modeling

Objectives

1. Understand dimensions, units, and dimensional homogeneity
2. Understand benefits of dimensional analysis
3. Know how to use the method of repeating variables (Π theorem)
4. Understand the concept of similarity and how to apply it to experimental modeling

Dimensions and Units

- Review
  - Dimension: Measure of a physical quantity, *e.g.*, length, time, mass
  - Units: Assignment of a number to a dimension, *e.g.*, (m), (sec), (kg)
  - 7 Primary Dimensions:

1. Mass	m	(kg)
2. Length	L	(m)
3. Time	t	(sec)
4. Temperature	T	(K)
5. Current	I	(A)
6. Amount of Light	C	(cd)
7. Amount of matter	N	(mol)

Dimensions and Units

- Review, continued
  - All non-primary dimensions can be formed by a combination of the 7 primary dimensions
  - Examples
    - {Velocity} = {Length/Time} = {L/t}
    - {Force} = {Mass Length/Time^2} = {mL/t^2}

Dimensional Homogeneity

- Law of dimensional homogeneity (DH): every additive term in an equation must have the same dimensions
- Example: Bernoulli equation

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

  - $\{p\}$ = {force/area}={mass x length/time^2 x 1/length^2} = {m/(t^{2}L)}
  - $\{1/2\ \rho V^2\}$ = {mass/length^3 x (length/time)^2} = {m/(t^{2}L)}
  - $\{\rho g z\}$ = {mass/length^3 x length/time^2 x length} ={m/(t^{2}L)}

Nondimensionalization of Equations

- Given the law of DH, if we divide each term in the equation by a collection of variables and constants that have the same dimensions, the equation is rendered nondimensional
- In the process of nondimensionalizing an equation, nondimensional parameters often appear, e.g., Reynolds number and Froude number

Nondimensionalization of Equations

- To nondimensionalize, for example, the Bernoulli equation, the first step is to list primary dimensions of all dimensional variables and constants

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

$\{p\} = \{m/(t^2L)\}$ $\{\rho\} = \{m/L^3\}$ $\{V\} = \{L/t\}$

$\{g\} = \{L/t^2\}$ $\{z\} = \{L\}$

  - Next, we need to select Scaling Parameters. For this example, select L, U_0, ρ_0

Nondimensionalization of Equations

- By inspection, nondimensionalize all variables with scaling parameters

$$p^\star = \frac{p}{\rho_0 U_0^2} \qquad \rho^\star = \frac{\rho}{\rho_0} \qquad V^\star = \frac{V}{U_0}$$

$$g^\star = \frac{gL}{U_0^2} \qquad z^\star = \frac{z}{L}$$

- Back-substitute p, ρ, V, g, z into dimensional equation

$$\rho_0 U_0^2 p^\star + \frac{1}{2}\rho_0 \rho^\star \left(U_0^2 {V^\star}^2\right) + \rho_0 \rho^\star g^\star U_0^2 z^\star = C$$

Nondimensionalization of Equations

- Divide by $\rho_0 U_0^2$ and set $\rho^* = 1$ (incompressible flow)

$$p^\star + \frac{1}{2}V^{\star 2} + g^\star z^\star = \frac{C}{\rho_0 U_0^2} = C^\star$$

- Since $g^* = 1/Fr^2$, where $Fr = \frac{U_0}{\sqrt{gL}}$

$$p^\star + \frac{1}{2}V^{\star 2} + \frac{1}{Fr^2} z^\star = C^\star$$

Nondimensionalization of Equations

- Note that convention often dictates many of the nondimensional parameters, e.g., $\frac{1}{2} \rho_0 U_0^2$ is typically used to nondimensionalize pressure.

$$p^\star = \frac{p - p_0}{\frac{1}{2}\rho U_0^2}$$

- This results in a slightly different form of the nondimensional equation

$$p^\star + V^{\star 2} + \frac{2}{Fr^2} z^\star = C^\star$$

- BE CAREFUL! Always double check definitions.

Nondimensionalization of Equations

- Advantages of nondimensionalization
  - Increases insight about key parameters
  - Decreases number of parameters in the problem
    - Easier communication
    - Fewer experiments
    - Fewer simulations
  - Extrapolation of results to untested conditions

Dimensional Analysis and Similarity

- Nondimensionalization of an equation is useful only when the equation is known!
- In many real-world flows, the equations are either unknown or too difficult to solve.
  - *Experimentation* is the only method of obtaining reliable information (including *numerical* experimentation)
  - In most experiments, geometrically-scaled models are used (time and money).
  - Experimental conditions and results must be properly scaled so that results are meaningful for the full-scale prototype.
  - ***Dimensional Analysis***

Dimensional Analysis and Similarity

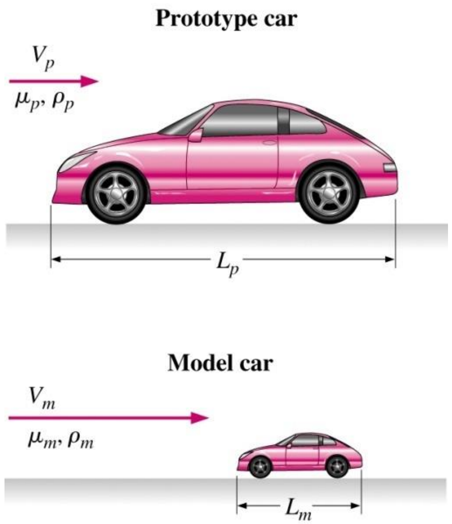
- Primary purposes of dimensional analysis
  - To **generate nondimensional parameters** that help in the design of experiments (physical and/or numerical) and in reporting of results
  - To **obtain scaling laws** so that *prototype* performance can be predicted from *model* performance.
  - To **predict trends** in the relationship between parameters.

Dimensional Analysis and Similarity

- **Geometric Similarity** - the model must be the same shape as the prototype. Each dimension must be scaled by the same factor.
- **Kinematic Similarity** - velocity at any point in the model must be proportional
- **Dynamic Similarity** - *all forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow.
- **Complete Similarity** is achieved only if all 3 conditions are met. This is not always possible, e.g., river hydraulics models.

Dimensional Analysis and Similarity

- Complete similarity is ensured if all independent Π groups are the same between model and prototype.
- What is Π?
  - We let uppercase Greek letter Π denote a nondimensional parameter, e.g.,Reynolds number Re, Froude number Fr, Drag coefficient, C_D, etc.



- Consider automobile experiment
- Drag force is $F = f(V_p, \rho, \mu, L)$
- Through dimensional analysis, we can reduce the problem to

$$\Pi_1 = f(\Pi_2) \rightarrow C_D = f(Re)$$

Method of Repeating Parameters

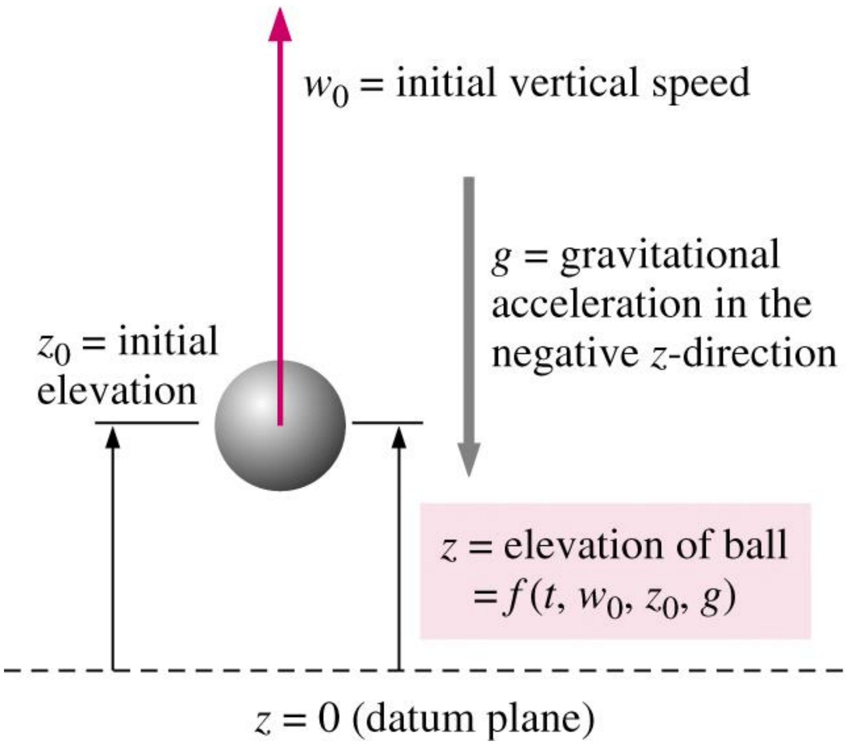
- Nondimensional parameters Π can be generated by several methods.
- We will use the **Method of Repeating Parameters** **(**Buckingham Π Theorem**)**
- Six steps
  1. List the parameters in the problem and count their total number n.
  2. List the primary dimensions of each of the n parameters
  3. Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's, $k = n - j$.
  4. Choose *j repeating parameters.*
  5. Construct the k Π's, and manipulate as necessary.
  6. Write the final functional relationship and check algebra.

Guidelines for choosing *Repeating Parameters*

1. Never pick the dependent variable. Otherwise, it may appear in all the Π's.
2. Chosen repeating parameters must not *by themselves* be able to form a dimensionless group. They must be **dimensionally independent**, otherwise it would be impossible to generate the rest of the Π's. Never pick parameters that are already dimensionless, nor pick two parameters with the same dimensions or with dimensions that differ by only an exponent.
3. Chosen repeating parameters must represent *all* the primary dimensions.
4. Choose dimensional constants over dimensional variables so that each Π contains a single dimensional variable.
5. Pick common parameters since they may appear in each of the Π's.
6. Pick simple parameters over complex parameters.

Example

Ball Falling in a Vacuum



- Step 1: List relevant parameters.

$$z = f(t, w_0, z_0, g) \Rightarrow n = 5$$

- Step 2: Primary dimensions of each parameter

z	t	w_0	z_0	g
$\{L^1\}$	$\{t^1\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{L^1t^{-2}\}$

- Step 3: As a first guess, reduction j is set to 2 which is the number of primary dimensions (L and t). Number of expected Π's is $k = n - j = 5 - 2 = 3$
- Step 4: Choose repeating parameters w_0 and z_0

Example, continued

- Step 5: Combine repeating parameters into products with each of the remaining parameters, one at a time, to create the Π's.
- $$\Pi_1 = z\, w_0^{a_1} z_0^{b_1}$$
  - a_1 and b_1 are constant exponents which must be determined.
  - Use the primary dimensions identified in Step 2 and solve for a_1 and b_1.

$$\{\Pi_1\} = \{L^0 t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1 (L^1 t^{-1})^{a_1} L^{b_1}\}$$

  - Time equation: $\{t^0\} = \{t^{-a_1}\} \rightarrow 0 = -a_1 \rightarrow a_1 = 0$
  - Length equation:

$$\{L^0\} = \{L^1 L^{a_1} L^{b_1}\} \rightarrow 0 = 1 + a_1 + b_1 \rightarrow b_1 = -1 - a_1 \rightarrow b_1 = -1$$

  - This results in $$\boxed{\Pi_1 = z w_0^0 z_0^{-1} = \frac{z}{z_0}}$$

Example, continued

- Step 5: continued
  - Repeat process for Π_2 by combining repeating parameters with t
  - $\Pi_2 = t\, w_0^{a_2} z_0^{b_2}$

$$\{\Pi_2\} = \{L^0 t^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{t^1 (L^1 t^{-1})^{a_2} L^{b_2}\}$$

  - Time equation:

$$\{t^0\} = \{t^1 t^{-a_2}\} \rightarrow 0 = 1 - a_2 \rightarrow a_2 = 1$$

  - Length equation:

$$\{L^0\} = \{L^{a_2} L^{b_2}\} \rightarrow 0 = a_2 + b_2 \rightarrow b_2 = -a_2 \rightarrow b_2 = -1$$

  - This results in

$$\Pi_2 = t w_0^1 z_0^{-1} = \frac{w_0 t}{z_0}$$

Example, continued

- Step 5: continued
  - Repeat process for Π_3 by combining repeating parameters with g
  - $\Pi_3 = g\, w_0^{a_3}\, z_0^{b_3}$

$$\{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$$

  - Time equation:

$$\{t^0\} = \{t^{-2} t^{-a_3}\} \rightarrow 0 = -2 - a_3 \rightarrow a_3 = -2$$

  - Length equation:

$$\{L^0\} = \{L^1 L^{a_3} L^{b_3}\} \rightarrow 0 = 1 + a_3 + b_3 \rightarrow b_3 = -1 - a_3 \rightarrow b_3 = 1$$

  - This results in

$$\Pi_3 = g w_0^{-2} z_0^1 = \frac{g z_0}{w_0^2}$$

$$\Pi_{3,modified} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = Fr$$

Example, continued

- Step 6:
  - Double check that the Π's are dimensionless.
  - Write the functional relationship between Π's

$$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \boxed{\frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{\sqrt{g z_0}}\right)}$$

  - Or, in terms of nondimensional variables

$$z^* = f(t^*, Fr)$$

- Overall conclusion: Method of repeating variables properly predicts the functional relationship between dimensionless groups.
- However, the method cannot predict the exact mathematical form of the equation.

General fluid dynamics problem

- $F = f_0(\rho, U_0, L, \mu, \kappa_s, \sigma_s, g, r_s, r_r)$
  - $\kappa_s = \rho\, (\partial p/\partial \rho)_s$ (similar to κ – isothermal bulk modulus of elasticity – in solids)

    $\kappa_{s\ \text{ideal gas}} = \gamma p$

    r_s and r_r are shape and roughness coefficients
- Select ρ, U_0 and L as *repeating parameters* to render the functional relation dimensionless
- $\Pi_0 = f\,(1, 1, 1, \Pi_1, \Pi_2, \Pi_3, \Pi_4, r_s, r_r) = f\,(\Pi_1, \Pi_2, \Pi_3, \Pi_4, r_s, r_r)$
- Π_i dimensionless; f_0 different from f

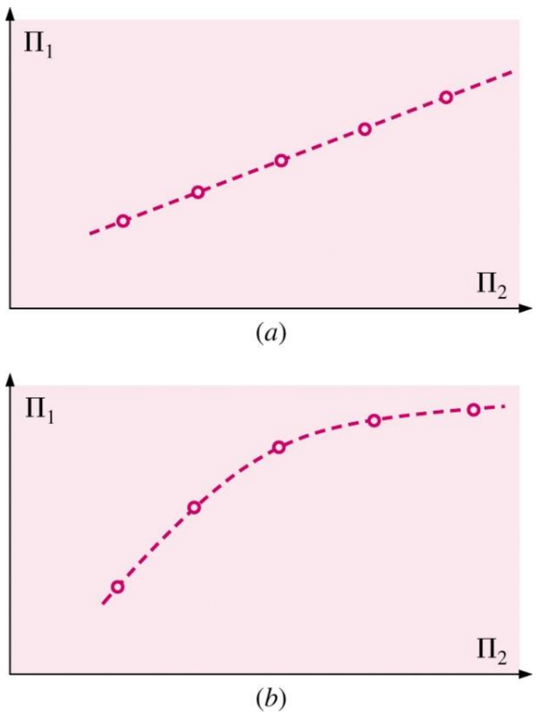
General fluid dynamics problem

$$Ne = \varphi\,(Re, Ma, We, Fr, r_s, r_r)$$

- Ne = Newton number = $F/(\rho U_0^2 L^2)$
- Re = Reynolds number = $\rho U_0 L/\mu$
- Ma = Mach number = $U_0/\sqrt{\kappa_s/\rho}$
- We = Weber number = $U_0/\sqrt{\sigma_s/(\rho L)}$
- Fr = Froude number = $U_0/\sqrt{gL}$

Experimental Testing and Similarity

- One of the most useful applications of dimensional analysis is in designing physical and/or numerical experiments, and in reporting the results.
- Setup of an experiment and correlation of data.



- Consider a problem with 5 parameters: one dependent and 4 independent.
- Full test matrix with 5 data points for each independent parameter would require 5^4=625 experiments!!
- If we can reduce to 2 Π's, the number of independent parameters is reduced from 4 to 1, which results in 5^1=5 experiments vs. 625!!

Experimental Testing and Similarity

Wanapum Dam on Columbia River

Physical Model at
Iowa Institute of Hydraulic Research

- Flows with free surfaces present unique challenges in achieving complete dynamic similarity.
- For hydraulics applications, depth is very small in comparison to horizontal dimensions. If geometric similarity is used, the model depth would be so small that other issues would arise
  - Surface tension effects (Weber number) would become important.
  - Data collection becomes difficult.
- *Distorted models* are therefore employed, which requires empirical corrections/correlations to extrapolate model data to full scale.

Incomplete Similarity

DDG-51 Destroyer

1/20th scale model

- For ship hydrodynamics, Fr similarity is maintained while Re is allowed to be different.
- Why? Look at complete similarity:

$$Re_p = \frac{V_p L_p}{\nu_p} = Re_m = \frac{V_m L_m}{\nu_m} \rightarrow \frac{L_m}{L_p} = \frac{\nu_m}{\nu_p} \frac{V_p}{V_m}$$

$$Fr_p = \frac{V_p}{\sqrt{g L_p}} = Fr_m = \frac{V_m}{\sqrt{g L_m}} \rightarrow \frac{L_m}{L_p} = \left(\frac{V_m}{V_p}\right)^2$$

- To match both Re and Fr, viscosity in the model test is a function of scale ratio! ***This is hardly feasible.***

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$

Similarity: reduction scales

- To be able to establish dynamic similarity (*complete* or *incomplete*) between fluid dynamics phenomena, there must exist:
  - A scale $\lambda = L_m/L_p$ for the reduction of lengths (geometric similarity)
  - A scale $\tau = t_m/t_p$ for the reduction of characteristic time scales (kinematic similarity)
  - A scale $\varphi = F_m/F_p$ to reduce corresponding forces

Incomplete (or partial) similarities

- Reynolds incomplete similarity

  $Ne_m = Ne_p$ if $Re_m = Re_p$

  It is found: $\tau = \lambda^2,\ \varphi = 1$ (same fluids)

- Froude incomplete similarity

  $Ne_m = Ne_p$ if $Fr_m = Fr_p$

  It is found: $\tau = \lambda^{1/2},\ \varphi = \lambda^3$ (same fluids)