Turbulence and CFD models: Theory and applications

Roadmap to Lecture 6

Part 2

1. Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation
2. Revisiting the closure problem
3. Two equations models – The $k - \epsilon$ model
4. Two equations models – The $k - \omega$ model
5. One equation model – The Spalart-Allmaras model

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Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- Let us recall the exact Reynolds stress transport equation,

$$\underbrace{\frac{\partial \tau_{ij}}{\partial t}}_{1} + \underbrace{\bar{u}_k \frac{\partial \tau_{ij}}{x_k}}_{2} = \underbrace{-\left(\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right)}_{3} + \underbrace{2\nu \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}}}_{4} + \dots$$

$$\dots + \underbrace{\frac{1}{\rho}\left(\overline{u'_i \frac{\partial p'}{\partial x_j}} + \overline{u'_j \frac{\partial p'}{\partial x_i}}\right)}_{5} + \underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ij}}{\partial x_k}\right)}_{6} + \underbrace{\frac{\partial}{\partial x_k}\left(\overline{u'_i u'_j u'_k}\right)}_{7}$$

1. Transient stress rate of change term.
2. Convective term.
3. Production term.
4. Dissipation rate.
5. Turbulent stress transport related to the velocity and pressure fluctuations.
6. Rate of viscous stress diffusion (molecular)
7. Diffusive stress transport resulting from the triple correlation of velocity fluctuations.

- Recall that in our notation $\tau_{ij} = \tau_{ij}^R$.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- These equations can be further simplified as follows,

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = \underbrace{-\left(\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right)}_{\text{Production}} + \underbrace{\epsilon_{ij}}_{\text{Dissipation}} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk}\right]$$

- Where,

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \qquad \Pi_{ij} = \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$$

$$\rho C_{ijk} = \rho \overline{u_i' u_j' u_k'} + \overline{p' u_i'}\delta_{jk} + \overline{p' u_j'}\delta_{ik}$$

- These are the **exact** Reynolds stress transport equations.
- To derive the **solvable** equations, we need to use approximations in place of the terms that contain fluctuating variables ($\epsilon_{ij}, \Pi_{ij}, \rho C_{ijk}$).
- The Reynolds stresses can be modeled using the Boussinesq approximation.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- Let us recall the **exact** turbulent kinetic energy equation TKE, which is obtained by taking the trace of the Reynolds stress transport equation,

$$\underbrace{\frac{\partial \tau_{ii}}{\partial t}}_{1} + \underbrace{\bar{u}_k \frac{\partial \tau_{ii}}{\partial x_k}}_{2} = \underbrace{2\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}}_{3} + \underbrace{\epsilon_{ii}}_{4} + \underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ii}}{\partial x_k}\right)}_{5} + \underbrace{\frac{2}{\rho}\overline{\left(u_i' \frac{\partial p'}{\partial x_i}\right)}}_{6} + \underbrace{\frac{\partial}{\partial x_k}(\overline{u_i' u_i' u_k'})}_{7}$$

1. Transient rate of change term.
2. Convective term.
3. Production term arising from the product of the Reynolds stress and the velocity gradient.
4. Dissipation rate.
5. Rate of viscous stress diffusion (molecular).
6. Turbulent transport associated with the eddy pressure and velocity fluctuations.
7. Diffusive turbulent transport resulting from the triple correlation of velocity fluctuations.

- And recall that,

$$k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$-(\overline{\mathbf{u}'\mathbf{u}'})^{\mathrm{tr}} = -(\overline{u_i' u_i'}) = \tau_{ii} = -2k$$

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- We can now substitute $\tau_{ii} = -2k$ and simplify to obtain the following equation,

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon + \frac{\partial}{\partial x_j} \left[\nu \frac{\partial k}{\partial x_j} - \frac{1}{2} \overline{u_i' u_i' u_j'} - \frac{1}{\rho} \overline{p' u_j'} \right]$$

- Where ϵ is the dissipation rate (per unit mass) and is given by the following relation,

$$\epsilon_{ii} = \epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}$$

- This is the **exact** turbulent kinetic energy transport equation.
- To derive the **solvable** equation, we need to use approximations in place of the terms that contain fluctuating quantities.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- The **solvable** turbulent kinetic energy equation TKE can be written as follows,

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Production}} - \underbrace{\epsilon}_{\text{Dissipation}} + \underbrace{\frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]}_{\text{Diffusion}}$$

- The Reynolds stresses can be modeled using the Boussinesq approximation.
- The term related to the turbulent transport and the pressure diffusion can be modeled as follows,

$$\frac{1}{2}\overline{u_i' u_i' u_j'} + \frac{1}{\rho}\overline{p' u_j'} = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$$

- The term related to the dissipation rate can be modeled by adding an additional transport equation, which will be derived later.
- All the approximations added are based on DNS simulations, experimental data, analytical solutions, or engineering intuition.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- The **exact** form of the Reynolds stress transport equation, turbulent kinetic energy transport equation, and other turbulent quantities transport equations that we will derive later (dissipation rate, specific rate of dissipation, and so on) share some similarities.
- Namely, a production term (eddy factory), a dissipation or destruction term (where eddies are destroyed – eddy graveyard –), and a turbulence diffusion term (transport, diffusion, and redistribution due to turbulence).
- Therefore, the transport equations of the turbulent quantities can be expressed in the following way,

$$\underbrace{\nabla_t \phi}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \phi}_{\text{Convection}} = \underbrace{P^{\phi}}_{\text{Production}} + \underbrace{\epsilon^{\phi}}_{\text{Dissipation}} + \underbrace{D^{\phi}}_{\text{Diffusion}} + \underbrace{S^{\phi}}_{\text{Source terms}}$$

- Where ϕ represents the transported turbulent quantity.

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4. Two equations models – The $k - \omega$ model
5. One equation model – The Spalart-Allmaras model

Revisiting the closure problem

- The **solvable** RANS equations can be written as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}\left(\mu + \mu_t\right) \nabla \bar{\mathbf{u}}\right]$$

Turbulent viscosity

- In this case, the **solvable** RANS equations were obtained after substituting the Boussinesq approximation into the **exact** RANS equations.
- The problem now reduces to computing the turbulent eddy viscosity in the momentum equation.
- Each turbulence model computes the turbulent eddy viscosity in a different way,

$$\mu_t = f(k, \epsilon, \omega, l, t, v, \ldots)$$

- At this point, let us explore the most widely used turbulence models.

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Two equations models – The $k - \epsilon$ model

- This is one of the most popular two-equation turbulence model.
- The initial development of this model can be attributed to Chou [1], circa 1945.
- Launder and Spalding [2] and Launder and Sharma [3] further developed and calibrated the model and created what is generally referred to as the Standard $k - \epsilon$ model.
- This is the model that we are going to address hereafter.
- There are many variations of this model.
- Each one designed to add new capabilities and overcome the limitations of the standard $k - \epsilon$ model.
- The most notable limitation is that it requires the use of wall functions.
- Variants of this model include the RNG $k - \epsilon$ model [3] and the Realizable $k - \epsilon$ model [4], just to name a few.

References:
[1] P. Y. Chou. On Velocity Correlations and the Solutions of the Equations of Turbulent Fluctuation. Quarterly of Applied Mathematics. 1945.
[2] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.
[3] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.
[4] V. Yakhot, S. A. Orszag. Renormalization Group Analysis of Turbulence I Basic Theory. Journal of Scientific Computing. 1986.
[5] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New - Eddy-Viscosity Model for High Reynolds Number Turbulent Flows - Model Development and Validation. Computers Fluids. 1995.

Two equations models – The $k-\epsilon$ model

- It is called $k-\epsilon$ because it solves two additional equations for modeling the turbulent viscosity, namely, the turbulent kinetic energy k and the turbulence dissipation rate ϵ .

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

- This model used the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- With the following closure coefficients,

$$C_{\epsilon_1} = 1.44 \qquad C_{\epsilon_2} = 1.92 \qquad C_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.3$$

- And auxiliary relationships,

$$\omega = \frac{\epsilon}{C_\mu k} \qquad l = \frac{C_\mu k^{3/2}}{\epsilon}$$

Two equations models – The $k-\epsilon$ model

- The closure equations of the standard $k-\epsilon$ model have been manipulated so there are no terms including fluctuating quantities (*i.e.*, velocity and pressure), and doble or triple correlations of the fluctuating quantities.
- Remember, the Reynolds stress tensor is modeled using the Boussinesq approximation.
- The turbulence dissipation rate is modeled using a second transport equation.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \overbrace{\tau^R : \nabla \bar{\mathbf{u}}}^{\text{Production}} - \overbrace{\epsilon}^{\text{Dissipation}} + \overbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]}^{\text{Diffusion}}$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = \underbrace{C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underbrace{C_{\epsilon_2} \frac{\epsilon^2}{k}}_{\text{Dissipation}} + \underbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]}_{\text{Diffusion}}$$

Two equations models – The $k-\epsilon$ model

- The transport equation of the turbulence dissipation rate ϵ used in this model can be derived by taking the following moment of the NSE equations,

$$2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial}{\partial x_j}\left[\mathcal{N}\left(u_i\right)\right]}=0$$

- Where the operator $\mathcal{N}(u_i)$ is equal to,

$$\mathcal{N}(u_i)=\rho\frac{\partial u_i}{\partial t}+\rho u_k\frac{\partial u_i}{\partial x_k}+\frac{\partial p}{\partial x_i}-\mu\frac{\partial^2 u_i}{\partial x_k x_k}$$

- The **exact** turbulence dissipation rate transport equation is far more complicated than the turbulent kinetic energy equation.
- This equation contains several new unknown double and triple correlations of fluctuating velocity, pressure, and velocity gradients.

Two equations models – The $k-\epsilon$ model

- There is a lot of algebra involved in the derivation of the **exact** turbulence dissipation rate transport equation. The final equation looks like this,

$$
\underbrace{\frac{\partial \epsilon}{\partial t}}_{1} + \underbrace{\overline{u}_j \frac{\partial \epsilon}{\partial x_j}}_{2} = \underbrace{-2\nu \frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}} \right)}_{3} \underbrace{-2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j}\overline{u_k' \frac{\partial u_i'}{\partial x_j}}}_{4}
$$

$$
\underbrace{-2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}}_{5} \underbrace{-2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m}\frac{\partial^2 u_i'}{\partial x_k \partial x_m}}}_{6}
$$

$$
\underbrace{+\nu \frac{\partial \left(\frac{\partial \epsilon}{\partial x_j} \right)}{\partial x_j}}_{7} \underbrace{-\nu \frac{\partial \left(\overline{u_j' \frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}} \right)}{\partial x_j}}_{8} \underbrace{-2\frac{\nu}{\rho}\frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}} \right)}{\partial x_j}}_{9}
$$

1. Transient rate of change term.
2. Convective term.
3. Production term that arises from the product of the gradients of the fluctuating and mean velocities.
4. Production term that generates additional dissipation based on the fluctuating and mean velocities.
5. Dissipation (destruction) associated with eddy velocity fluctuating gradients.
6. Dissipation (destruction) arising from eddy velocity fluctuating diffusion.
7. Viscous diffusion.
8. Diffusive turbulent transport resulting from the eddy velocity fluctuations.
9. Dissipation of turbulent transport arising from eddy pressure and fluctuating velocity gradients.

Two equations models – The $k-\epsilon$ model

- To derive the **solvable** transport equation of the turbulence dissipation rate, we need to use approximations in place of the terms that contain fluctuating quantities (velocity, pressure, and so on).
- The following approximations can be added to the **exact** turbulence dissipation rate transport equation.

$$\underbrace{-2\nu \frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} + \overline{\frac{\partial u'_k}{\partial x_i}\frac{\partial u'_k}{\partial x_j}} \right)}_{3} \underbrace{-2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j} \overline{u'_k \frac{\partial u'_i}{\partial x_j}}}_{4} \longrightarrow C_{\epsilon 1} \frac{\epsilon}{k} \tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j}$$

Production

$$\underbrace{-2\nu \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_k}{\partial x_m}}}_{5} \underbrace{-2\nu^2 \overline{\frac{\partial^2 u'_i}{\partial x_k \partial x_m}\frac{\partial^2 u'_i}{\partial x_k \partial x_m}}}_{6} \longrightarrow -C_{\epsilon 2}\frac{\epsilon^2}{k}$$

Dissipation

$$\underbrace{\nu \frac{\partial \left(\frac{\partial \epsilon}{\partial x_j} \right)}{\partial x_j}}_{7} \underbrace{-\nu \frac{\partial \left(\overline{u'_j \frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}} \right)}{\partial x_j}}_{8} \underbrace{-2\frac{\nu}{\rho} \frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u'_j}{\partial x_m}} \right)}{\partial x_j}}_{9} \longrightarrow \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right]$$

Diffusion

Two equations models – The $k-\epsilon$ model

- By substituting the previous approximations in the **exact** turbulence dissipation rate transport equation we derive the **solvable** equation.
- It is not easy to elucidate the behavior of each term appearing in the **exact** turbulence dissipation rate transport equation.
- All the approximations added are based on DNS simulations, experimental data, analytical solutions, or engineering intuition.
- The **solvable** turbulence dissipation rate transport equation takes the following form,

$$\frac{\partial \epsilon}{\partial t}+\bar{u}_j\frac{\partial \epsilon}{\partial x_j}=\underbrace{C_{\epsilon_1}\frac{\epsilon}{k}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Production}}-\underbrace{C_{\epsilon_2}\frac{\epsilon^2}{k}}_{\text{Dissipation}}+\underbrace{\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right]}_{\text{Diffusion}}$$

Two equations models – The $k-\epsilon$ model

- The standard $k-\epsilon$ model use wall functions.
- The wall boundary conditions for the solution variables are all taken care of by the wall functions implementation.
- Therefore, when using commercial solvers (Fluent in our case) you do not need to be concerned about the boundary conditions at the walls.
- Using the standard walls functions approach developed by Launder and Spalding [1], the numerical values of the boundary conditions at the walls are computed as follows,

$$\frac{\partial k}{\partial n} = 0 \qquad \epsilon_P = \frac{C_\mu^{3/4} k_P^{3/2}}{\kappa y_P}$$

Where the subscript P means cell center

- The free-stream values can be computed using the method introduced in Lecture 4.
- It is strongly recommended to not initialize these quantities with the same value or with values close to zero (in particular the turbulence dissipation rate).

[1] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.

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Two equations models – The $k-\omega$ model

- This is probably the most widely used two-equation turbulence model.
- The initial development of this model can be attributed to Kolmogorov [1], circa 1942. This was the first two-equation model of turbulence.
- The method was further developed and improved by Saffman [2], Launder and Spalding [3], Wilcox [4,5], Menter [6] and many more.
- There are many variations of this model. Hereafter, we will address the Wilcox 1988 $k-\omega$ model, which probably is the first formulation of the modern $k-\omega$ family of turbulence models.
- Each variation is designed to add new capabilities and overcome the limitations of the predecessor formulations.
- The most notable drawback is its limitation to resolve streamline curvature.
- This family of models is y^+ insensitive.
- Variants of this model include the Wilcox 1998 $k-\omega$, Wilcox 2006 $k-\omega$, and Menter 2003 $k-\omega$ SST.

References:
[1] A. N. Kolmogorov. Equations of Turbulent Motion in an Incompressible Fluid. Physics. 1941.
[2] P. Saffman. A Model for Inhomogeneous Turbulent Flow. Proceedings of the Royal Society of London. 1970.
[3] B. E. Launder, D. B. Spalding. Mathematical Models of Turbulence. Academic Press. 1972.
[4] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[5] D. C. Wilcox. Turbulence Modeling for CFD. DCW Industries, 2010.
[6] F. Menter, M. Kuntz, R. Langtry. Ten Years of Industrial Experience with the SST Turbulence Model. Turbulence, Heat and Mass Transfer. 2003.

Two equations models – The $k-\omega$ model

- It is called $k-\omega$ because it solves two additional equations for modeling the turbulence, namely, the turbulent kinetic energy k and the specific turbulence dissipation rate ω.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \sigma^* \nu_t \right) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \sigma \nu_t \right) \nabla \omega \right]$$

- This model used the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{k}{\omega}$$

- With the following closure coefficients,

$$\alpha = 5/9 \qquad \beta = 3/40 \qquad \beta^* = 9/100 \qquad \sigma = 1/2 \qquad \sigma^* = 1/2$$

- And auxiliary relationships,

$$\epsilon = \beta^* \omega k \qquad\qquad l = \frac{k^{1/2}}{\omega}$$

Two equations models – The $k-\omega$ model

- The closure equations of the Wilcox (1988) $k-\omega$ model have been manipulated so there are no terms including fluctuating quantities (*i.e.*, velocity and pressure), and doble or triple correlations of the fluctuating quantities.
- Remember, the Reynolds stress tensor is modeled using the Boussinesq approximation.
- The specific turbulence dissipation rate is modeled using a second transport equation.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \overbrace{\tau^R : \nabla \bar{\mathbf{u}}}^{\text{Production}} - \overset{\text{Dissipation}}{\beta^* k \omega} + \overbrace{\nabla \cdot \left[\left(\nu + \sigma^* \nu_t \right) \nabla k \right]}^{\text{Diffusion}}$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \underbrace{\alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underset{\text{Dissipation}}{\beta \omega^2} + \underbrace{\nabla \cdot \left[\left(\nu + \sigma \nu_t \right) \nabla \omega \right]}_{\text{Diffusion}}$$

Two equations models – The $k - \omega$ model

- The transport equation for the specific turbulence dissipation rate ω can be derived from the transport equation of the turbulence dissipation rate ϵ .
- The model can be thought as the ratio of ϵ to k.
- To derive the **solvable** equations of the Wilcox (1988) $k - \omega$ turbulence model, we can substitute the relation $\epsilon = \beta^* \omega k$ into the **solvable** equations of the $k - \epsilon$ turbulence model.
- The production, dissipation, and diffusion terms are given by,

$$P^{\omega} = \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j} = \alpha \frac{\omega}{k} P^k \qquad \epsilon^{\omega} = \beta \omega^2$$

$$D^{\omega} = \frac{\partial}{\partial x_j} \left[\left(\nu + \sigma \nu_T \right) \frac{\partial \omega}{\partial x_j} \right]$$

- The closure coefficients need to be recalibrated.

Two equations models – The $k-\omega$ model

- By using the following equations, it is possible to derive an **exact** transport equation for the specific turbulence dissipation rate ω .

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\epsilon+\frac{\partial}{\partial x_j}\left[\nu\frac{\partial k}{\partial x_j}-\frac{1}{2}\overline{u_i'u_i'u_j'}-\frac{1}{\rho}\overline{p'u_j'}\right]$$

$$\frac{\partial \epsilon}{\partial t}+\overline{u}_j\frac{\partial \epsilon}{\partial x_j}=-2\nu\frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}+\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}}\right)-2\nu\frac{\partial^2\overline{u}_i}{\partial x_k\partial x_j}\overline{u_k'\frac{\partial u_i'}{\partial x_j}}-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}$$

$$-2\nu^2\overline{\frac{\partial^2 u_i'}{\partial x_k\partial x_m}\frac{\partial^2 u_i'}{\partial x_k\partial x_m}}+\nu\frac{\partial\left(\frac{\partial \epsilon}{\partial x_j}\right)}{\partial x_j}-\nu\frac{\partial\left(\overline{u_j'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)}{\partial x_j}-2\frac{\nu}{\rho}\frac{\partial\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}}\right)}{\partial x_j}$$

$$\epsilon=\beta^*\omega k$$

- The new **exact** transport equation for the specific turbulence dissipation rate ω can be derived from the turbulence dissipation rate equation ϵ , therefore, they share many similarities.

Two equations models – The $k - \omega$ model

- As for the turbulence dissipation rate equation ϵ, there is a lot of algebra involved.
- Hereafter, we show the most important steps.
- By using the product rule, we can write $\epsilon = \beta^* \omega k$ as follows,

$$\frac{d\epsilon}{dt} = \beta^* k \frac{d\omega}{dt} + \beta^* \omega \frac{dk}{dt} \quad \Longrightarrow \quad \frac{d\omega}{dt} = \frac{1}{\beta^* k} \frac{d\epsilon}{dt} - \frac{\omega}{k} \frac{dk}{dt}$$

- Where d/dt is the material derivative (dependent of the mean velocity),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}$$

- By substituting the following relations into $d\omega/dt$,

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \overline{u}_j \frac{\partial \epsilon}{\partial x_j}$$

Exact transport equation of turbulence dissipation rate

$$\frac{dk}{dt} = \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j}$$

Exact transport equation of turbulence kinetic energy

- And doing a lot algebra, we obtain the **exact** equations of ω .

Two equations models – The $k-\omega$ model

- The final **exact** equation of the specific turbulence dissipation rate ω, can be written as follows,

$$
\begin{aligned}
\frac{d\omega}{dt} = \frac{\omega}{k}\left[-\tau_{ij} - 2\nu\frac{\overline{u'_{i,k}u'_{j,k}} + \overline{u'_{k,i}u'_{k,j}}}{\beta^*\omega}\right]\frac{\partial \overline{u}_i}{\partial x_j} \\
-2\nu\frac{\overline{u'_k u'_{i,j}}}{\beta^* k}\frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j} \\
-\left[2\nu\frac{\overline{u'_{i,k}u'_{i,m}u'_{k,m}} + \nu\overline{u'_{i,km}u'_{i,km}}}{\beta^*\omega} - \beta^*\omega^2\right] \\
+\frac{\partial}{\partial x_j}\left[\nu\frac{\partial \omega}{\partial x_j} - \nu\frac{\overline{u'_j u'_{i,m}u'_{i,m}}}{\beta^* k}\right] + \frac{1}{2}\omega\frac{\overline{u'_j u'_i u'_i}}{k} - 2\nu\frac{\overline{p'_{,m}u'_{j,m}}}{\beta^*\rho k} + \omega\frac{\overline{p' u'_j}}{\rho k} \\
+\frac{1}{k}\left[2\nu\frac{\partial k}{\partial x_j} - \frac{1}{2}\overline{u'_j u'_i u'_i} - \frac{1}{\rho}\overline{p' u'_j}\right]\frac{\partial \omega}{\partial x_j} \\
+\frac{1}{k^2}\left[\frac{\omega}{2}\overline{u'_j u'_i u'_i} - \frac{1}{\beta^*}\nu\overline{u'_j u'_{i,m}u'_{i,m}} + \frac{\omega}{\rho}\overline{p' u'_j} - \frac{2}{\beta^*}\frac{\nu}{\rho}\overline{p'_{,m}u'_{j,m}}\right]\frac{\partial k}{\partial x_j}
\end{aligned}
$$

- As for the **exact** turbulence dissipation rate transport equation, it is not easy to elucidate the behavior of each term appearing in this equation.
- As this equation was derived from **exact** turbulence dissipation rate transport equation, we can use similar approximations.

Two equations models – The $k-\omega$ model

- The $k-\omega$ family of turbulence models are y^+ insensitive.
- These models work by blending the viscous sublayer formulation and the logarithmic layer formulation based on the y^+.
- Unlike the standard $k-\epsilon$ model and some other models, the $k-\omega$ models can be integrated through the viscous sublayer without the need for wall functions.
- The wall boundary conditions for the turbulent variables can be computed as follows,

$$k = 0 \qquad \omega = \frac{6\nu}{\beta_0 d^2} \qquad \beta_0 \approx 0.075$$

d is the distance to the first cell center normal to the wall

- The free-stream values can be computed using the method introduced in Lecture 4.
- It is strongly recommended to not initialize these quantities with the same value or with values close to zero (in particular the specific turbulence dissipation rate).

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
2. ~~Revisiting the closure problem~~
3. ~~Two equations models – The $k - \epsilon$ model~~
4. ~~Two equations models – The $k - \omega$ model~~
5. **One equation model – The Spalart-Allmaras model**

One equation model – The Spalart-Allmaras model

- The Spalart-Allmaras model [1,2] is a one-equation model that solves a model transport equation for the modified turbulent kinematic viscosity.
- By far this is the most popular and successful one-equation model.
- It also has been adopted as the foundation for DES [3].
- The Spalart-Allmaras model was designed specifically for aerospace applications involving wall-bounded flows.
- In its original form, the Spalart-Allmaras model is a wall resolving method, requiring the use of fine meshes in order to resolve the viscous sublayer.
- Over the years this method has been improved. Each variation is designed to add new capabilities and overcome the limitations of the predecessor formulations.
- The most notable drawback is its limitation to deal with massive flow separation.
- Variants of this model include the addition of rotation/curvature corrections, trip terms, production limiters, strain adaptive formulations, wall roughness corrections, compressibility corrections, extension to y^+ insensitive treatment, and so on.

References:
[1] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1992.
[2] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1994.
[3] M. Shur, P. R. Spalart, M. Strelets, A. Travin. Detached-Eddy Simulation of an Airfoil at High Angle of Attack. 1999.

One equation model – The Spalart-Allmaras model

- In the Spalart-Allmaras model (SA), a closed equation for the turbulent eddy viscosity is artificially created that fits well a range of experimental and empirical data.
- To accomplish this the SA ν_T equation is built up term by term in a series of calibrations involving flows of increasing complexity.

$$\underbrace{\nabla_t \nu_T}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \nu_T}_{\text{Convection}} = \underbrace{P^{\nu_T}}_{\text{Production}} + \underbrace{\epsilon^{\nu_T}}_{\text{Dissipation}} + \underbrace{D^{\nu_T}}_{\text{Diffusion}} + \underbrace{S^{\nu_T}}_{\text{Source terms}}$$

- The resulting model has gone through a number of developmental iterations beyond its original form and has been widely tested for different external aerodynamics applications.
- It is beyond the scope of this discussion to delve into the calibration the have gone into producing each term in the model and the choice of parameter values.
- The interested reader should take a look at the following link:
    - https://turbmodels.larc.nasa.gov/spalart.html

One equation model – The Spalart-Allmaras model

- The closure equations of the standard SA model are as follows,

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = c_{b1} \left(1 - f_{t2}\right) \tilde{S} \tilde{\nu} - \left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2} \right] \left(\frac{\tilde{\nu}}{d} \right)^2$$

$$+ \frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[\left(\nu + \tilde{\nu}\right) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}$$

- Where $\tilde{\nu}$ is the modified eddy viscosity.
- In this model used the following relation for the kinematic eddy viscosity,

$$\nu_T = \tilde{\nu} f_{v1}$$

- Where f_{v1} can be interpreted as a wall damping function [1].

[1] G. Mellor, H. Herring. Two methods of calculating turbulent boundary layer behavior based on numerical solutions of the equations of motion. 1968.

One equation model – The Spalart-Allmaras model

- With the following closure coefficients and auxiliary relationships,

$$c_{b1} = 0.1355 \qquad c_{b2} = 0.622 \qquad c_{v1} = 7.1 \qquad \sigma = 2/3$$

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{(1 + c_{b2})}{\sigma} \qquad c_{w2} = 0.3 \qquad c_{w3} = 2 \qquad \kappa = 0.41$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \qquad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \qquad f_w = g \left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right]^{1/6}$$

$$f_{t2} = c_{t3} e^{-c_{t4}\chi^2} \qquad c_{t3} = 1.2 \qquad c_{t4} = 0.5$$

$$\chi = \frac{\tilde{\nu}}{\nu} \qquad g = r + c_{w2}\left(r^6 - r\right) \qquad r = \min\left[\frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}, 10\right]$$

$$\tilde{S} = \Omega + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2} \qquad \Omega = \sqrt{2W_{ij}W_{ij}} \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

One equation model – The Spalart-Allmaras model

- In the previous relationships, W_{ij} is the rotation tensor and d is the distance from the closest wall.
- Notice that the modified eddy viscosity equation depends on the distance from the closest wall, as well as on the gradient of the modified eddy viscosity gradient.
- Since $d \rightarrow \infty$ far from the walls, this model also predicts no decay of the eddy viscosity in a uniform stream.
- Inspection of the transport equation reveals that κd has been used as length scale.
- The length scale κd is also used in the term $\tilde{S}$, which is related to the vorticity.
- To avoid possible numerical problems, the vorticity parameter $\tilde{S}$ must never be allowed to reach zero or go negative. In references [1] a limiting method is reported.
- Many implementations of the SA model ignore the term f_{t2}, which was added to provide more stability when the trip term is used.
- Based on studies described in [2], the use of this form as opposed to the SA version with the trip term probably makes very little difference.
- The form of the Spalart-Allmaras model with the trip term included is given in reference [3].

[1] S. Allmaras, F. Johnson, P. Spalart. Modifications and Clarifications for the Implementation of the Spalart-Allmaras Turbulence Model. 2012.
[2] C. Rumsey. Apparent Transition Behavior of Widely-Used Turbulence Models. 2007.
[3] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1994.

One equation model – The Spalart-Allmaras model

- The closure equations SA model have been derived using empirical relationships, dimensional analysis, and experimental and numerical data.

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = \underbrace{c_{b1} \tilde{S} \tilde{\nu}}_{\text{Production}} - \underbrace{c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2}_{\text{Dissipation}} + \underbrace{\frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \overbrace{\frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}}^{\text{Extra diffusion source term - Wake profile spreading}}}_{\text{Diffusion}}$$

- Remember, the standard SA model is wall resolving.
- The wall boundary conditions for the turbulent variables can be computed as follows,

$$\tilde{\nu} = 0 \qquad \nu_t = 0$$

- The freestream conditions can be computed as follows,

$$\tilde{\nu}_{\text{farfield}} = 3\nu_\infty \quad to \quad 5\nu_\infty$$

$$\nu_{t_{\text{farfield}}} = 0.210438\nu_\infty \quad to \quad 1.29423\nu_\infty$$

or

$$\tilde{\nu} = \nu_t = \frac{C_\mu k^2}{\epsilon}$$

$$\tilde{\nu} = \nu_t = \frac{k}{\omega}$$