Transition to Turbulence in Shear Flows

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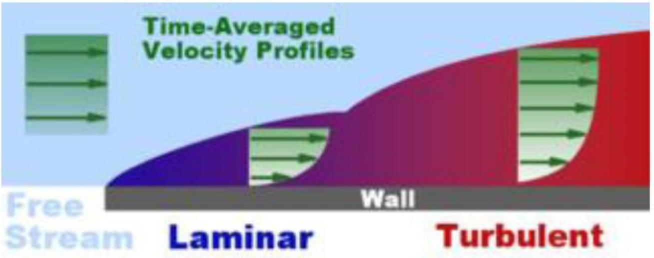
Transition: a burning question for 100+ years!

What happens/why?



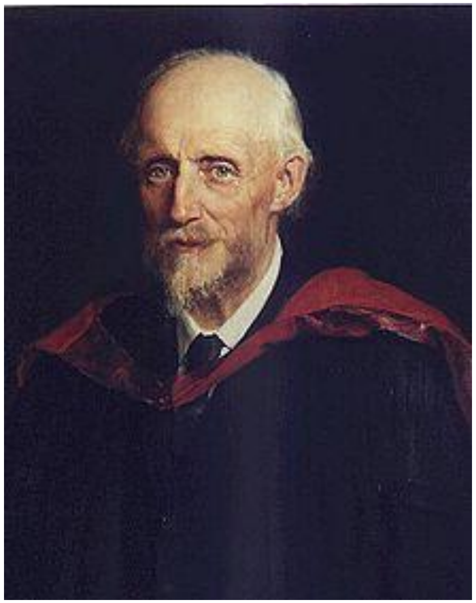
http://en.wikipedia.org/wiki/Boundary_layer_transition

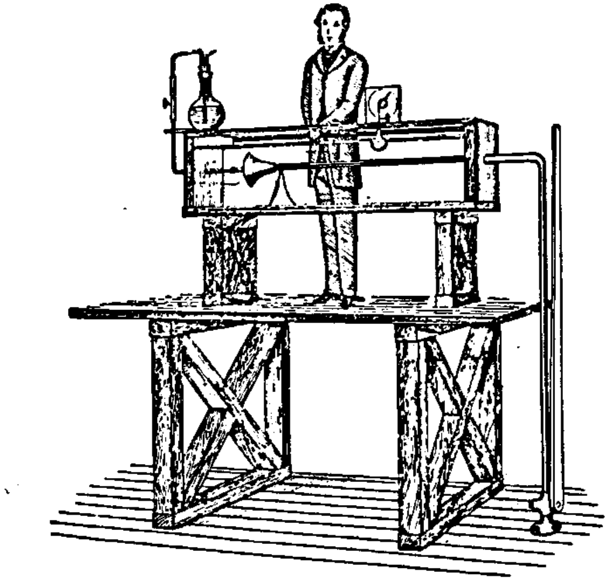
'… the concept of boundary layer transition is a complex one and still lacks a complete theoretical exposition.'

Objectives

1. What is "transition to turbulence" and why is it important?
2. Early attempts at describing transition analytically in *parallel* shear flows (Rayleigh, Orr, Sommerfeld)
3. Partial experimental confirmations (Tollmien-Schlichting waves)
4. Something does not work … back to square one! Transient growth and the "optimal perturbations"
5. Still having problems: nonlinear transients …
6. And if we reversed the problem? Using chaos theory …

1. What is “transition to turbulence” and why is it important?





Osborne Reynolds (1842-1912)

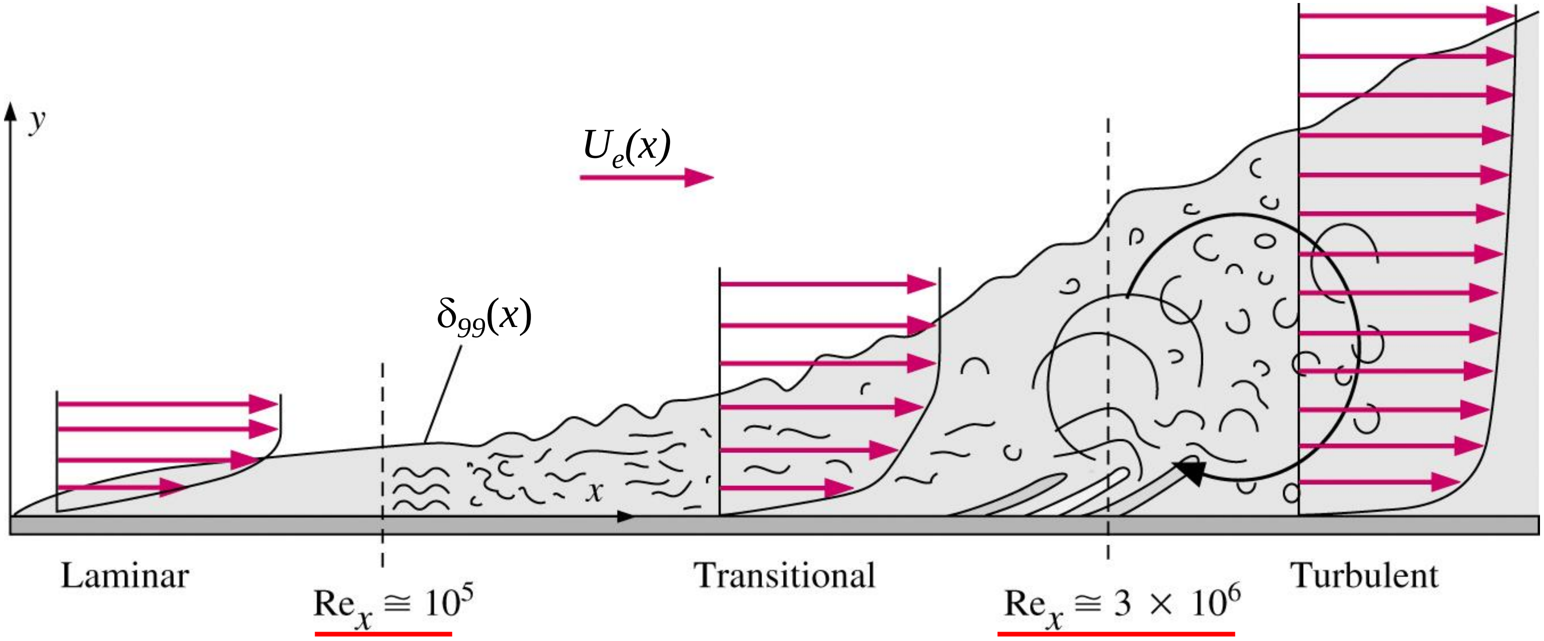
What is transition to turbulence?

Phenomenon which progressively brings a given flow – take a simple Blasius boundary layer as an example – from a laminar (orderly) state to a new state which is 3D, chaotic, possibly stochastic, vortical, …

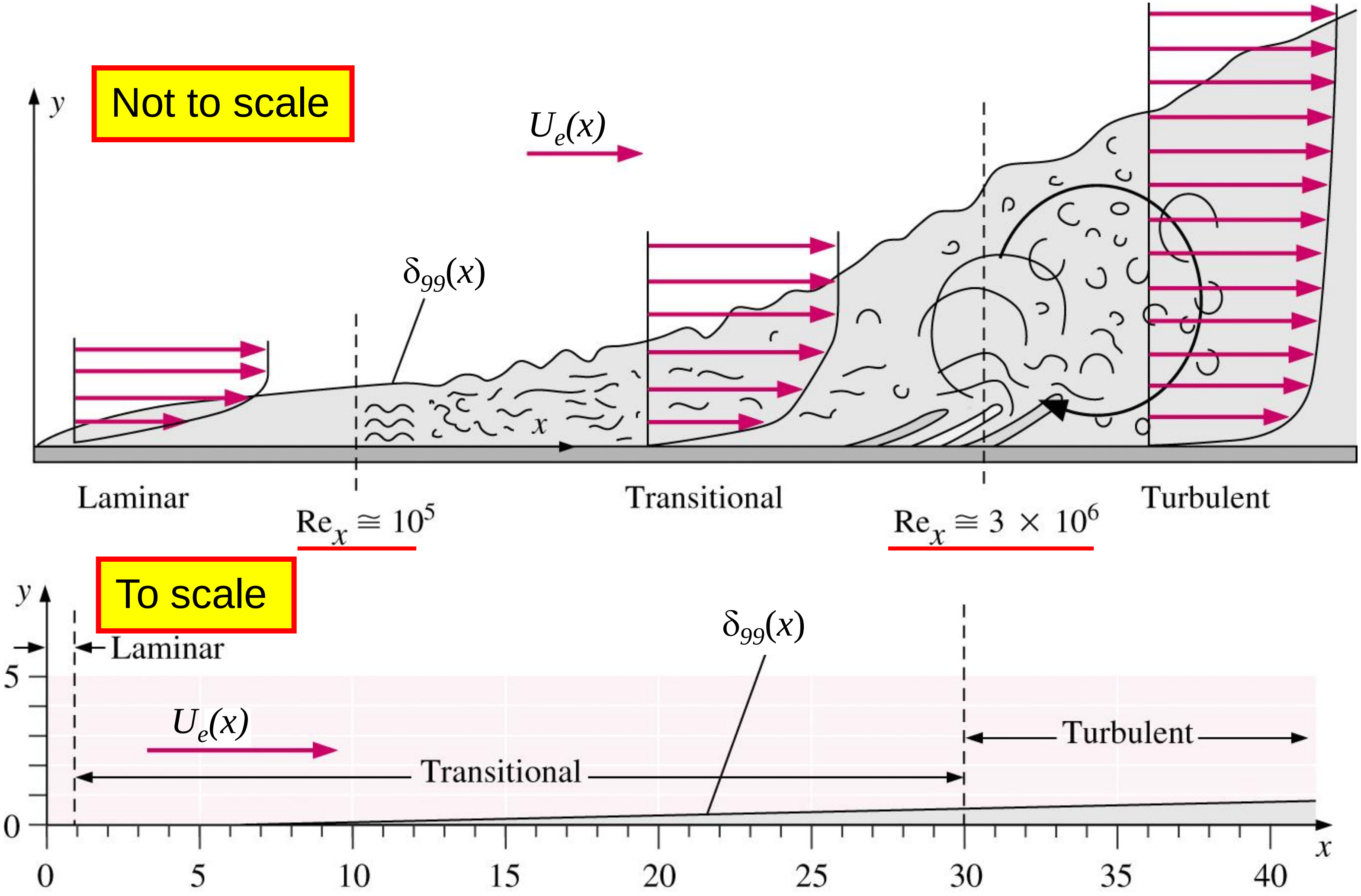
Transition corresponds to the breaking of (more than one) symmetries of an initially well organized flow state.

In a boundary layer transition is triggered by exogeneous disturbances.

What is transition to turbulence?



What is transition to turbulence?



Laminar flow: the boundary layer approx.

- Incompressible laminar boundary layer equations

Continuity
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

x-momentum
$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = U_e\frac{\partial U_e}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2}$$

y-momentum
$$\frac{\partial P}{\partial y} = 0$$

The Blasius boundary layer

- Incompressible laminar boundary layer equations with no external pressure gradient

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

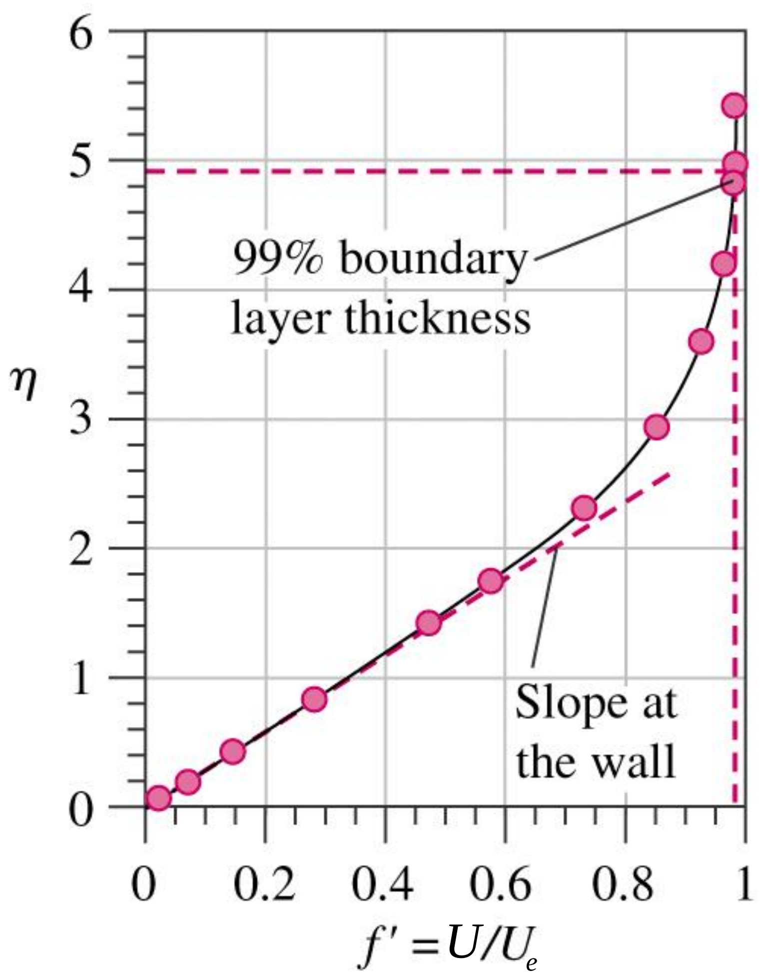
x-momentum

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = U_e\frac{\partial U_e}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2}$$

y-momentum

$$\frac{\partial P}{\partial y} = 0$$

Blasius Similarity Solution



- Blasius introduced similarity variables

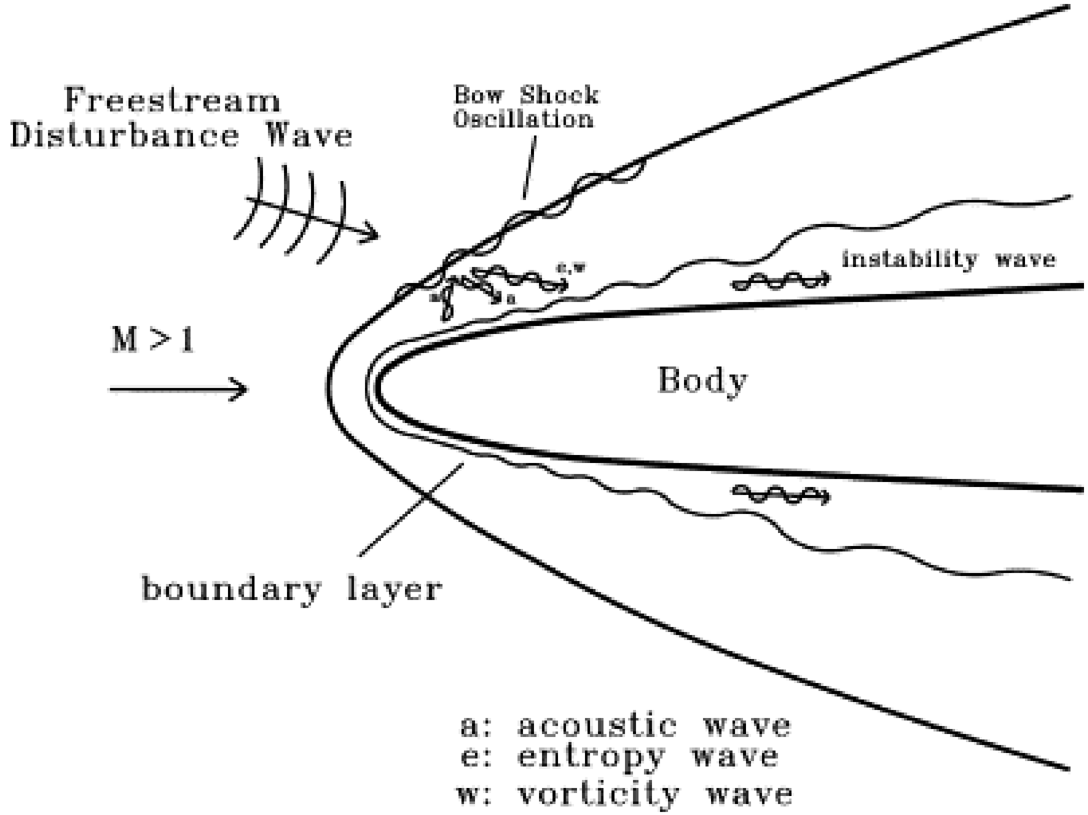
$$f' = \frac{U}{U_e} \qquad \eta = y\sqrt{\frac{U_e}{\nu x}}$$

- This reduces the BLE to

$$2f''' + ff'' = 0$$
$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

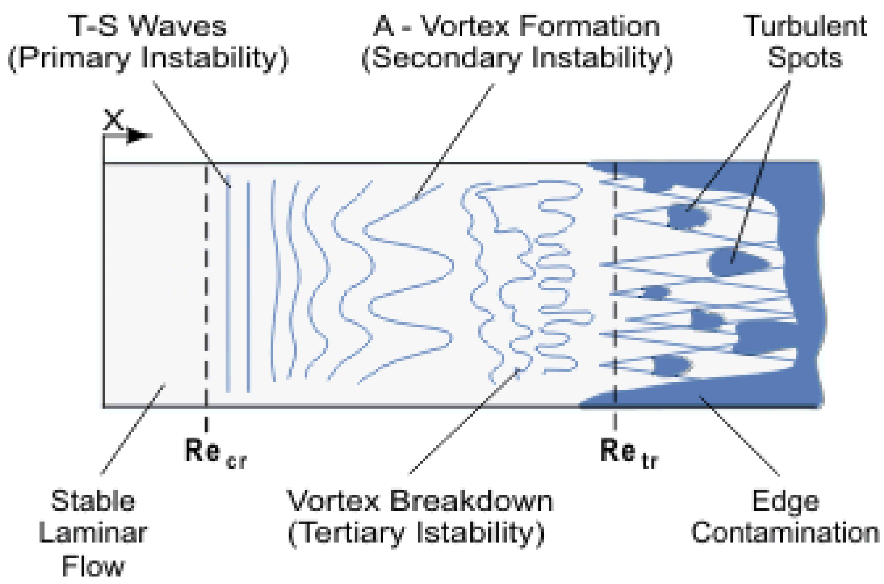
- This ODE can be solved using Runge-Kutta technique
- Result is a boundary layer profile which holds at every station along the flat plate

The triggering of instabilities



The creation of disturbance waves in the boundary layer from (the possible interaction of) exogenous disturbances is called receptivity.

Experimental observations (1)



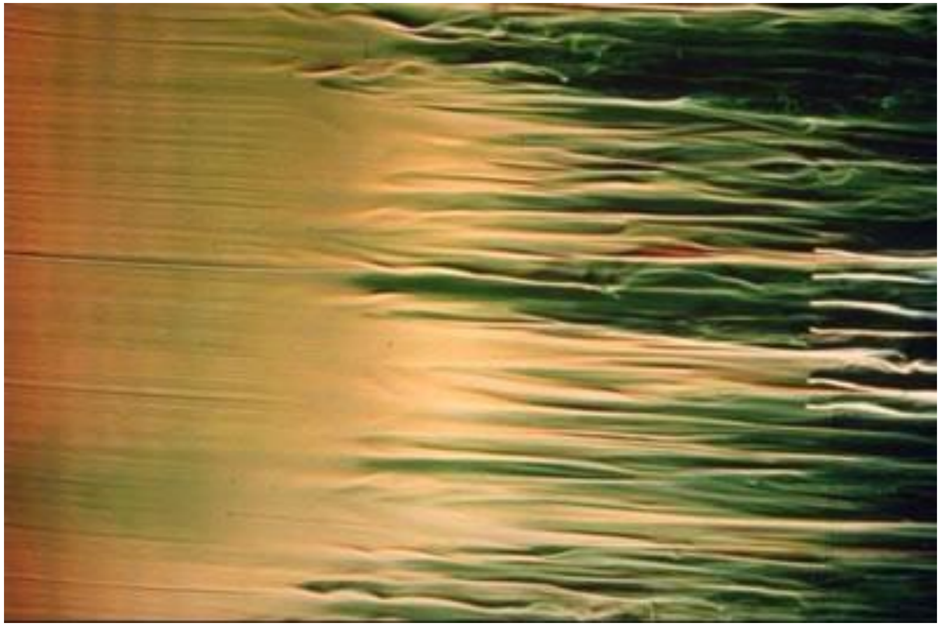


Image: ONERA DAFE, Paris

Experimental observations (2)

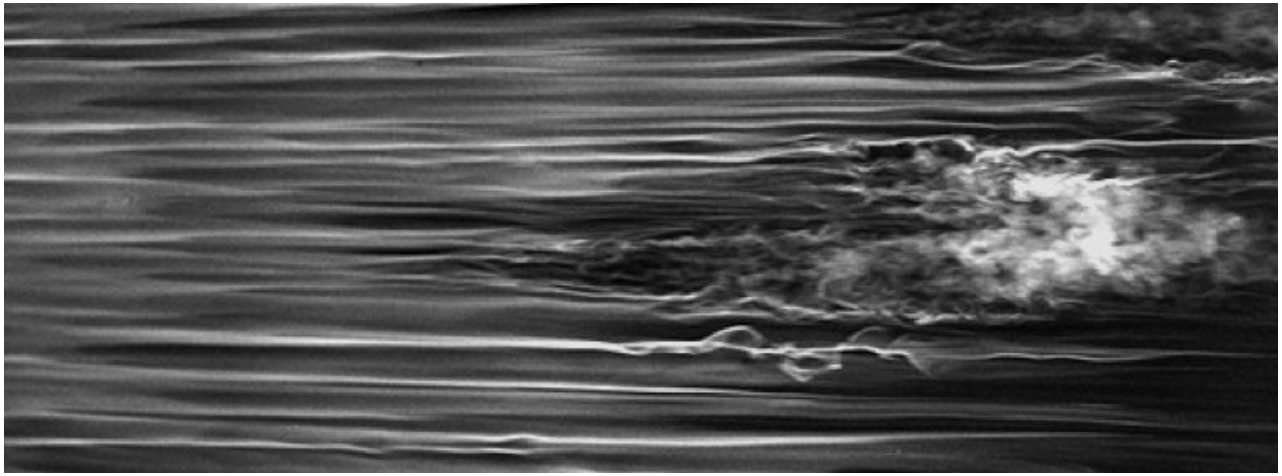
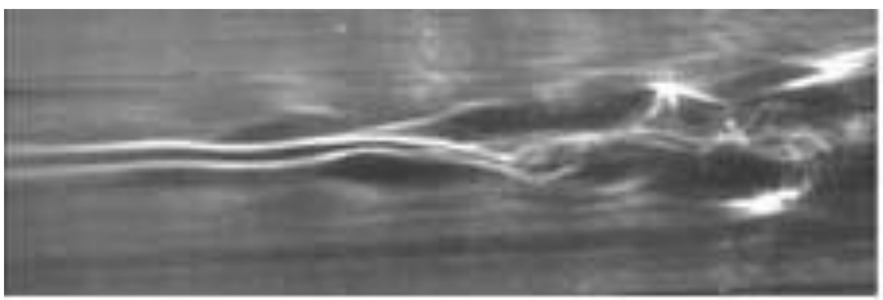
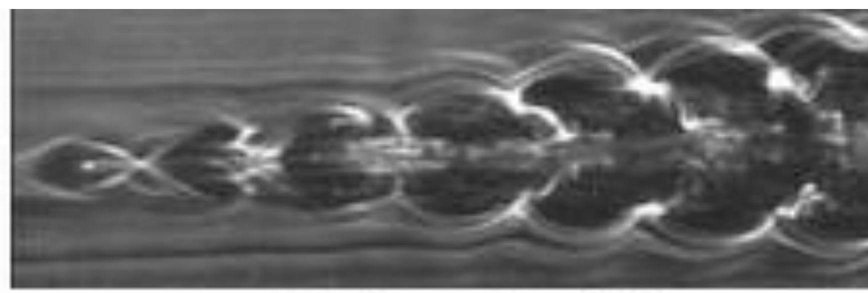


Image: Dept. Of Mechanics, KTH, Stockholm



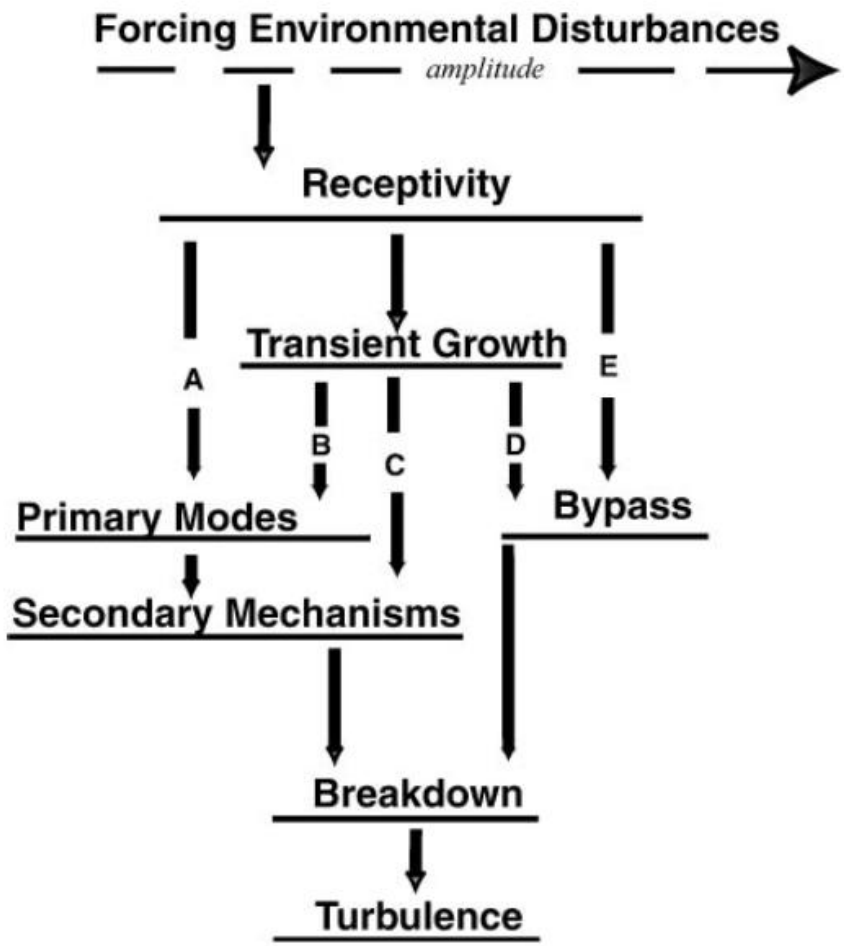
Sinuous Instability



Varicose Instability

Image: Fluid Dynamics Laboratory Tokyo Metropolitan University

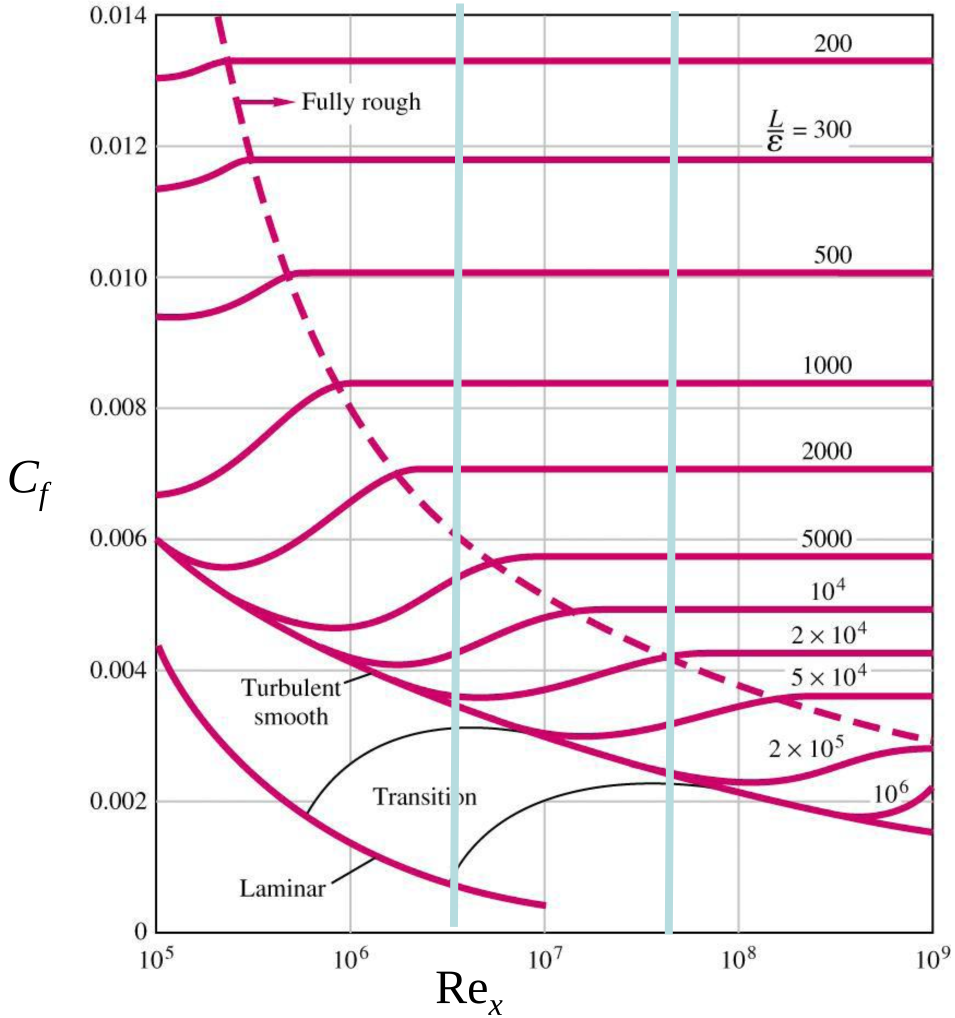
Interpretation



The receptivity process defines the type of disturbance waves which will emerge.

Morkovin, 1994

Why is turbulent transition important?



$$\mathrm{Re}_x = U_\infty x / \nu$$

$$C_f = 2 \, \tau_w / (\rho \, U_\infty{}^2)$$

Why is turbulent transition important?

Aeronautics: delaying transition over wings is fundamental to reduce fuel consumption, CO_2 emissions and operating costs.

It has been estimated (Joslin, 1998) that aircraft laminar flow control over wings, tail, nacelles, etc. can reduce DOC by a few percentage points, leading to savings of several M$/year.

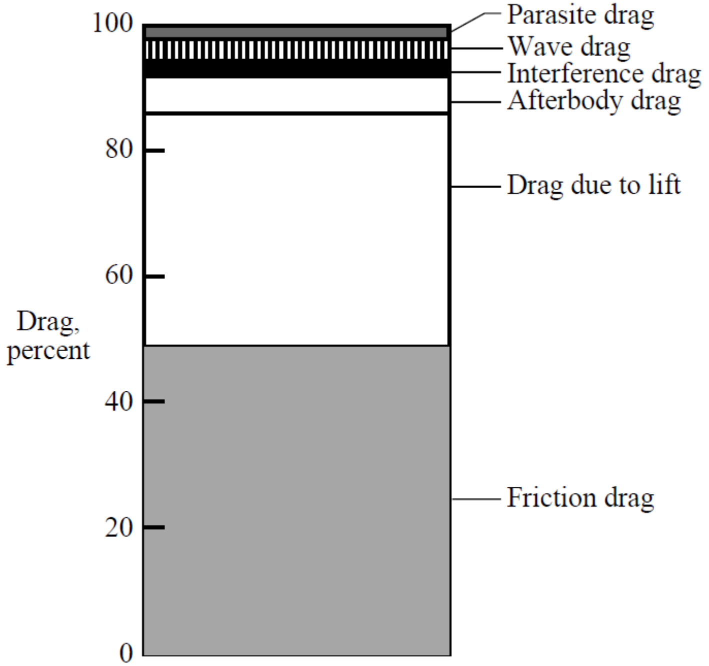
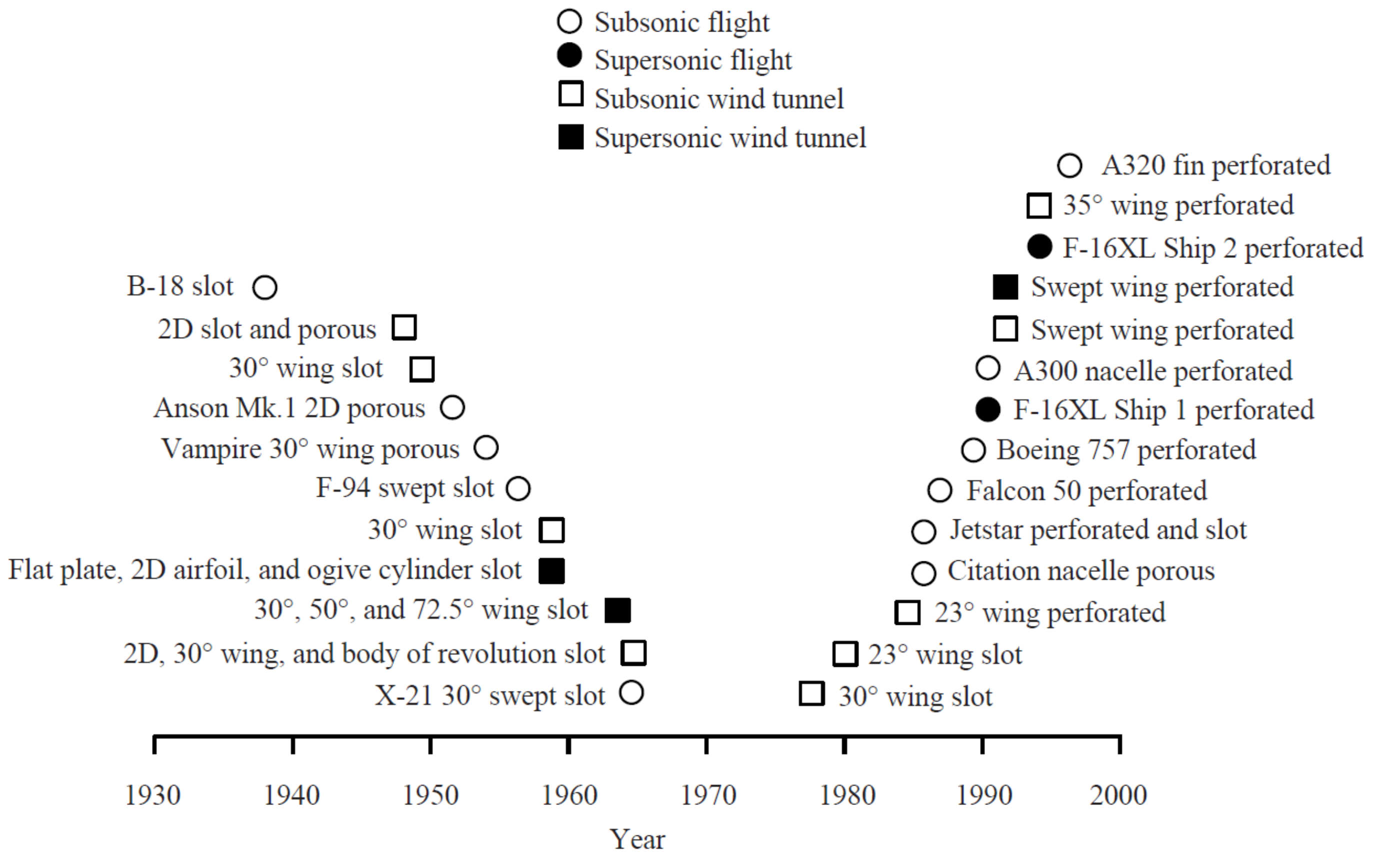


Figure 3. Aircraft drag breakdown. (From Thibert, Reneaux, and Schmitt 1990.)

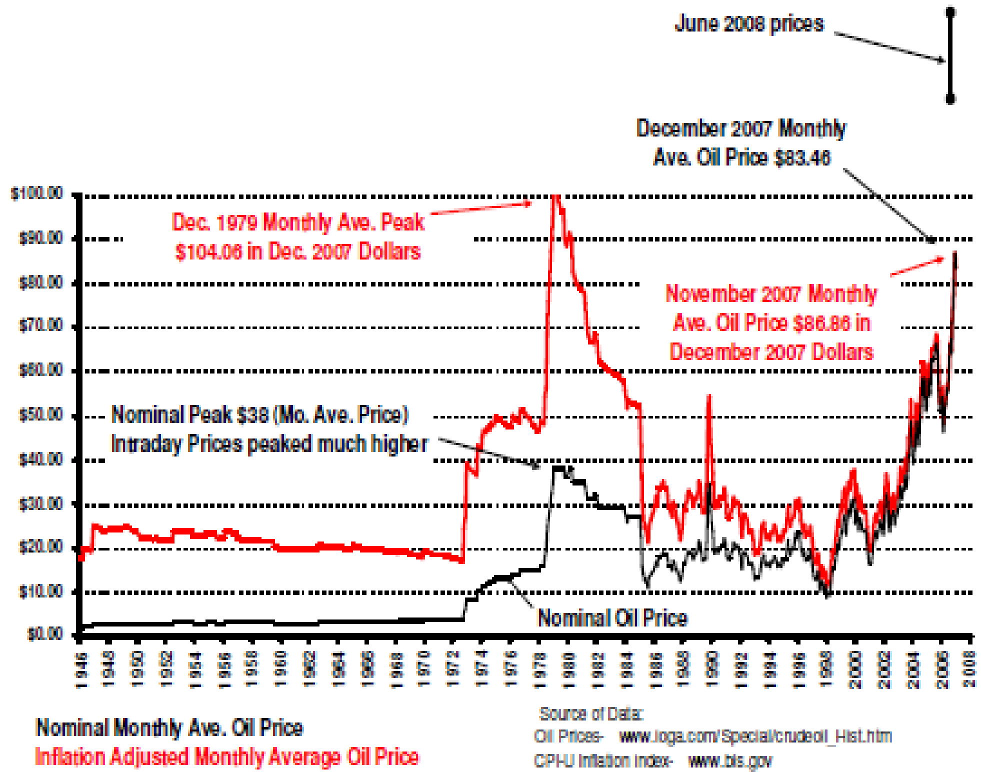
Laminar flow control



Overview of Laminar Flow Control Projects.

Joslin, 1998

Laminar flow control



Inflation-adjusted monthly crude oil prices, 1946 - present
(from www.InflationData.com updated 16 January 2008, with June 2008 data added by author)

Green, 2008

Laminar flow control

38th Fluid Dynamics Conference and Exhibit
23 - 26 June 2008, Seattle, Washington

AIAA 2008-3738

Laminar Flow Control – Back to the Future?

John E. Green[1]
Aircraft Research Association Ltd., Bedford UK, MK41 7PF

In the 21st Century, reducing the environmental impact of aviation will become an increasingly important priority for the aircraft designer. Among the various environmental impacts, emission of CO_2 can be expected to emerge as the most important in the long term and reducing fuel burn to become the overriding environmental priority. Increasing fuel costs and the world's limited oil reserves will add to the pressure to reduce fuel burn. Starting from the limitations imposed on the aircraft designer by the laws of physics – the Breguet Range Equation, the Second Law of Thermodynamics, the behaviour of real, viscous fluids – the paper discusses the technological and design options available to the designer. Improvements in propulsion and structural efficiency have valuable contributions to make but it is in drag reduction through laminar flow control that the greatest opportunity lies. The physics underlying laminar flow control is discussed and the key features and limitations of natural, hybrid and full laminar flow control are explained. Experience to date in this field is briefly reviewed, with particular attention drawn to the substantial body of work in the 1950s and 1960s that demonstrated the potential of full laminar flow control by boundary-layer suction. The case is argued for revisiting the design of an aircraft with full laminar flow control, taking into account the advances over the past half century in all aspects of aircraft engineering, notably in propulsion and materials. With approximately half the thrust provided by the boundary layer suction system, this aircraft presents a completely new challenge in airframe-propulsion integration. We understand the physics of boundary layer control, we know that an aircraft with full laminar flow is potentially much more fuel efficient than the alternatives, what is needed now is a wholehearted attack on the engineering obstacles in its path.

Flow control techniques

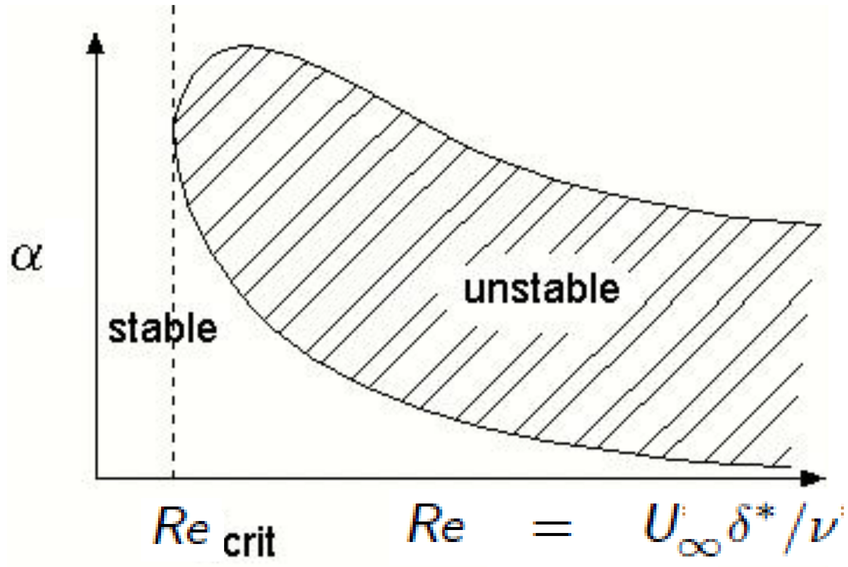
Active techniques

Blowing and/or suction
Wall motion
Wall heating/cooling
MEMS
Synthetic jets
EMHD
Plasma flow control
…

Passive techniques

Shaping
Compliant coatings
Turbulators/roughness
Porous surfaces
Poroelasticity
Riblets
Super-hydrophobicity (in H_2O)
…

2. Early attempts at describing transition analytically in *parallel* shear flows (Rayleigh, Orr, Sommerfeld)



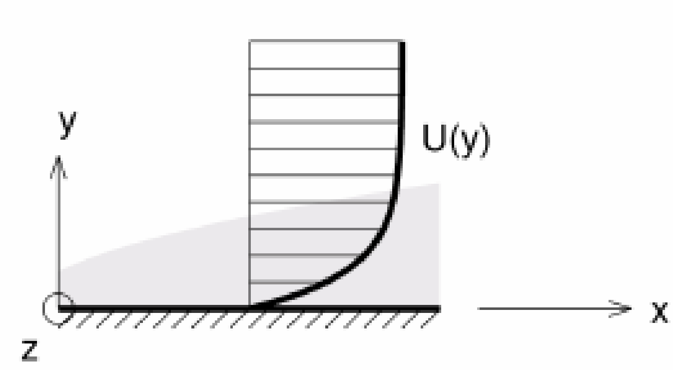
The (incompressible) disturbance equations

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i(x_i, 0) = u_i^0(x_i)$$

$$u_i(x_i, t) = 0 \quad \text{on solid boundaries}$$



$$Re = U_\infty \delta^* / \nu$$

$$u_i = U_i + u_i' \quad \text{decomposition}$$

$$p = P + p'$$

Introduce decomposition, drop primes, subtract eq's for $\{U_i, P\}$

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

The (incompressible) disturbance equations

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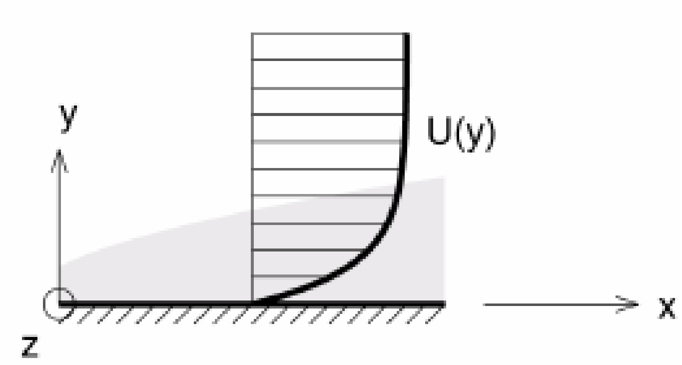

$$Re = U_\infty \delta^* / \nu$$

$$u_i = U_i + u_i' \quad \text{decomposition}$$

$$p = P + p'$$

Introduce decomposition, drop primes, linearize

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

The (incompressible) disturbance equations

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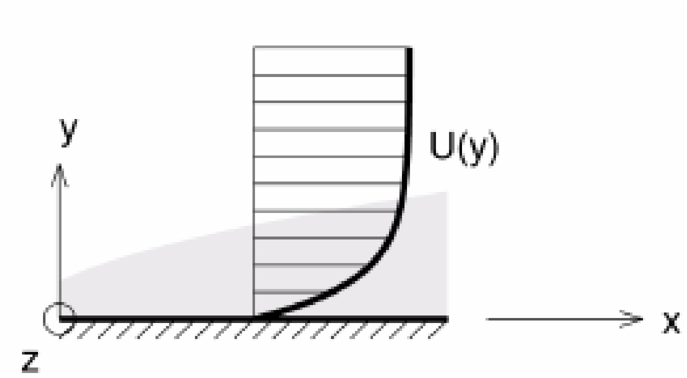


$$Re = U_\infty \delta^* / \nu$$

$$u_i = U_i + u_i' \quad \text{decomposition}$$

$$p = P + p'$$

Linearised Navier-Stokes equations,

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Stability definition

$$E(t) = \frac{1}{2}\int_{\Omega} u_i(t)u_i(t)\, d\Omega$$

Stable : $\lim_{t\to\infty} \frac{E(t)}{E(0)} \to 0$

Conditionally stable : $\exists\, \delta > 0 :\ E(0) < \delta \Rightarrow \text{stable}$

Globally stable : Conditionally stable with $\delta \to \infty$

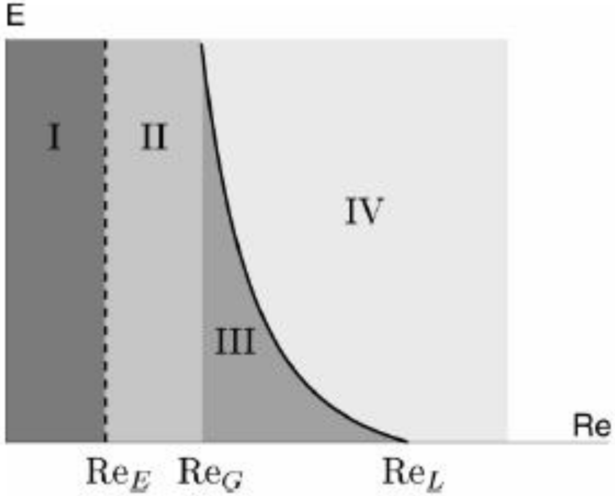
Monotonically stable : Globally stable and $\frac{dE}{dt} \leq 0 \quad \forall t > 0$

Stability definition

Re_E : $Re < Re_E$ flow monotonically stable

Re_G : $Re < Re_G$ flow globally stable

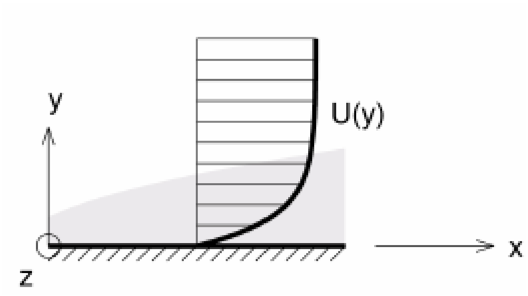
Re_L : $Re < Re_L$ flow linearly stable ($\delta \to 0$)



Initial energy E vs the Reynolds number Re

Local stability of the Blasius boundary layer

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + &= -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$



Initial conditions :

$$\{u, v, w\}(x, y, z, t = 0) = \{u_0, v_0, w_0\}(x, y, z)$$

Boundary conditions : depend on flow case

$$\{u, v, w\}(x, y = y_1, z, t) = 0$$ solid boundaries

Semi-infinite domain :

$$\{u, v, w\}(x, y \to \infty, z, t) \to 0$$ free stream

Local stability of the Blasius boundary layer

We can reduce the original 4 eq's & 4 unknowns to a system of 2 eq's and 2 unknowns
This is in two steps

1. Take the divergence of the momentum equations. This yields

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}.$$

2. The new pressure equation is introduced in the momentum equation for v. This yields

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right] v = 0.$$

The three-dimensional flow is then analyzed introducing the normal vorticity

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^2 \right] \eta = -U' \frac{\partial v}{\partial z}.$$

with the boundary conditions

$$v = v' = \eta = 0 \qquad \text{at a solid wall and in the far field}$$

Local stability of the Blasius boundary layer

Assume wave-like solutions:

$$v(x, y, z, t) = \tilde{v}(y) \exp i(\alpha x + \beta z - \omega t)$$

Introduce the ansatz in the equations for $\{v, \eta\}$. This yields

$$\begin{aligned} \left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} &= 0 \\ \left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right]\eta &= -i\beta U' \tilde{v} \end{aligned}$$

Here, $k^2 = \alpha^2 + \beta^2$ and $D^i = \partial^i / dy^i$.

Orr-Sommerfeld modes : $\{\tilde{v}_n, \tilde{\eta}_n^p, \omega_n\}_{n=1}^{N}$

Squire modes : $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^{M}$

Notations

$$\omega = \alpha c$$

$$v = \text{Real}\{|\tilde{v}(y)|\ e^{i\phi(y)}\ e^{i[\alpha x + \beta z - \alpha(c_r + i c_i)t]}\}$$

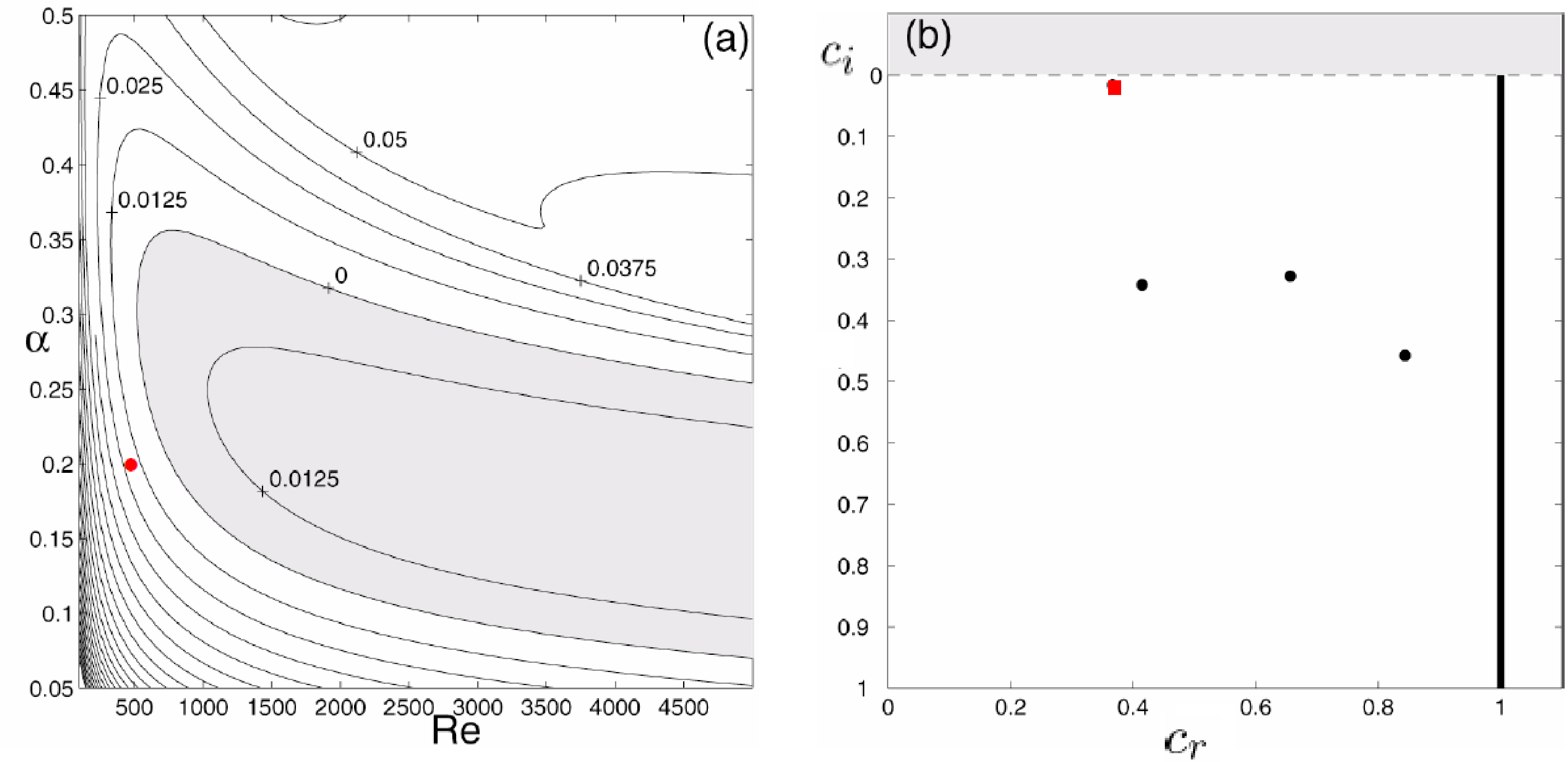
$$= |\tilde{v}(y)|\ e^{\alpha c_i t}\ \cos[\alpha(x - c_r t) + \beta z + \phi(y)]$$

ω	angular frequency
c_r	phase speed
c_i	temporal growthrate
α	streamwise wavenumber
β	spanwise wavenumber

Some old and useful results

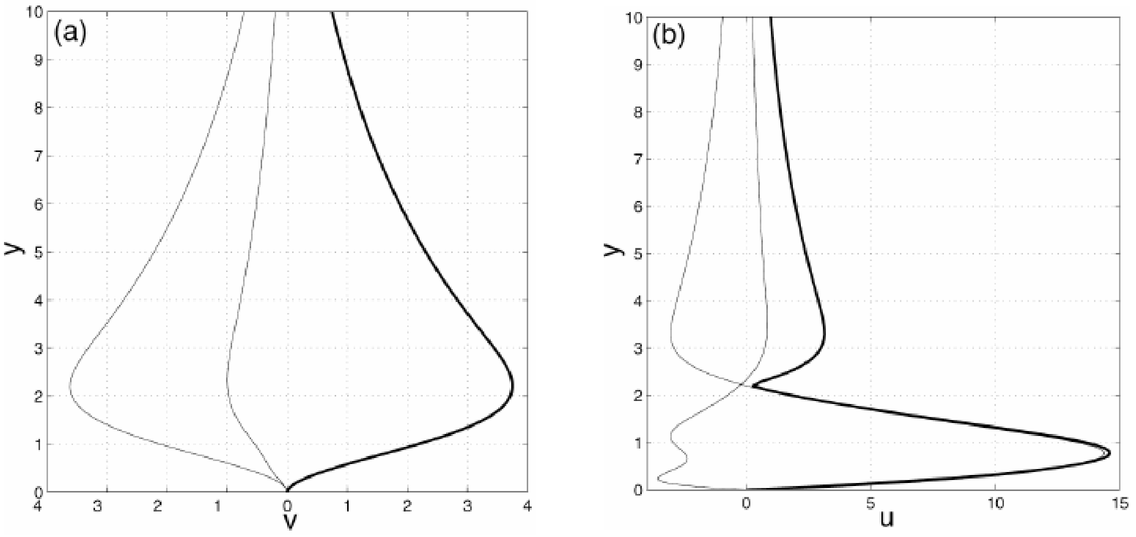
- Squire modes are always damped
- For each 3D mode there exist always a 2D mode more amplified (Squire theorem)
- Inviscid result: necessary condition for instability is the existence of an inflection point in the base flow profile U(y) corresponding to a maximum of vorticity (Rayleigh and Fjørtoft theorems)

Numerical results (OS equation)



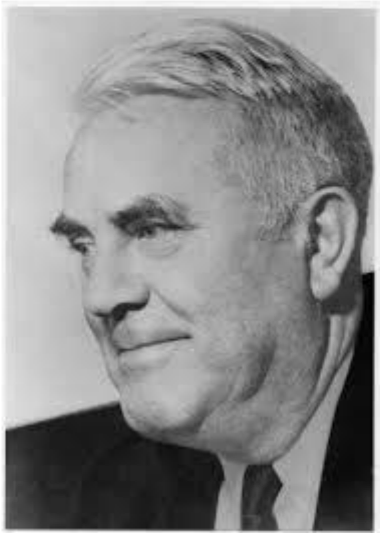
Numerical results (OS equation)

- Re = 500
- $\alpha = 0.2$
- TS-mode



3. Partial experimental confirmations (Tollmien-Schlichting waves)

Walter Tollmien (1900-1968)



Hermann Schlichting (1907-1982)

Experimental results (wind tunnel)

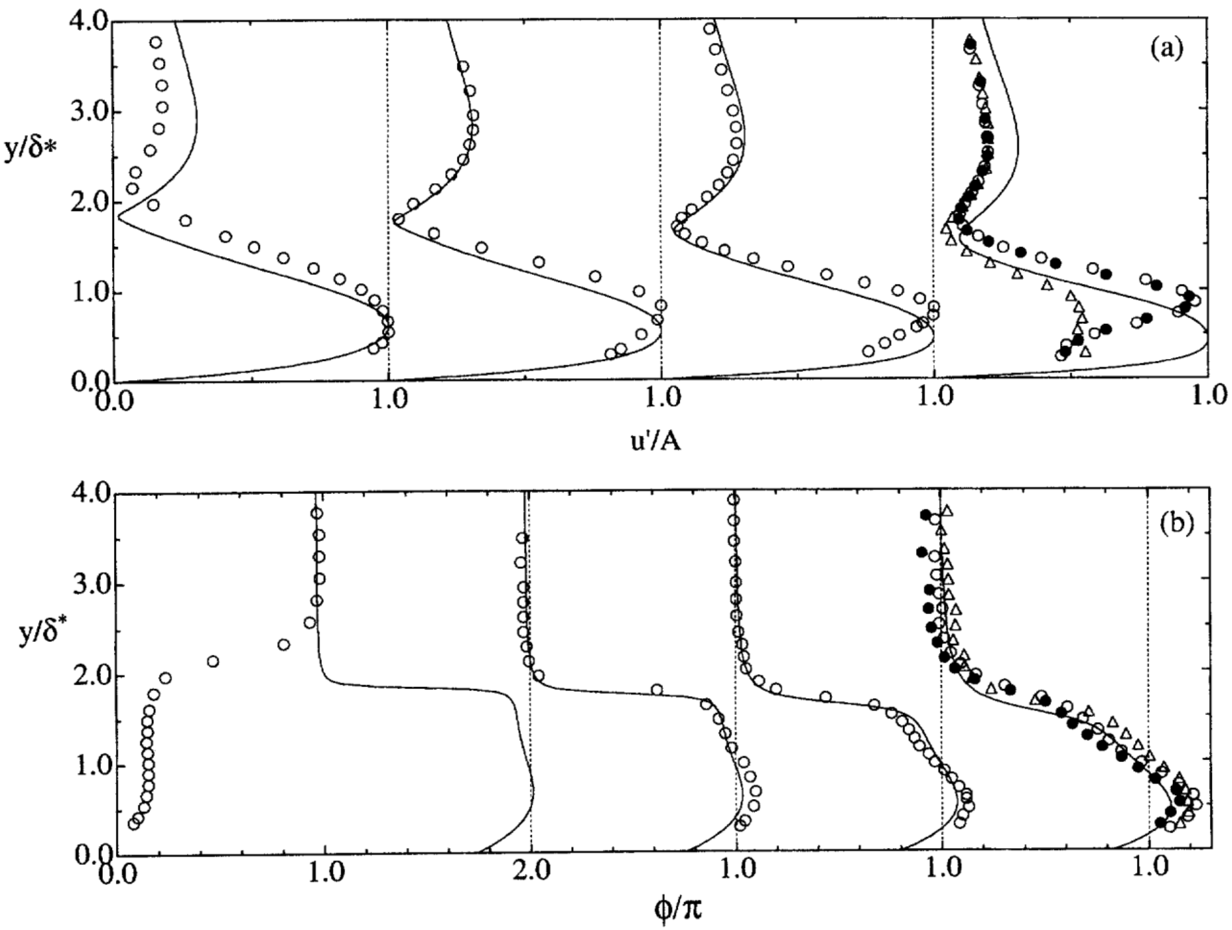
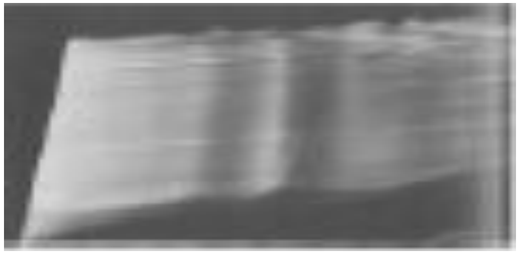


Fig. 2. – Amplitude (a) and phase (b) profiles of the generated TS-wave in experiment A (F = 455). The x-positions are from left to right x=100, 125, 160 and 200 mm ($Re_{\delta *}$=370, 405, 460 and 520). Labels: z=0 mm (○); z=-2.75 mm (●); z=5.25 mm (△); Linear PSE-calculations (—). Note that each amplitude profile at the last x-position is normalized to a value of 0.2 at the outer maximum.

Very well-controlled experimental conditions



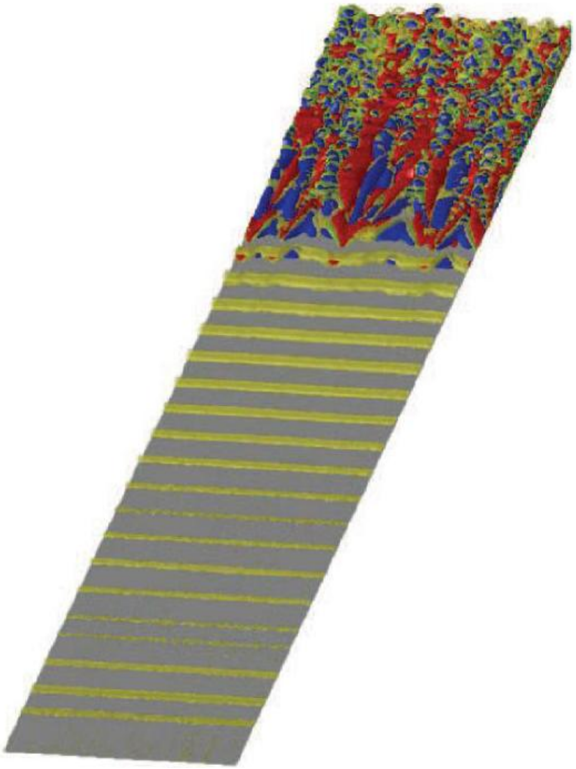
Bakchinov et al., 1998 (very low free stream Tu)

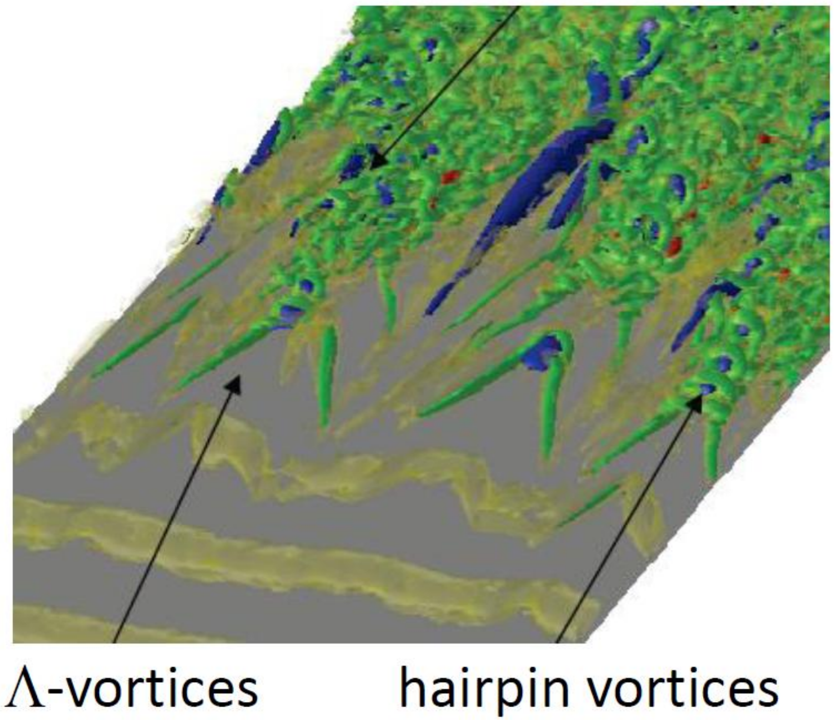
Numerical results (CFD)

- 2D TS waves

<u>**SUPERCRITICAL TRANSITION**</u>

(for 'small' disturbance levels)



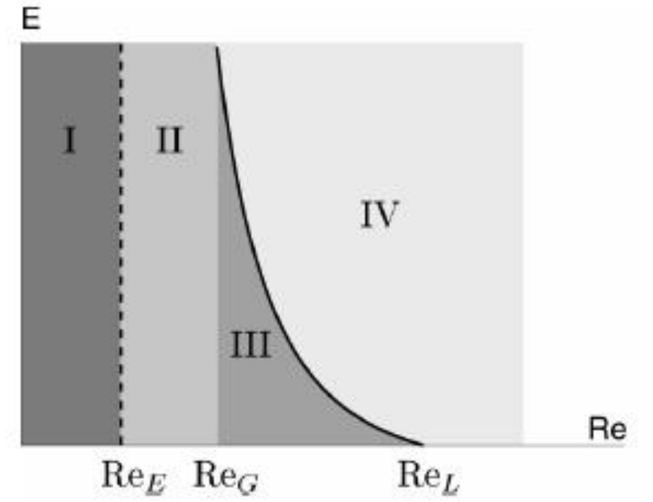


Philipp Schlatter, 2009

Most experiments disagree …

In reality, there is large disagreement between different experimental installations and theory, for all shear flows ...

	Poiseuille	Couette	Blasius
Re_L	5772	∞	519
Re_{trans}	~2000	~420	~400

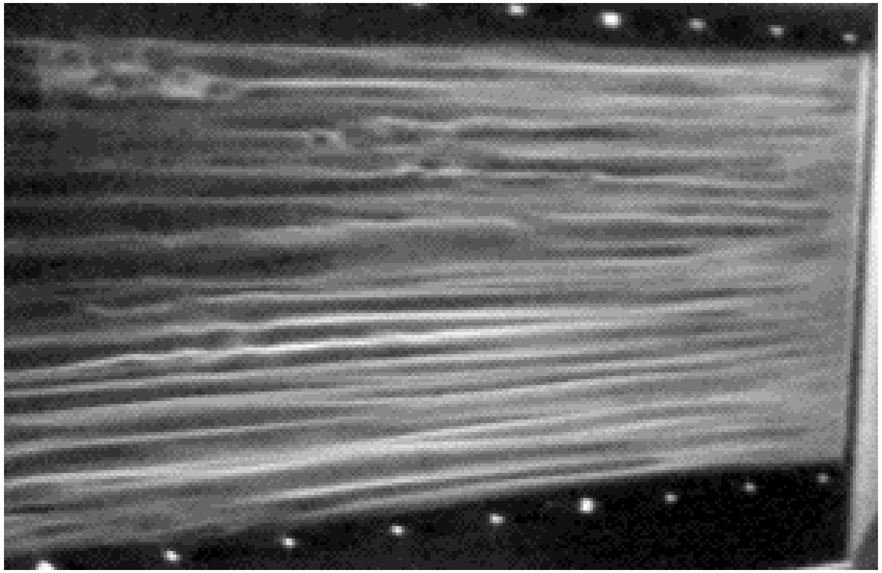


A strong dependence on initial conditions (exogeneous disturbances) is present.

Streaks

… and for large environmental disturbances TS waves are overruled by streaky structures, which dominate the transition process.

Hence, it is crucial to address the receptivity phase.



Alfredsson & Matsubara, 1996

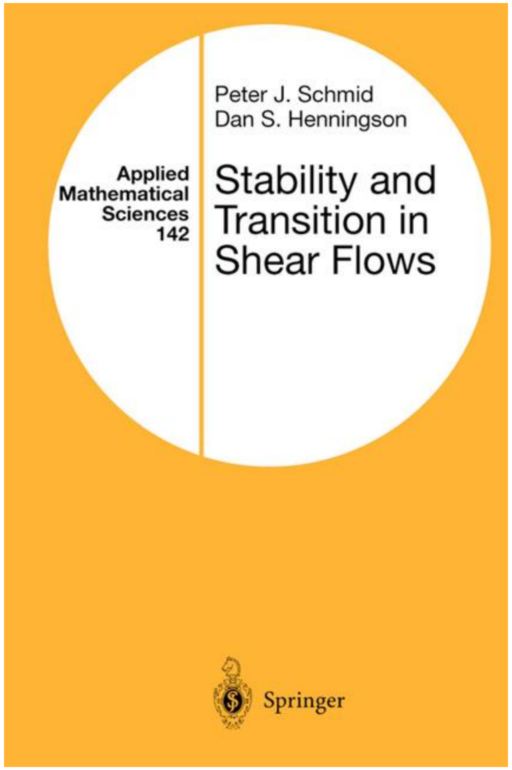
U_∞ = 2 m/s
free stream Tu = 6%

How can we describe the streaks?

Modern theories (1990s) say: "**forget the asymptotic, long-time growth of *modal* (classical) stability analysis and focus on the short time transient behaviour even in nominally subcritical ($Re < Re_L$) conditions!**"

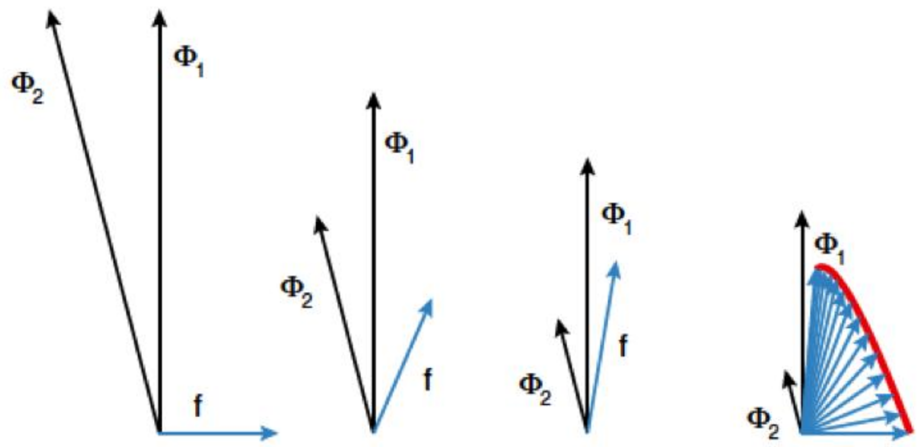
New theories take several names: *transient growth theory, optimal perturbations, nonnormal analysis, pseudospectra, etc.*

4. Something does not work … back to square one! Transient growth and the “optimal perturbations”



Eigenvectors of modal theories are not orthogonal!

→ transient (short-time) amplification is possible!



Superposition of decaying non-orthogonal eigenmodes Φ_1 and Φ_2

How do we recover the most dangerous dynamics over short time scales?

Optimal perturbations

To find the **most destabilizing perturbations** in subcritical conditions we can resort to a *constrained optimization* analysis based on adjoint equations. The advantages of this approach are that:

- No problems with the continuous spectrum (since we do not consider a generic disturbance as an eigenvector expansion)
- Can use discrete adjoint (transposing the direct equations in discrete form), avoinding lengthy derivations of the adjoint continuous equations
- Can extend to nonlinear regime

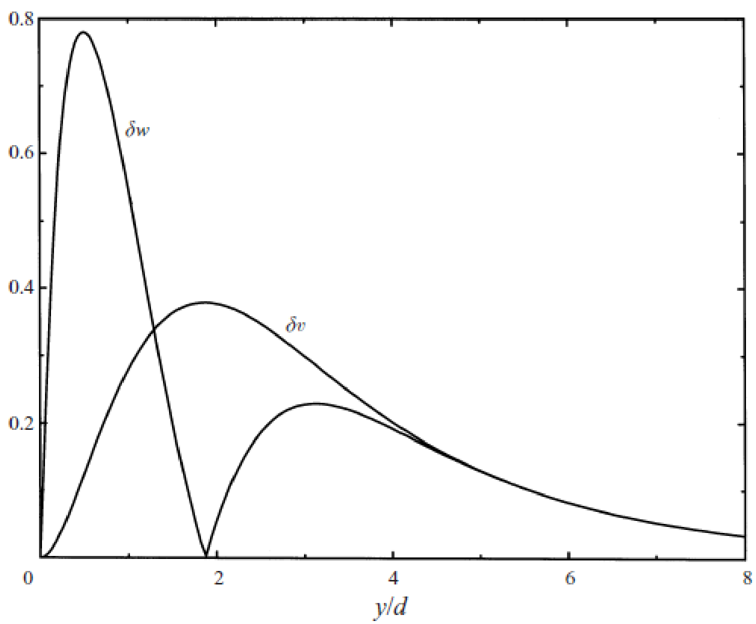
Sketch of the adjoint approach

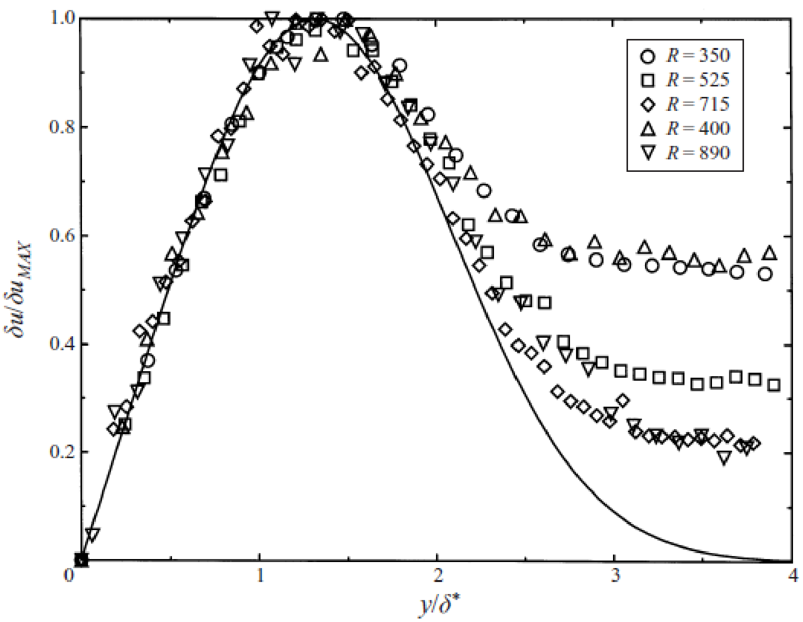
DIRECT EQUATIONS

$$\begin{array}{ccc} \mathbf{q}(t=0) & \xrightarrow{\quad q_t = Lq \quad} & \mathbf{q}(t=T) \\ \uparrow & & \downarrow \\ \mathbf{a}(t=T) & \xleftarrow{\quad -a_t = L^\dagger a \quad} & \mathbf{a}(t=0) \end{array}$$

ADJOINT EQUATIONS

Linear optimals (at the leading edge)





Optimal input: **vortex**

Ensuing output: **streak**

Luchini, 2000

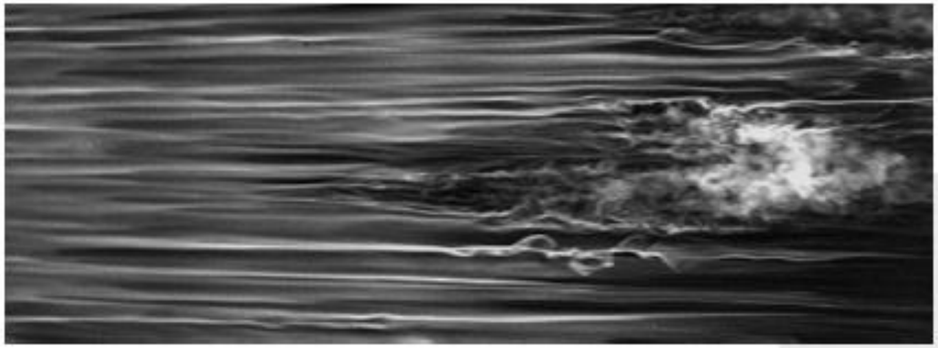
PROBLEM SOLVED???

PROBLEM SOLVED???

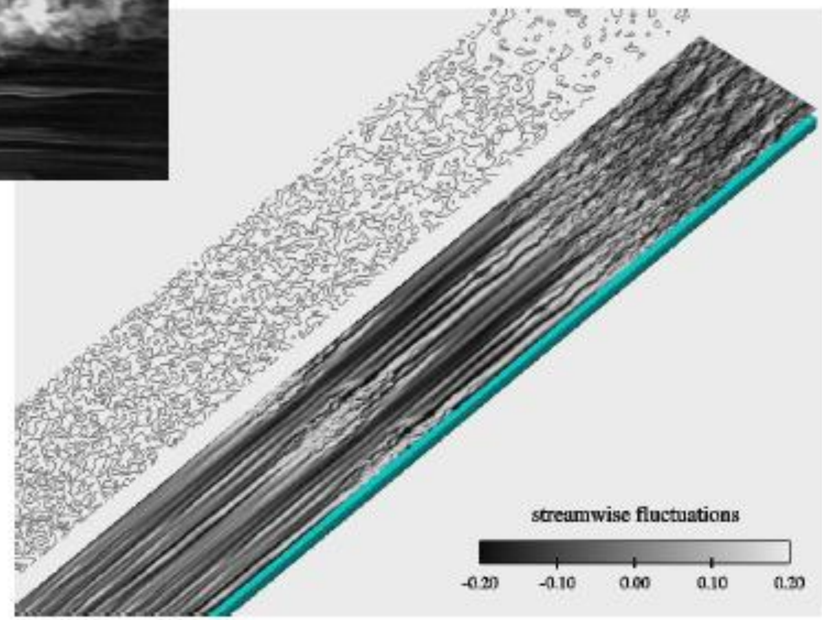
Low Tu: 2D TS waves, spanwise oscillations, Λ vortices, breakdown …

High Tu: linear streaks, elongated in x ($\alpha = 0$), nonlinear amplification, secondary wavy instability of the streaks, turbulent spots …

- Emmons (1951) spots, induced by free-stream turbulence



Matsubara & Alfredsson, 2005



SUBCRITICAL (BYPASS) TRANSITION
(for 'large' Tu disturbance levels)

Zaki & Durbin, 2005

Limitations of linear approach

HOWEVER:

Even when we let **small-amplitude input streak**s evolve nonlinearly with a DNS, they still need a **very large amplitude** before they undergo a secondary instability, much larger than that observed experimentally. Breakdown to turbulence is not the same as observed in experiments …

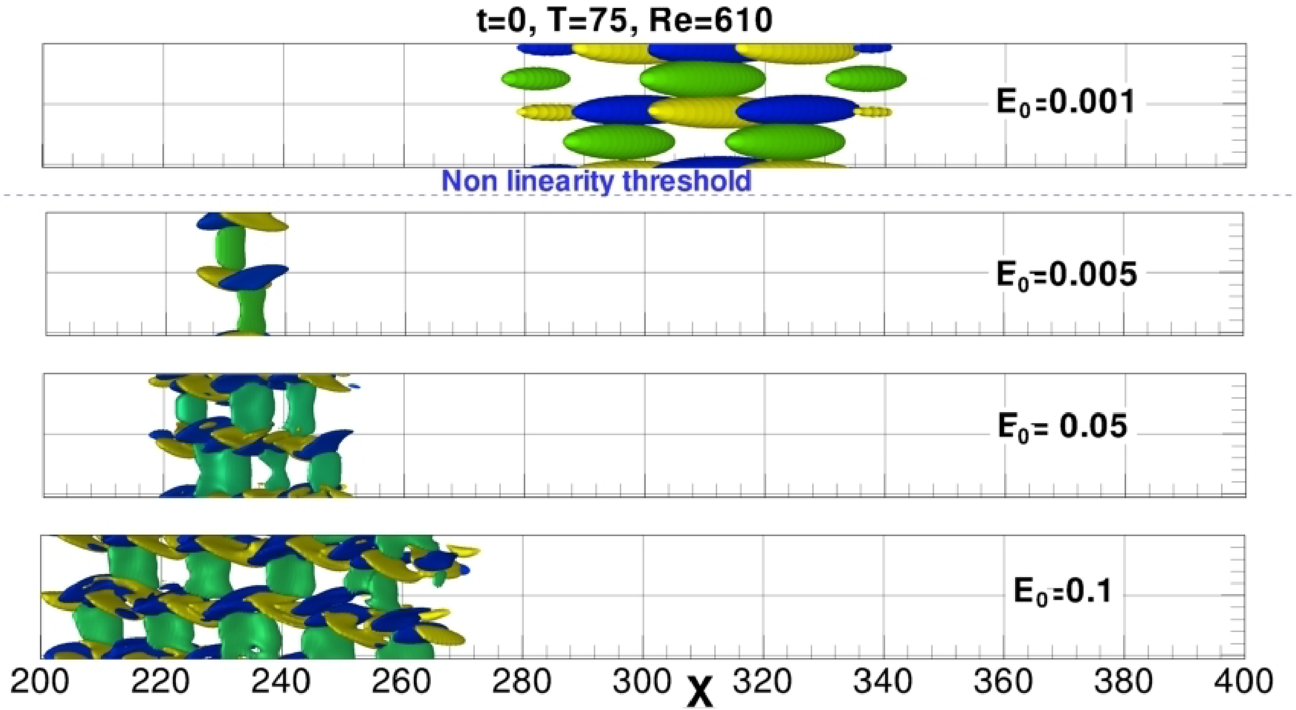
Andersson et al. 2001

Linear optimal disturbances do not tell the whole story!

The point is:

α = 0 streaks are not good at triggering transition

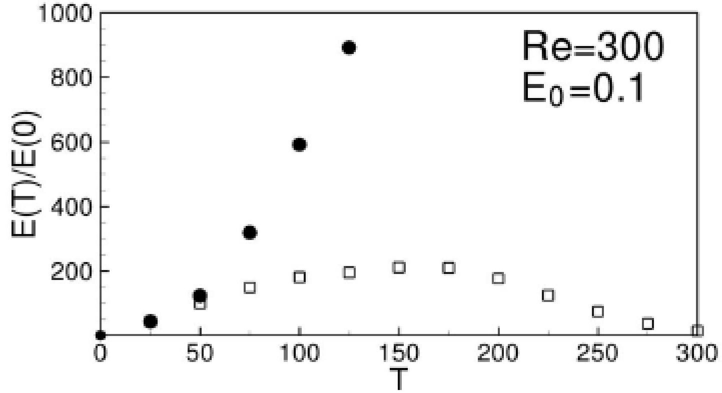
5. Still having problems: nonlinear transients …

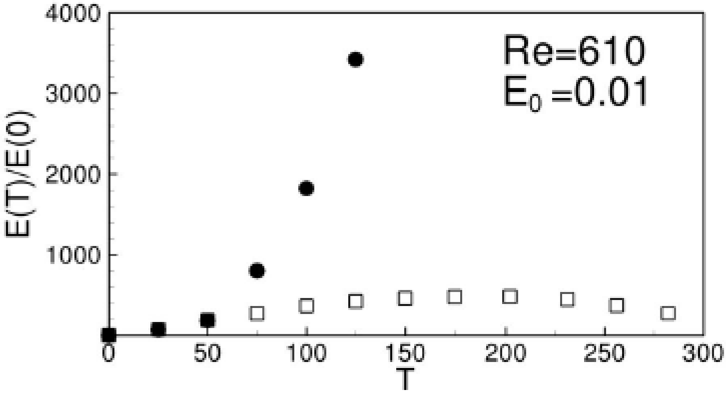


Linear *versus* nonlinear

Apply direct-adjoint optimization technique to identify **localized** **nonlinear** optimals, **not** infinitely elongated along the streamwise direction x ($\alpha \neq 0$).

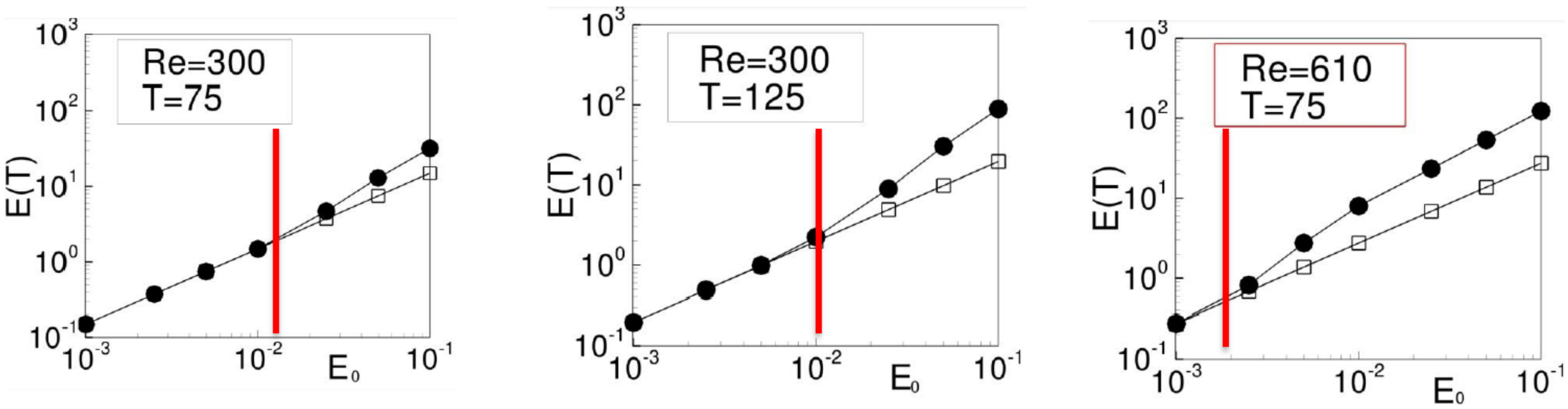
Cherubini et al. 2010, 2011





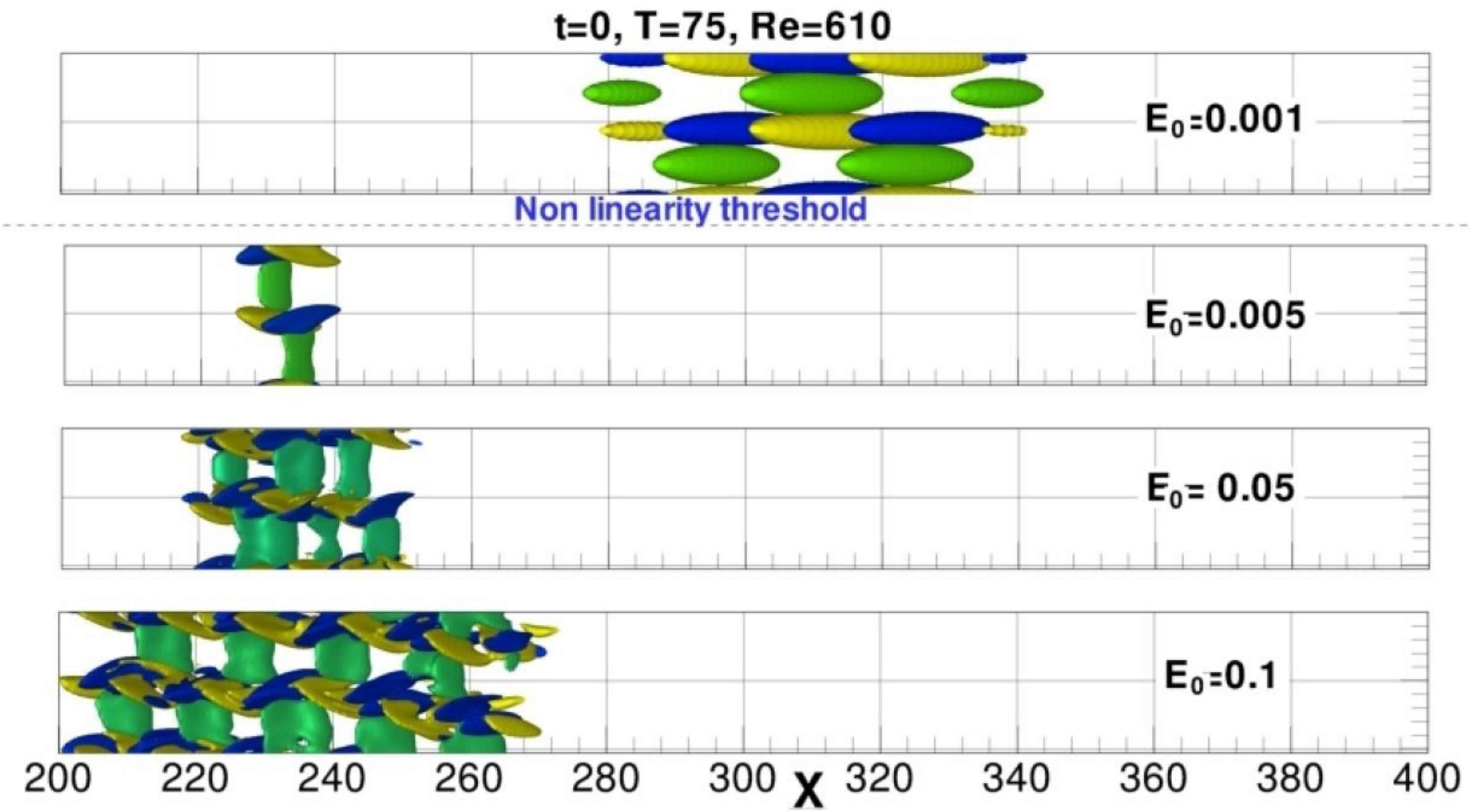
For target time T sufficiently large nonlinear optimals produce much larger gains

Linear *versus* nonlinear



For given Re and T, a *threshold* on E_0 exists above which nonlinear effects become important

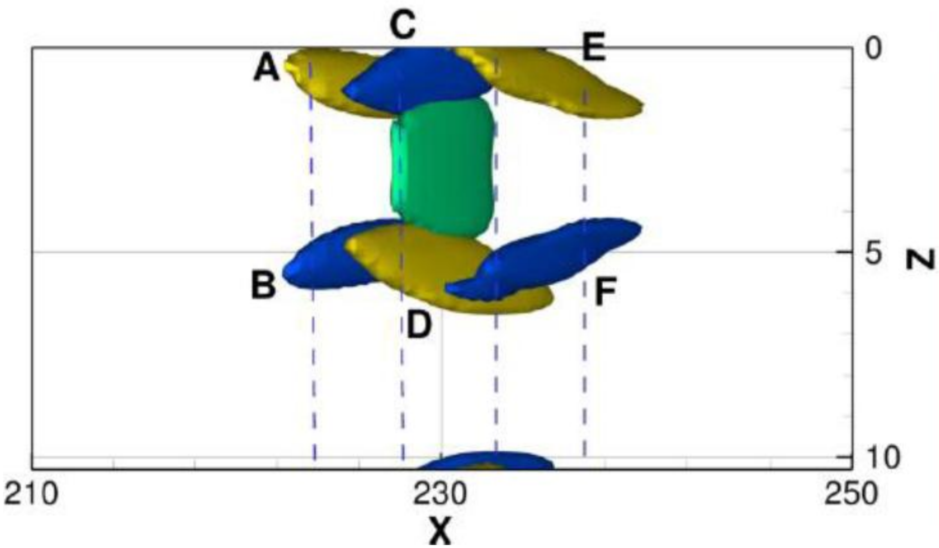
Dependence of nonlinear optimals on E_0



Above the *threshold* the same basic building block reappears ...

The *minimal seed*

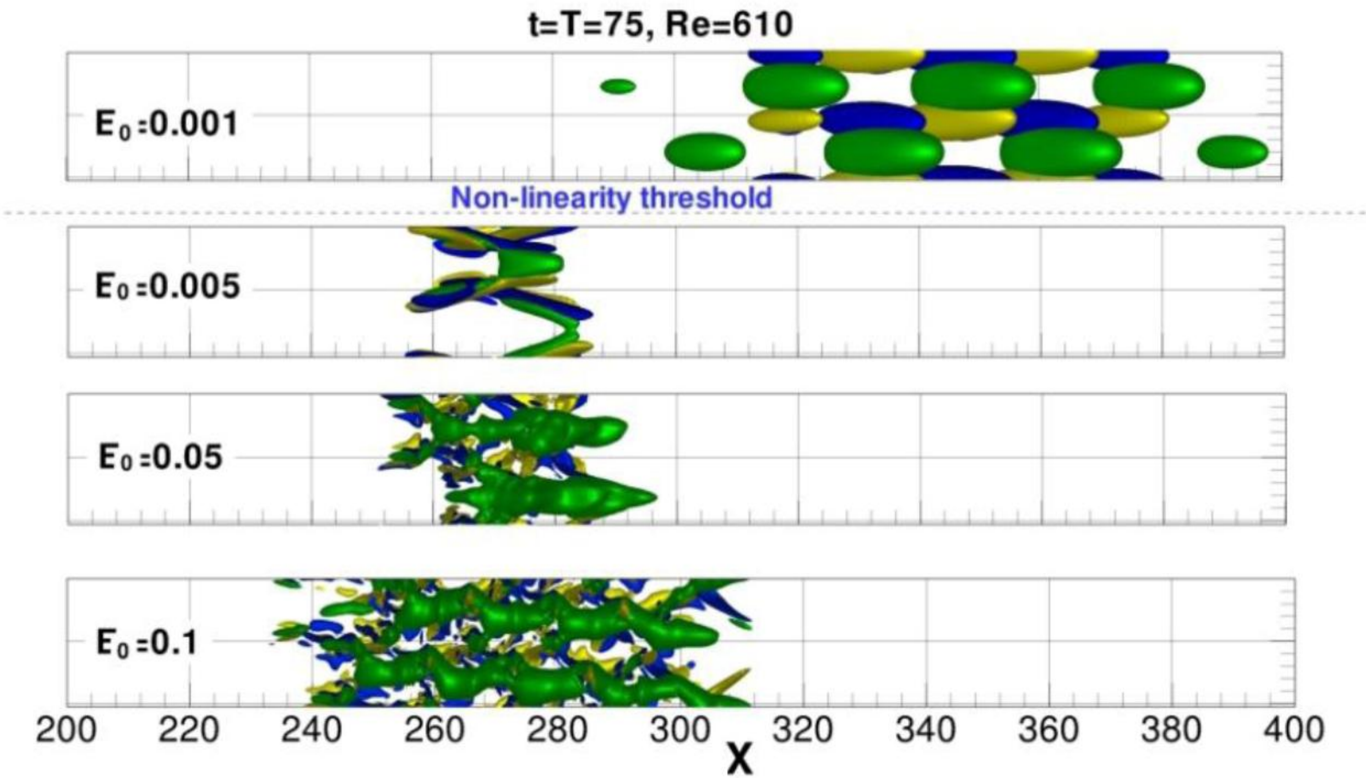
Optimal initial perturbation at $T = 75$, $E_0 = 0.01$ and $Re = 610 \rightarrow$ *alternated vortices inclined in x and tilted upstream (yellow and blue), which lay on the flanks of a region of high negative streamwise disturbance (green).*



Large differences w.r.t. the linear optimal:

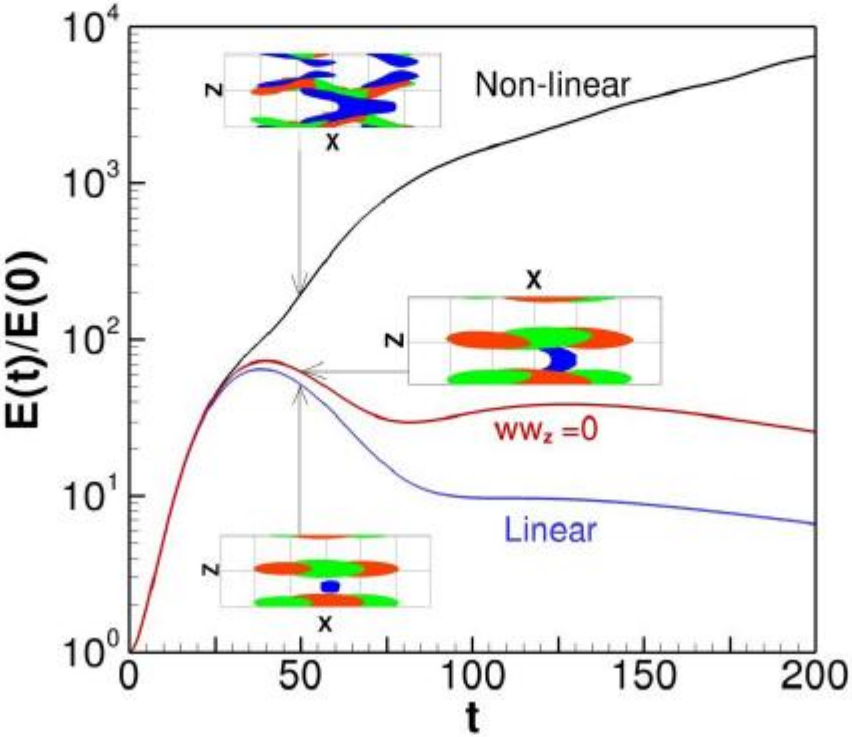
- it is localized in x and z
- vortices are streamwise-inclined
- u' is the largest component ($|u'_{max}| = 0.018$)
- regions with high negative u' are associated with high positive v'

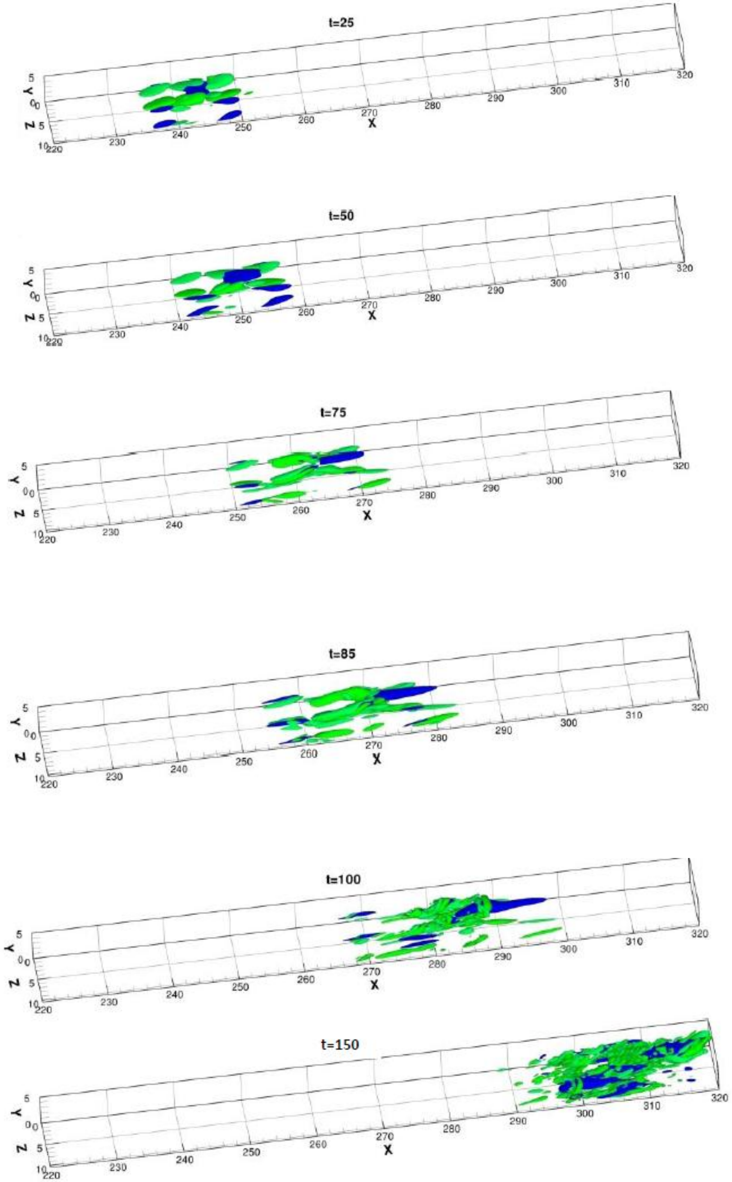
What happens at the target time T?



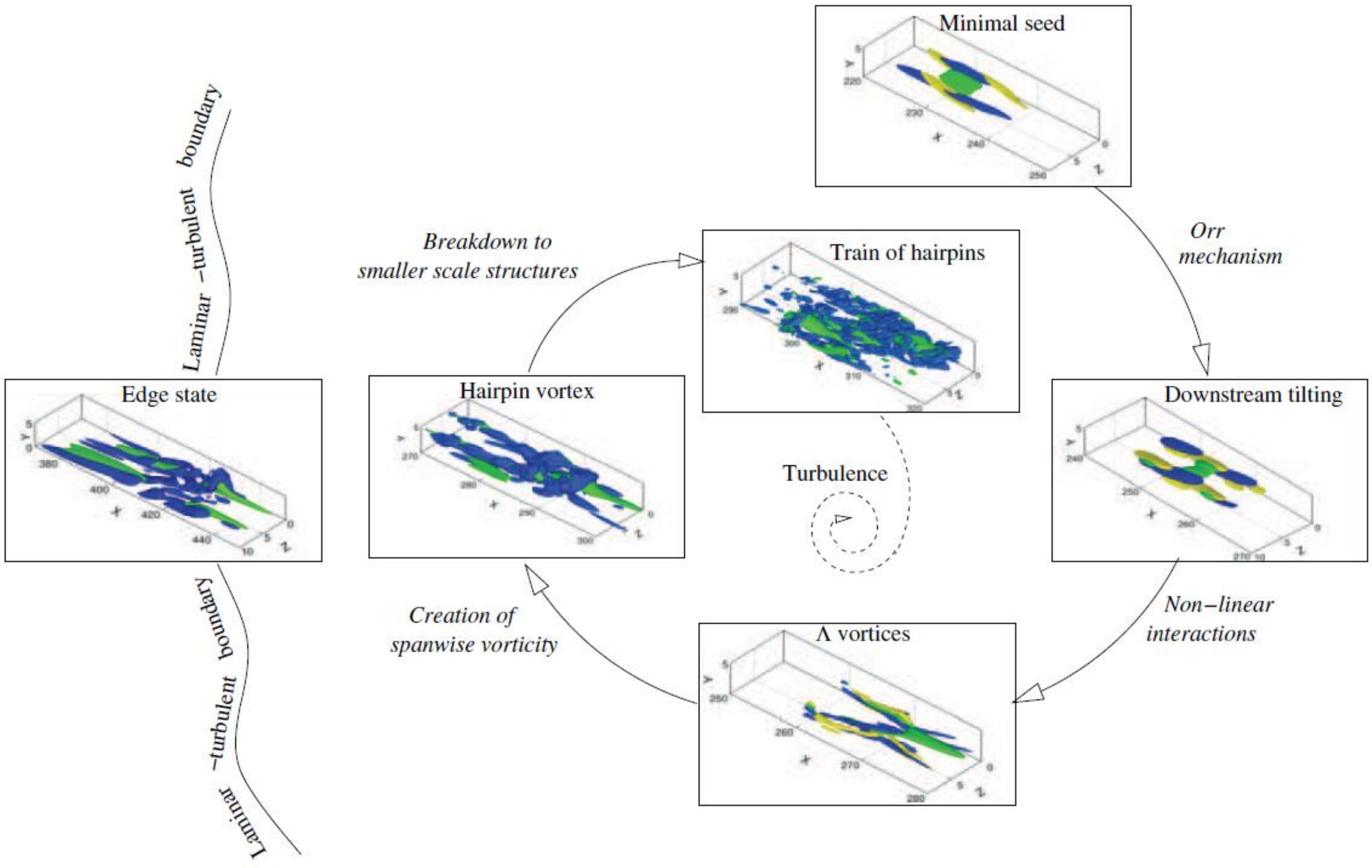
Beyond the non-linearity *threshold* Λ-vortices appear; their interactions lead the flow to turbulence when several *minimal seeds* are present in the initial field

Path to turbulence of the minimal seed

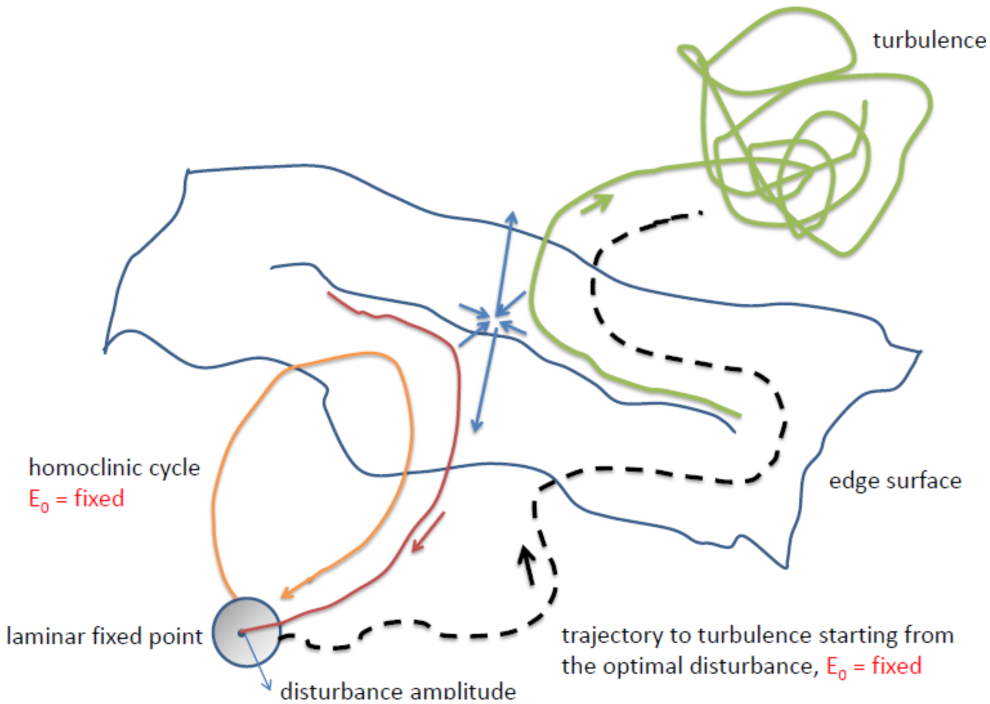




The disturbance regeneration cycle



6. And if we reversed the problem? Using chaos theory …



Henri Poincaré (1854-1912)

Recurrent patterns

Current wisdom holds that a “small” set of *recurrent patterns* are sufficient to develop a predictive tool for non-equilibrium turbulent flows.

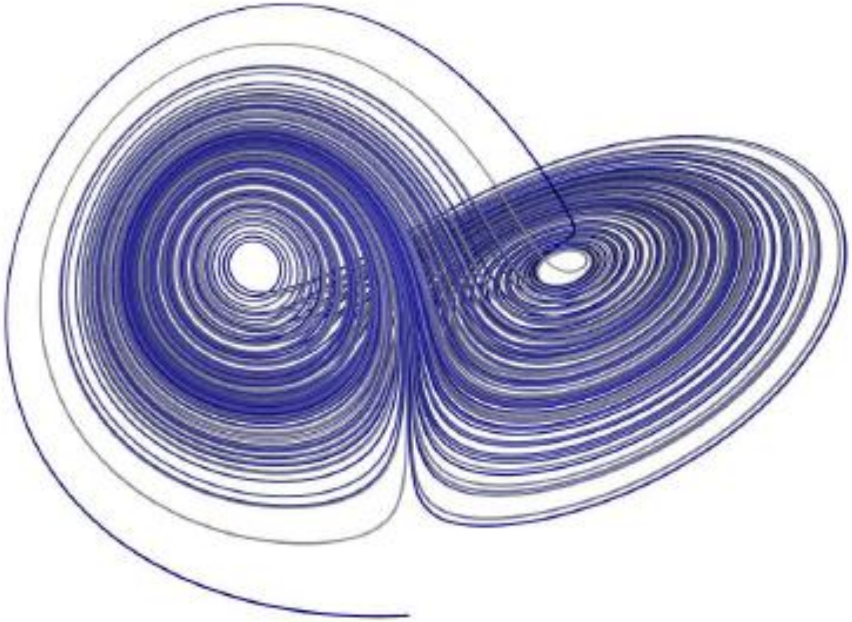
This idea has roots in the *prehistory* of chaos theory!

Lorenz attractor
(*J. Atmos. Sci.* 1963)

No steady states
No limit cycles
Sensitive dependence on IC

↓

Local unpredictability



Hopf theory of chaos

If turbulence can be interpreted as the wandering of the flow system's trajectory in phase space among mutually repelling states (Cvitanović refers to this as *Hopf theory of chaos*) it may be possible to

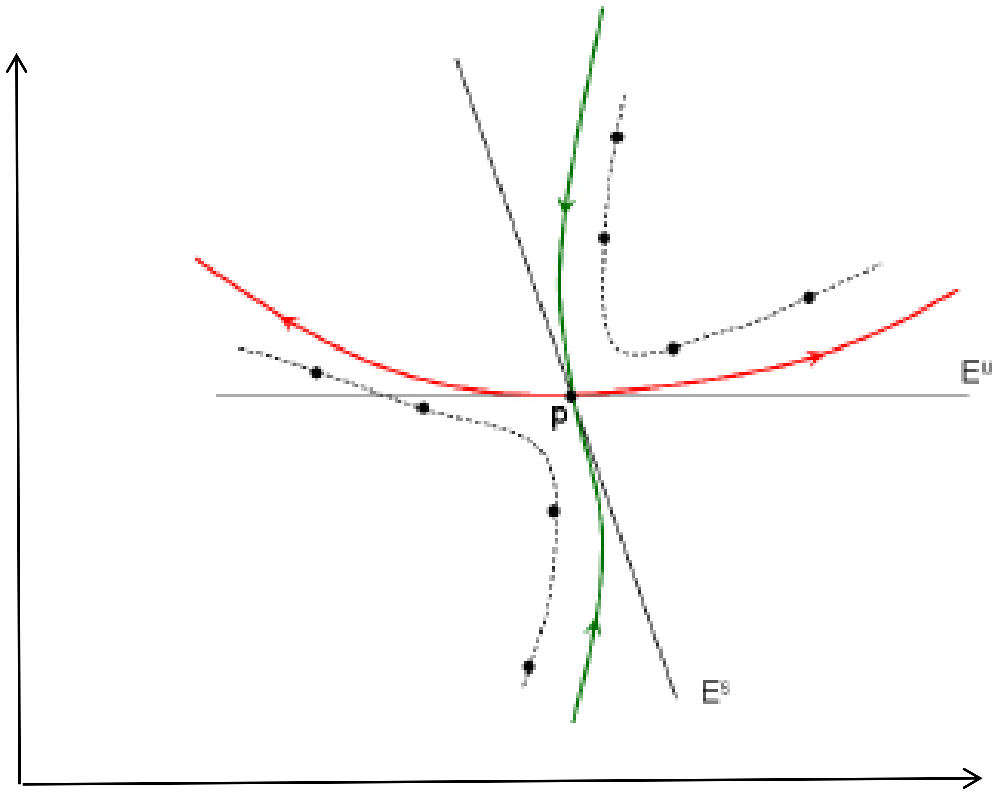
1. identify the set of *recurrent patterns* pertinent to each flow configuration and Reynolds number, &
2. Compute sensible global averages (→ global predictability) possibly retaining only the more *meaningful* patterns (i.e. the least unstable ones?)

Both tasks are difficult …

(Lan & Cvitanović, ***Phys. Rev E*** 2003, had some success with the 1D Kuramoto-Sivashinsky equation)

Heinz Hopf (1894-1971)

Repellors = saddle points



In some *relevant* phase (hyper-)space …

Continuation technique

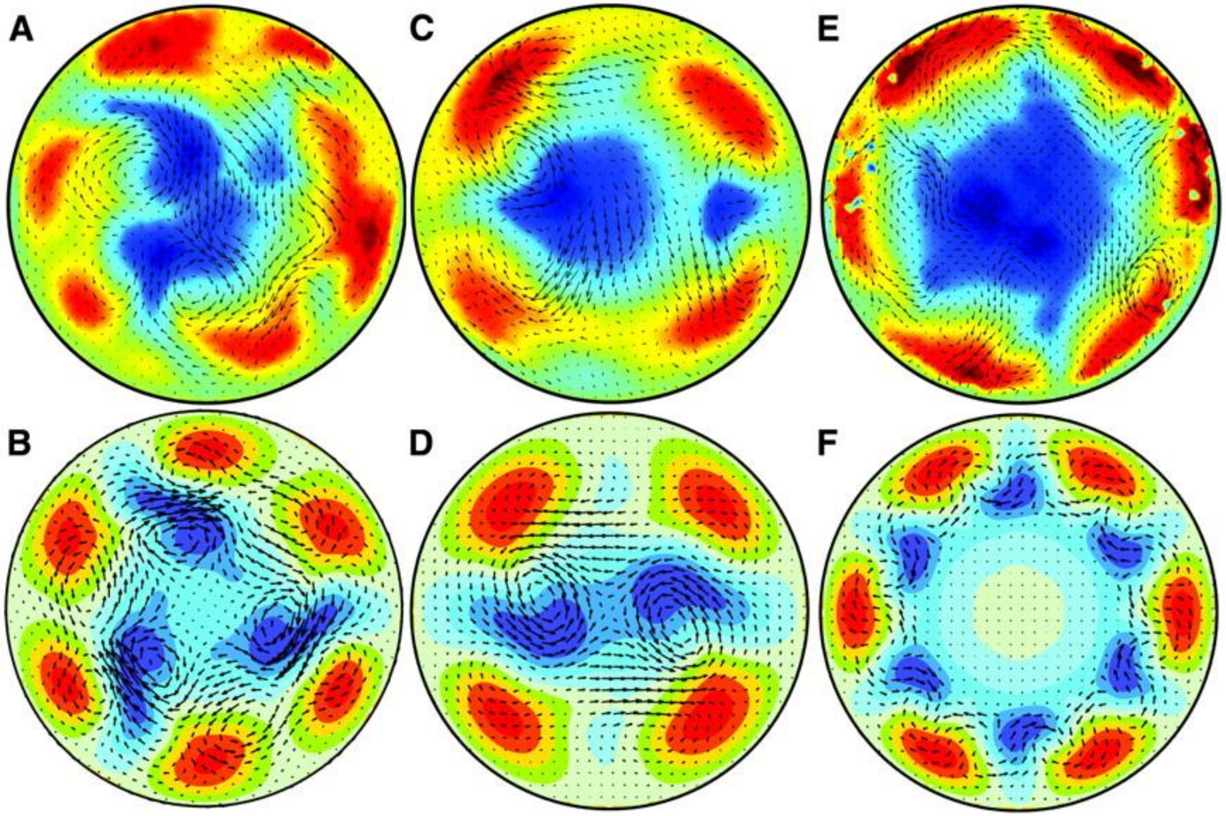
Looking for **unstable** **TW** solutions

Why???

Continuation technique

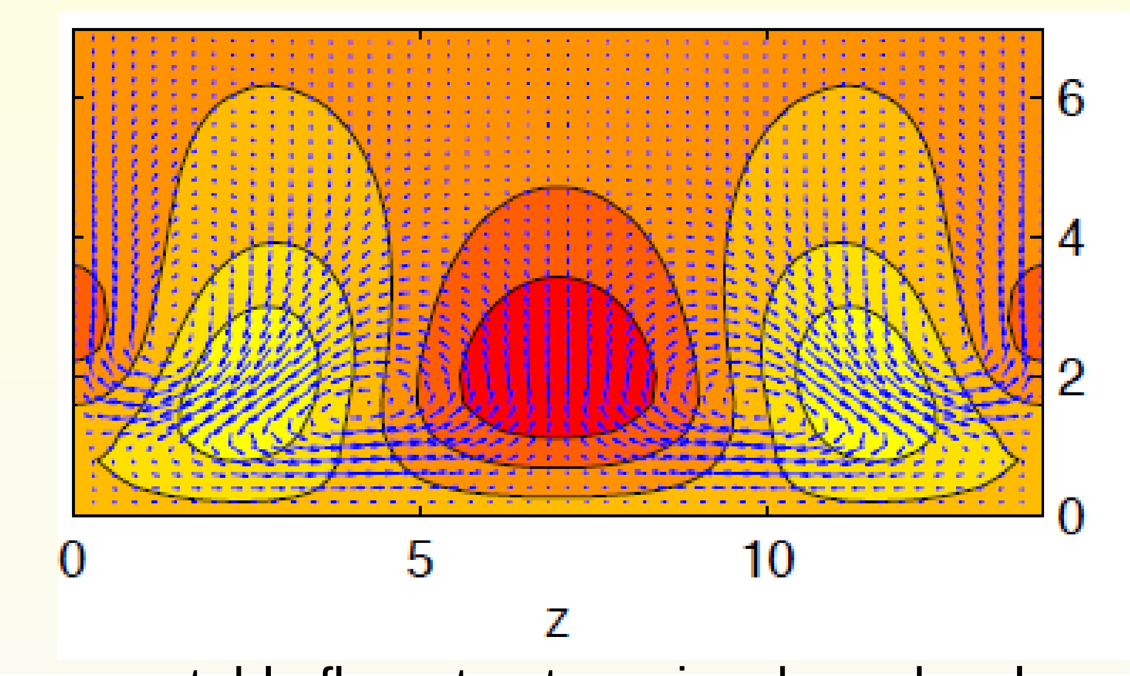
Looking for **unstable** TW solutions

Why???



Hof et al. 2004

Unstable structures in the boundary layer

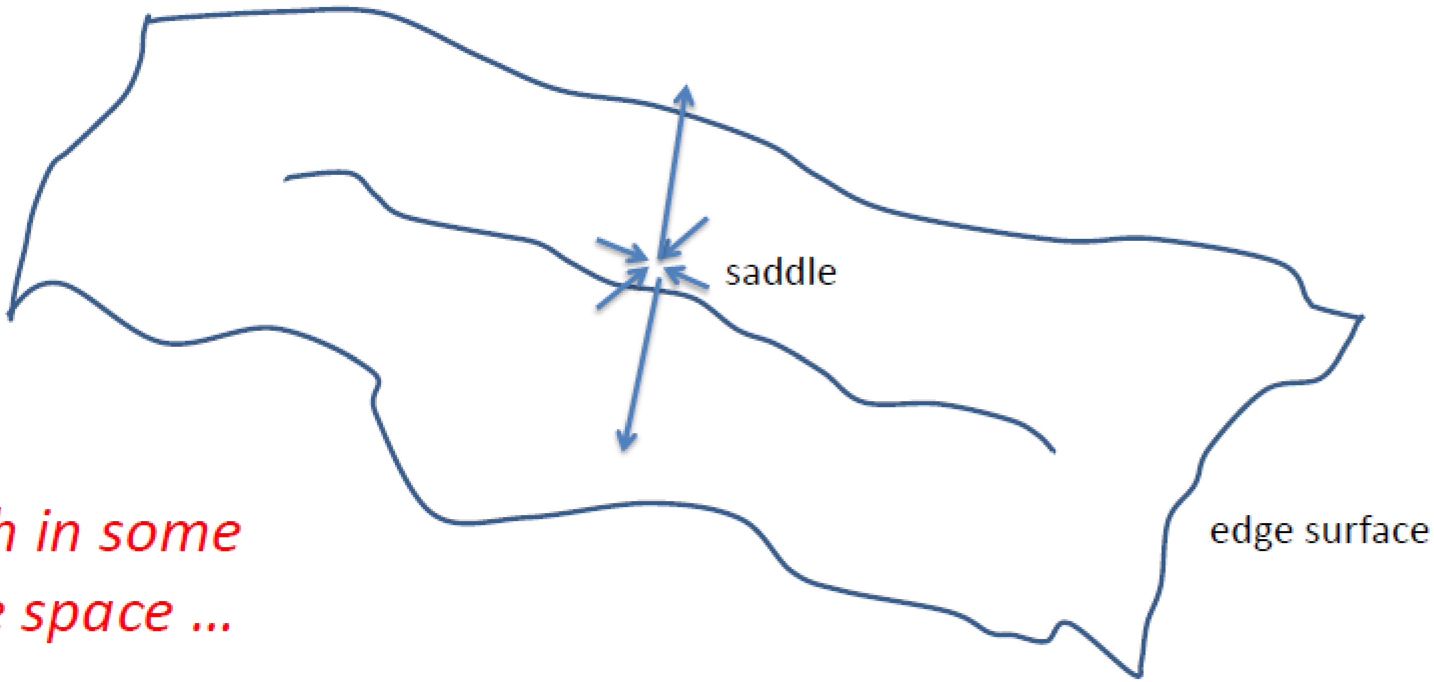


Typical nonlinear, unstable flow structures in a boundary layer are **TW** which, in the cross-flow plane, are constituted by two pairs of spanwise periodic vortices

Wedin et al. 2013

Chaos and the *edge* state

What about ECS, saddles, edge states, etc.?

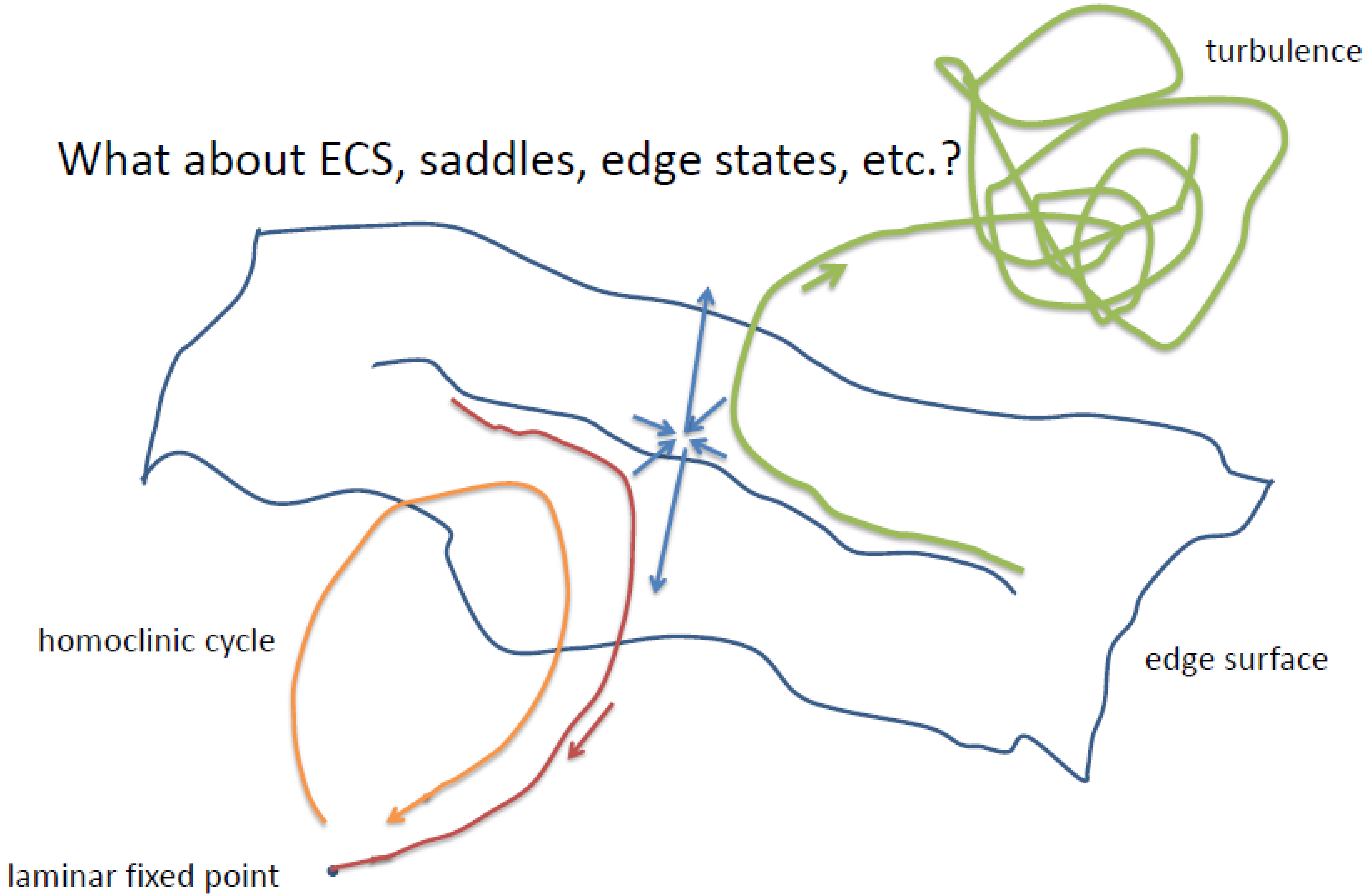


Sketch in some phase space …

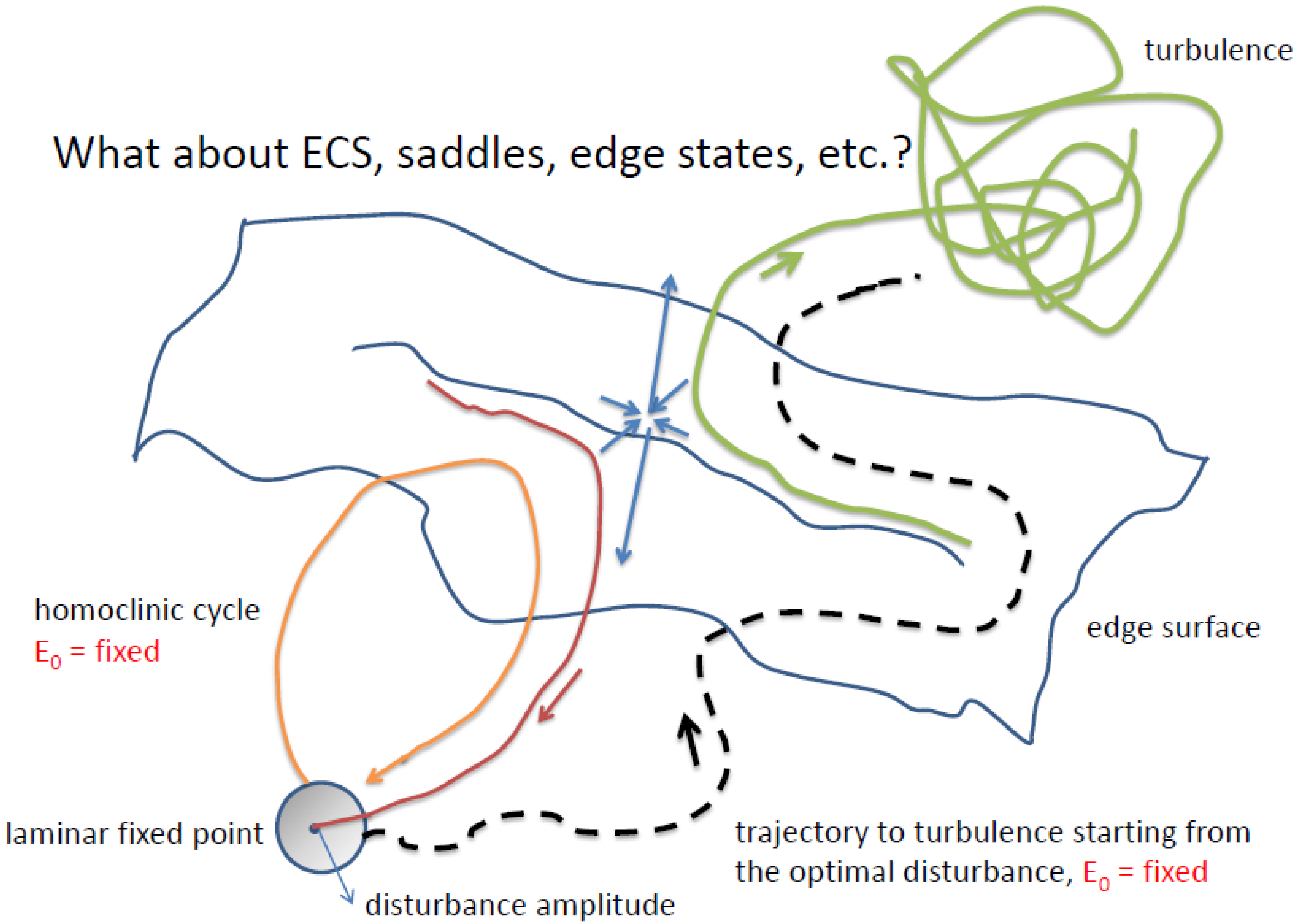
laminar fixed point •

Chaos and the *edge* state

What about ECS, saddles, edge states, etc.?



Chaos and the *edge* state



The laminar-turbulent (*edge*) boundary

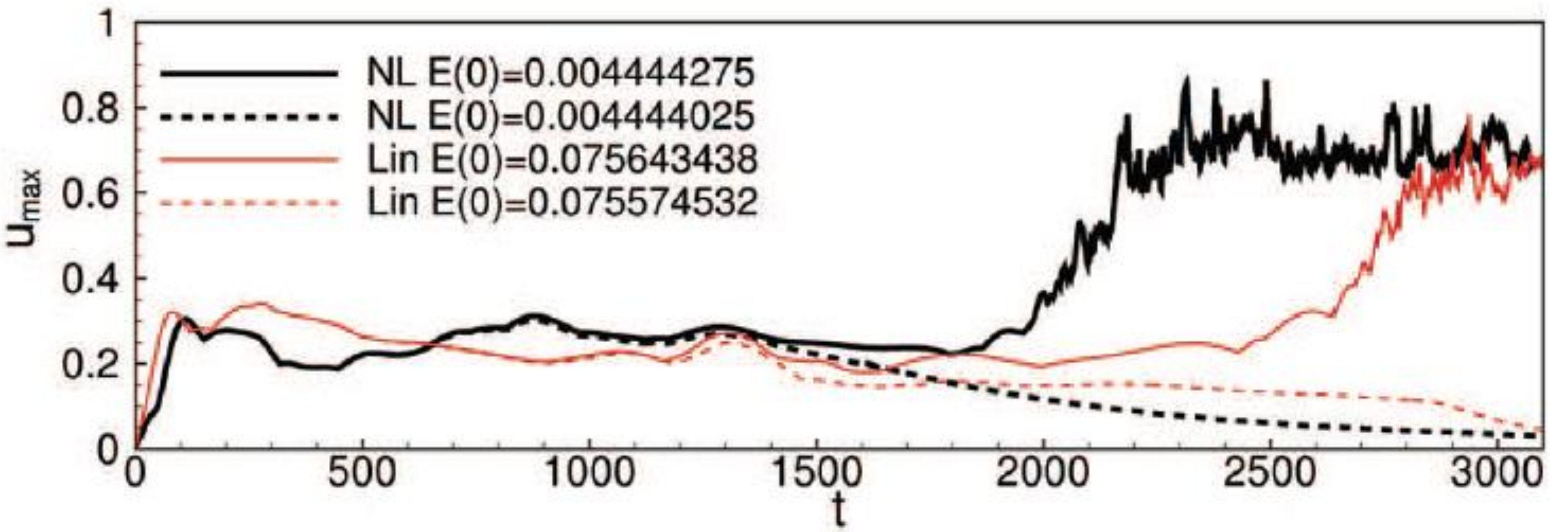


FIG. 1. (Color online) Streamwise disturbance velocity peaks versus time for DNSs initialized by the linear (thin gray lines, red online) and the nonlinear optimal perturbation (black thick lines) for different values of the initial energy.

Cherubini et al. 2011

The laminar-turbulent (*edge*) boundary

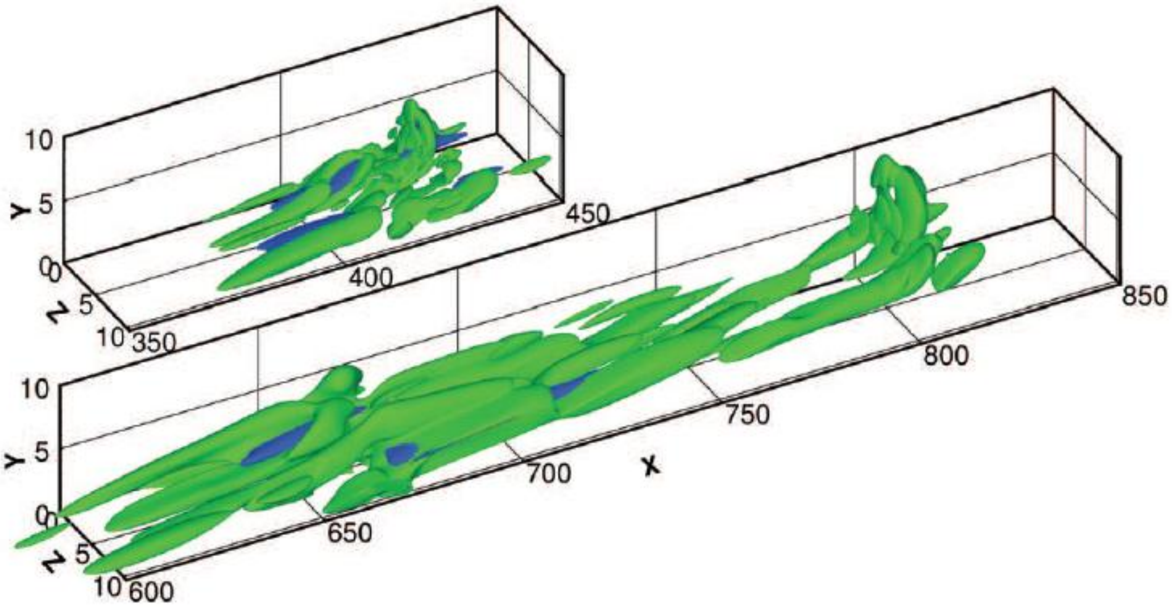


FIG. 2. (Color online) Snapshots of the streamwise component of the perturbation (darker surfaces, blue online, for $u = -0.13$) and of the Q-criterion (lighter surfaces, green online) at $t = 300$ and $t = 700$ (top and bottom, respectively) obtained by the DNS initialized with the nonlinear optimal perturbation with $E_0 = 0.004444275$.

Cherubini et al. 2011

Current research goal

Attack transition to turbulence in shear flows from two sides:

the laminar side

looking at how disturbances to some organized/laminar base state disrupt it

and the turbulent side

progressively reducing the amplitude of initial disturbances in a shear flow until the state sits – for as long as possible – onto an unstable, laminar-turbulent (*edge*) boundary.

A few suggested readings

Corbett P. & Bottaro A., "Optimal perturbations for boundary layers subject to stream-wise pressure gradient", *Phys. Fluids*, Vol. 12, 2000, pp.120-130

Biau D. & Bottaro, A., "Optimal perturbations and minimal defects: Initial paths of transition to turbulence in plane shear flows", *Phys. Fluids*, Vol. 16, 2004, pp. 3515-3529

Zuccher S., Bottaro A. & Luchini P., "Algebraic growth in a Blasius boundary layer: Nonlinear optimal disturbances", *Eur. J. Mech. B/Fluids*, Vol. 25, 2006, pp. 1-17

Biau D. & Bottaro A., "An optimal path to transition in a duct", *Phil. Transact. Royal Soc.*, Vol. 367, 2009, pp. 529-544

Cherubini S., De Palma P., Robinet J.-Ch. & Bottaro A. , "The *minimal seed* of turbulent transition in the boundary layer", *J. Fluid Mech.*, Vol. 689, 2011, pp. 221-253

Cherubini S., De Palma P., Robinet J.-Ch. & Bottaro A., "Edge states in a boundary layer", *Phys. Fluids*, Vol. 23, 2011, 051705

All papers are available at: http://www.dicca.unige.it/bottaro/papers.html