

Crash introduction to the lifting-line theory

LLT working example

Crash introduction to the lifting-line theory

- One way to predict the lift distribution over a three-dimensional wing is by using the LLT method.
- By using this method, we can easily compute the lift coefficient, induced drag, induced angle, induced drag factor (and Oswald span efficiency), slope of the lift curve, and lift distribution of the wing.
- To do so, we must know,
 - The geometrical parameters of the wing (aspect ratio, taper ratio, wingspan, geometrical/aerodynamic twist, and so on).
 - The aerodynamic characteristics of the wing cross-section (airfoil), which we can compute using XFOIL.
 - We will need to compute the following aerodynamic information of the cross-sections:
 - Slope of the lift curve.
 - Zero-lift AOA.
 - If you are using control surfaces or high lift devices, we should know the respective aerodynamic performance of the modified airfoil.
 - The Reynolds number of each cross section, as it is advisable to use the aerodynamic quantities for the Reynolds number that each cross-section effectively sees.

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- In few words, the LLT method consist in:
 - Sub-dividing the wing in several sections where we know the aerodynamic characteristics of the cross-sections in function of the Reynolds number.
 - These aerodynamic characteristics are linearly interpolated from section to section.
 - The aerodynamic characteristics of the cross-sections are corrected by adding the induced angle.
 - Using the coefficients obtained using the LLT equations, we can compute lift coefficient, induced drag coefficient, and lift distribution.
- Despite the considerable simplifications taken in this theory, it predicts with reasonably good accuracy the aerodynamic characteristics of wings in the linear regime.
- The LLT method is a very handy tool for designing the wing during the preliminary or conceptual phase.

Crash introduction to the lifting-line theory

- Before using the LLT method, we should be aware of its limitations (classical formulation*):
 - It is a linear theory, that is, does not predict stall.
 - Does not predict viscous effects.
 - It is not accurate when used with wings with small aspect ratios (approximately less than 2) and sweep angle of about $\Lambda_{c/4} > 15^\circ$.
 - Does not take into account dihedral/anedral angle effects nor non-planar lifting surfaces (winglets and wingtip devices).
 - It is formulated for a single lifting surface.
 - Cannot be used above the drag divergence Mach number.
- However, over the years, the LLT method has been revised so it can address all the previous limitations.
- A final remark, the LLT method is not only limited to the aeronautical field.
 - It has been used with relatively good success in car aerodynamics, naval engineering, and wind energy generation (among many fields).

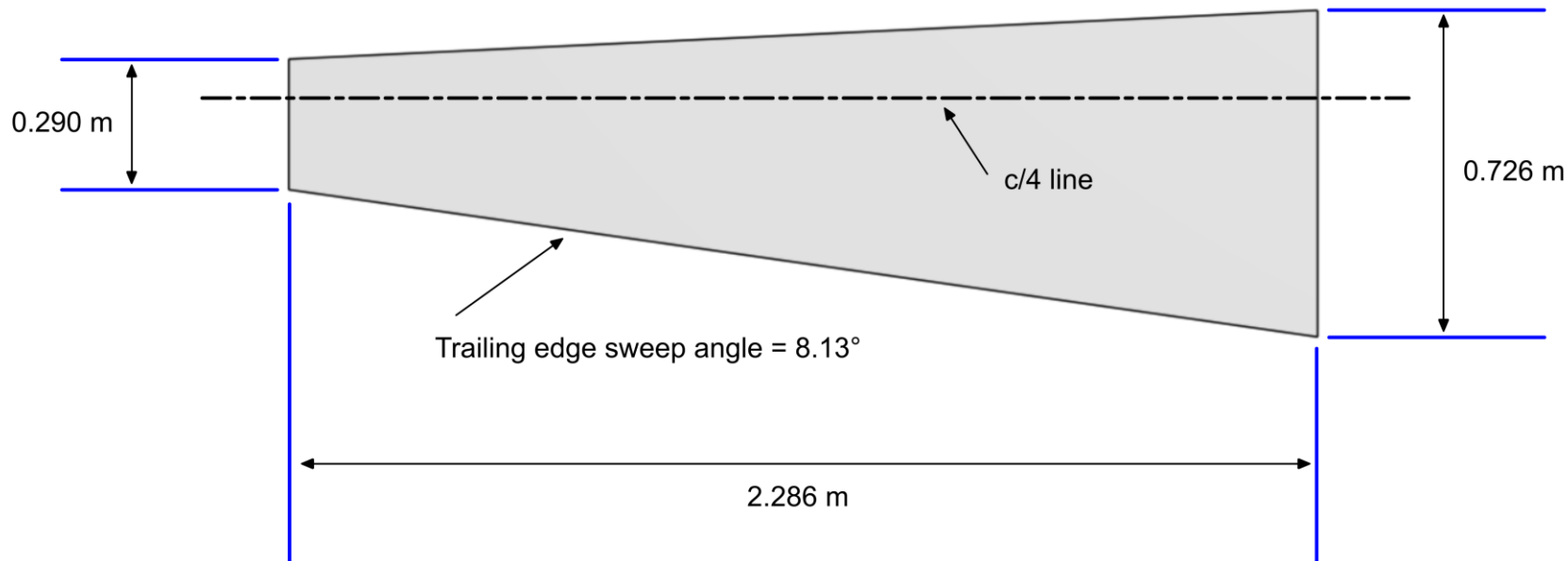
* L. Prandtl. Applications of modern hydrodynamics to aeronautics. NACA Report 116, 1921.

Crash introduction to the lifting-line theory

- More advanced/refined variants of the LLT method have been formulated to address:
 - Wing stall and viscous effects prediction (non-linear LLT).
 - Wings with sweep angle and low aspect ratios.
 - Non-planar lifting surfaces and wings dihedral/anedral angles.
 - Multiple lifting surfaces (configurations with wings and horizontal stabilizer).
 - Unsteady behavior (flapping wings, variable sweep angle, control surfaces actuation).
 - Wing-fuselage interaction.
 - Aeroelasticity and rotordynamics.
 - Compressibility effects.
- Variations of the LLT that take into account non-linear effects and arbitrary wing geometries:
 - Multhopp, Sivells-Neely, Jones, Weissinger, Schrenk, Tani, Philips-Snyder, Reissner (among many)
- Hereafter, we will work with the basic LLT formulation, that is, the linear formulation.

LLT working example

- LLT step-by-step working example. Hereafter, you are asked to:
 - Compute C_L and $C_{D_{ind}}$ of the wing illustrated below.
 - If the wing setting angle is 2° , and the airplane weights 4000N, and is flying at sea level, at what velocity it should fly to generate the required lift force?
 - For this exercise, you can divide the wing in four spanwise stations.



The wing cross-section is a NACA 65-210 airfoil.

$$AR = 9$$

$$TR = 0.4$$

$$S = 1.161288$$

$$\Lambda_{TE} = 8.13^\circ$$

$$\Lambda_{c4} = 0^\circ$$

Do you agree with this value



$$\alpha_{ZL} = -1.2$$

$$c_{l_\alpha} = a_0 = 2\pi$$

LLT working example

- To compute the lift and drag coefficients, we use the monoplane equation which is derived from the LLT,

$$\underbrace{\sum_{n=1}^N A_n \sin n\phi (\mu n + \sin\phi)}_{\text{Wake model (3D)}} = \underbrace{\mu(\alpha - \alpha_{ZL}) \sin\phi}_{\text{Section model (2D)}}$$

- The airfoil information is contained in the term,

$$\begin{array}{l} \mu(\alpha - \alpha_{ZL}) \\ \downarrow \\ \mu = \frac{ca_0}{8s} \quad \nearrow \quad c = c_{root}(1 + (\lambda - 1)\cos\phi) \\ \longrightarrow \quad \text{Wing semi-span or } b/2 \end{array}$$

- In viscous computations, due to taper, different cross section can have different Reynolds numbers.

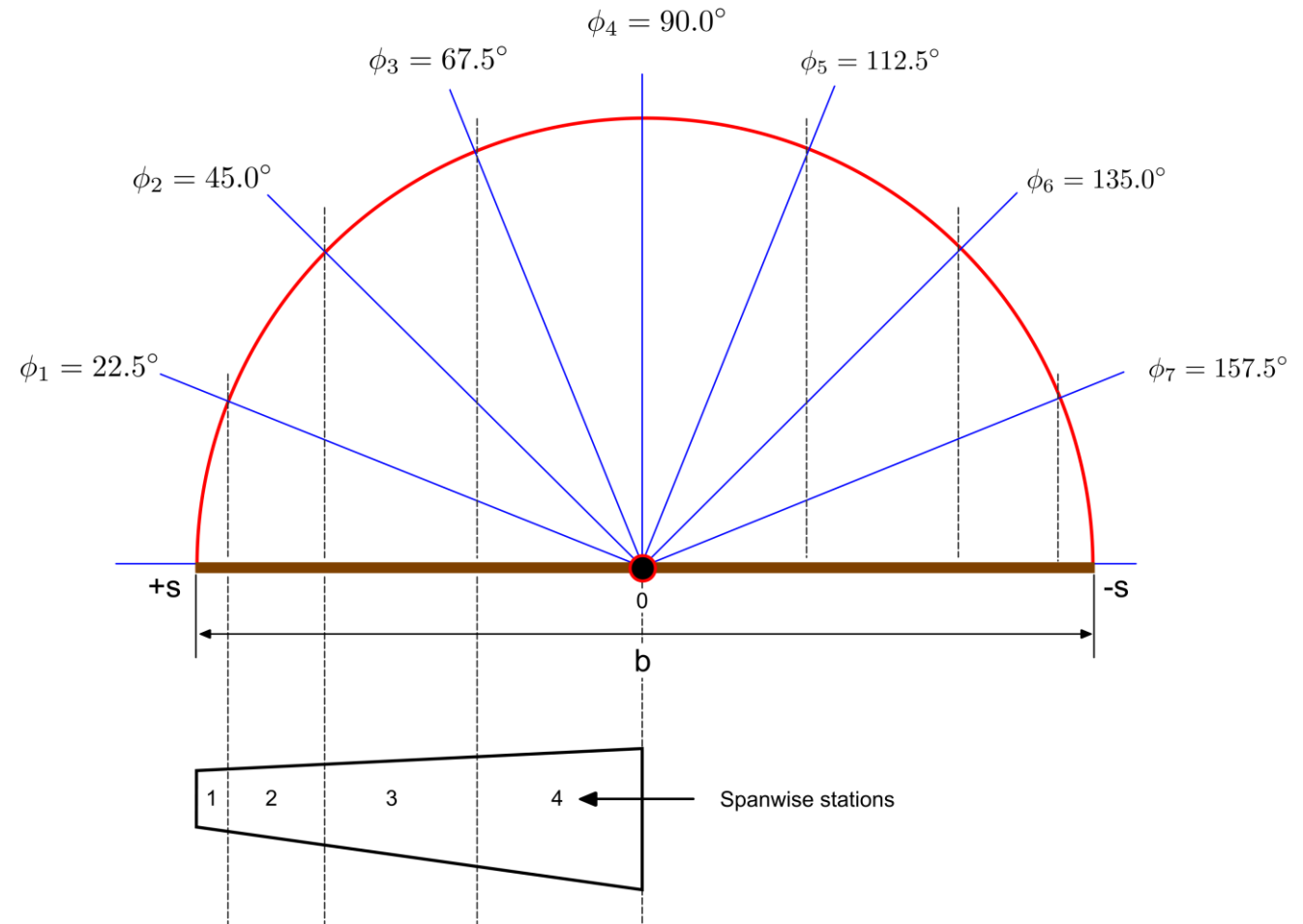
LLT working example

- We now divide the wing into spanwise stations.
- In this case we will use four stations.
- To do so, we introduce the following transformation,

$$s = \frac{b}{2} \cos(\phi)$$

- At this point we know all the geometrical information.
- We are now ready to solve the monoplane equation for this geometry.
- In this case, the local chord in each station can be computed as,

$$c = c_{root} (1 + (\lambda - 1) \cos \phi)$$



LLT working example

- We can start by computing all the coefficients appearing in the general monoplane equation.

$$\sum_{n=1}^N A_n \sin n\phi (\mu n + \sin\phi) = \mu(\alpha - \alpha_{ZL}) \sin\phi$$

- Let us use excel to assembly this table (we will only use odd terms, *i.e.*, 1, 3, 5, 7, ...),

$\phi (^{\circ})$		$\phi (rad)$	s	c(s)	$\cos(\phi)$	$\sin(\phi)$	$\sin(3\phi)$	$\sin(5\phi)$	$\sin(7\phi)$	μ
$\pi/8$	22.5	0.392699	1.055994	0.323189	0.92388	0.382683432	0.923879533	0.923879533	0.382683432	0.111120695
$\pi/4$	45	0.785398	0.808223	0.417701	0.707107	0.707106781	0.707106781	-0.707106781	-0.707106781	0.143549823
$3\pi/8$	67.5	1.178097	0.437407	0.55915	0.382683	0.923879533	-0.382683432	-0.382683432	0.923879533	0.192083442
$\pi/2$	90	1.570796	0	0.726	0	1	-1	1	-1	0.24933275



$$0 \leq \phi \leq \pi/2$$



$$0 \leq y \leq +s$$



$$c = c_{root}(1 + (\lambda - 1)\cos\phi)$$



$$\mu = \frac{ca_0}{8s} = \frac{ca_0}{2ARc_{root}(1 + \lambda)}$$

LLT working example

- Our objective is to compute the coefficients A_N contained in the monoplane equation.
- As we are interested in a symmetrical lift distribution, we will only compute the odd terms in the monoplane equation.
- After computing all the coefficients, we will end up with the following linear system that we can solve using any method (direct or iterative).

Geometrical and aerodynamic information twist is contained here

$$\begin{pmatrix} \sin\phi_1(\mu + \sin\phi_1) & \sin 2\phi_1(2\mu + \sin\phi_1) & \cdots & \sin N\phi_1(N\mu + \sin\phi_1) \\ \sin\phi_2(\mu + \sin\phi_2) & \sin 2\phi_2(2\mu + \sin\phi_2) & \cdots & \sin N\phi_2(N\mu + \sin\phi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \sin\phi_N(\mu + \sin\phi_N) & \sin 2\phi_N(2\mu + \sin\phi_N) & \cdots & \sin N\phi_N(N\mu + \sin\phi_N) \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} = \begin{pmatrix} \overbrace{\mu(\alpha - \alpha_{ZL_1})} \sin\phi_1 \\ \mu(\alpha - \alpha_{ZL_2}) \sin\phi_2 \\ \vdots \\ \mu(\alpha - \alpha_{ZL_N}) \sin\phi_N \end{pmatrix}$$

$$Ax = b$$

$$x = A^{-1}b$$

LLT working example

- At this point, our task is to solve the following linear system,

A				x	=	b
0.188971	0.66154	0.866864	0.444115	A_1		0.003859
0.601505	0.804515	-1.00753	-1.21054	A_3		0.009212
1.031015	-0.57407	-0.72109	2.095787	A_5		0.016106
1.249333	-1.748	2.246664	-2.74533	A_7		0.022629

$$\sum_{n=1}^N A_n \sin n\phi(\mu n + \sin\phi) = \mu(\alpha - \alpha_{ZL})\sin\phi$$

- Solving for the vector x, we obtain all the coefficients of the monoplane equations are equal to,

x	
A_1	0.016459
A_3	7.32E-05
A_5	0.000858
A_7	-9.7E-05

LLT working example

- Now that we have the solution for $A_1, A_3, A_5,$ and $A_7,$ we can compute the lift and induced drag coefficients.
- The lift coefficient can be computed as follows,

$$C_L = A_1 \pi AR$$

- The induced drag coefficient be computed as follows,

$$C_{D_{ind}} = \frac{C_L^2}{\pi AR} (1 + \delta)$$

- where δ is the induced drag factor and can be computed as follows,

$$\delta = \frac{3A_3^2}{A_1^2} + \frac{5A_5^2}{A_1^2} + \frac{7A_7^2}{A_1^2} + \dots$$

- An alternative way to compute the induced drag coefficient is as follows,

$$C_{D_{ind}} = \frac{C_L^2}{\pi AR e} = k C_L^2 \quad \text{where} \quad e = \frac{1}{1 + \delta} \quad \text{and} \quad k = \frac{1}{\pi AR e}$$

Oswald span efficiency

LLT working example

- Now that we have the solution for the coefficients A_1 , A_3 , A_5 , and A_7 .
- Using this information, we can compute the lift and induced drag coefficients.
- We can compute more coefficients but in practice ten is more than enough, and four is the bare minimum but a close enough solution.
- At this point, the lift coefficient can be computed as follows,

$$C_L = A_1 \pi AR$$

- The induced drag coefficient can be computed as follows,

$$C_{D_{ind}} = \frac{C_L^2}{\pi AR} (1 + \delta)$$

- Where δ is the induced drag factor.

LLT working example

- The induced drag factor δ can be computed as follows,

$$\delta = \frac{3A_3^2}{A_1^2} + \frac{5A_5^2}{A_1^2} + \frac{7A_7^2}{A_1^2} + \dots$$

- An alternative way to compute the induced drag coefficient is as follows,

$$C_{D_{ind}} = \frac{C_L^2}{\pi A R e} = k C_L^2 \quad \text{where} \quad e = \frac{1}{1 + \delta} \quad \text{and} \quad k = \frac{1}{\pi A R e}$$

↑
Oswald span efficiency

LLT working example

- The lift coefficient distribution can be computed as follows,

$$C_l(\phi) = \frac{2\Gamma(\phi)}{U_\infty c}$$

- where $\Gamma(\phi)$ is equal to,

$$\Gamma(\phi) = 4sU_\infty \sum_{n=1}^N A_n \sin n\phi$$

- And remember that the forces can be computed as follows,

$$L = \frac{1}{2} \times \rho \times V^2 \times S_{ref} \times C_L$$

$$D = \frac{1}{2} \times \rho \times V^2 \times S_{ref} \times C_D$$

LLT working example

- Notice that in the equations used, the lift coefficient only depends on the coefficient A_1 ,

$$C_L = A_1 \pi AR$$

- Whereas the induced drag factor δ depends in the rest of the coefficients

$$\delta = \frac{3A_3^2}{A_1^2} + \frac{5A_5^2}{A_1^2} + \frac{7A_7^2}{A_1^2} + \dots$$

- Also notice that as we are interested in a symmetrical lift distribution, we will only need to compute the odd terms in the monoplane equation.

LLT working example

- Recall that $\alpha = 2^\circ$, then our final solution is,

$$C_L = 0.2863$$

$$C_{D_{ind}} = 0.002941$$

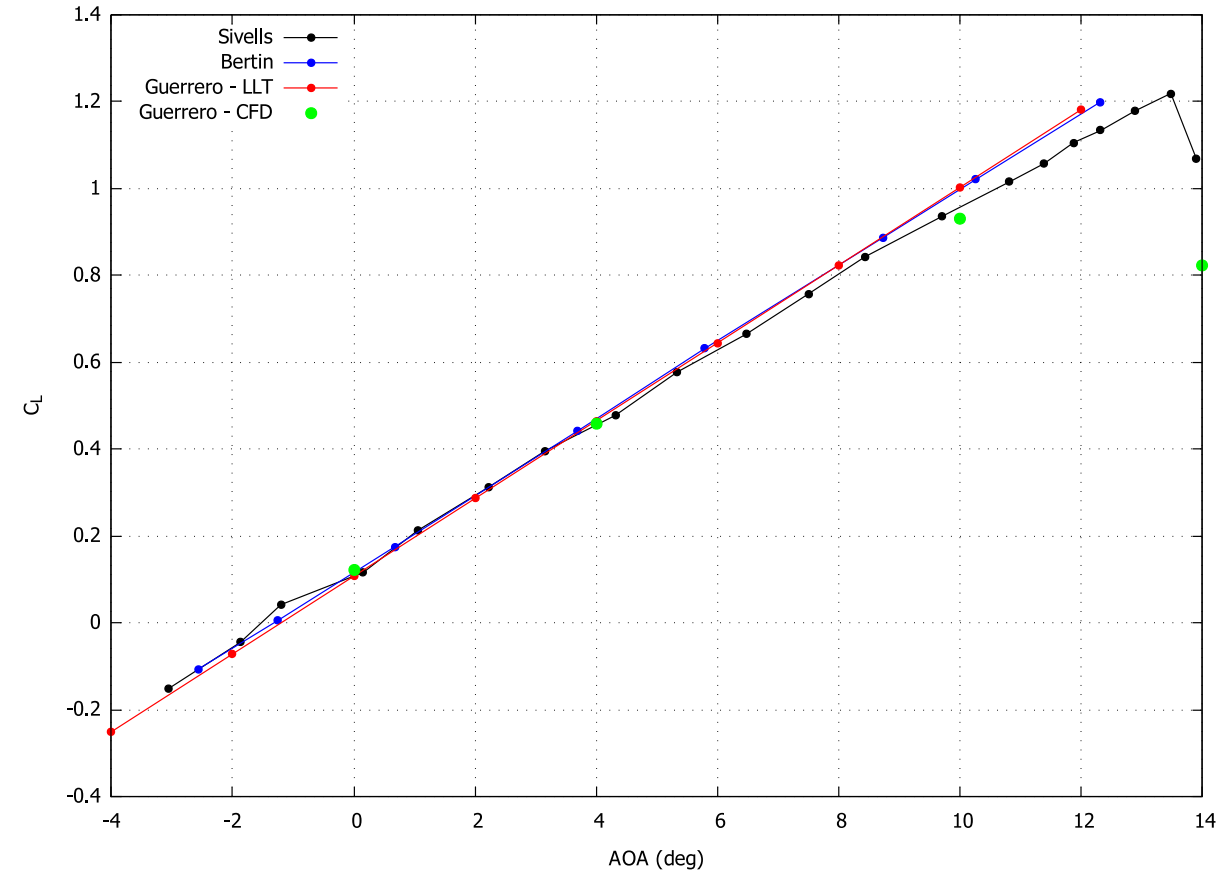
$$\delta = 0.013885$$

$$C_{L_\alpha} = \frac{C_L}{\alpha - \alpha_{ZL}} = 5.1277$$

$$V_\infty = \sqrt{\frac{L = W}{0.5\rho S C_L}} = 140.1302 \text{ m/s}$$

LLT working example

- Your LLT recipe:
 - Compute all geometrical properties of the wing.
 - Get all airfoil characteristics.
 - Remember different wing sections might see different Reynolds number.
 - Divide the wing in spanwise stations and compute all the coefficients of the monoplane equation.
 - Solve the linear system for the coefficients A_N .
 - Use A_N to compute the lift coefficient, drag coefficient and lift coefficient distribution.



- Data source:
 - J. Sivells. Experimental and calculated characteristics of three wings of NACA 64-210 and 65-210 airfoil sections with and without washout. NACA Report 1422, 1947.
 - Aerodynamics for Engineers (6th Edition). J. Bertin, R. Cummings. Pearson, 2013.

LLT working example

- Comparison of different methods and software.

