# LES equations – Filtered Navier-Stokes equations

- As for the incompressible RANS/URANS equations, in LES the starting point are the exact incompressible Navier-Stokes equations or NSE.
- Using vector notation, the NSE can be written as follows,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- Using index notation, the NSE can be written as follows,

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j}$$

# LES equations – Filtered Navier-Stokes equations

- Let us derive the filtered Navier-Stokes equations or FNS.
- The first step is to apply the filtering operator directly to the primitive variables.
- By doing so, we obtain the following set of equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}}$$

Notice that the overbar indicates filtering

$$\overline{\mathbf{u}\mathbf{u}} \neq \bar{\mathbf{u}}\bar{\mathbf{u}}$$

This is a property of the filtering operator

- In this set of equations, we cannot solve the system for both $\bar{\mathbf{u}}$ and $\overline{\mathbf{u}\mathbf{u}}$ .
- Therefore, we need to manipulate this set of equations to get a new set of equations expressed only in function of $\bar{\mathbf{u}}$ .
- Remember, the overbar indicates filtering.

# LES equations – Filtered Navier-Stokes equations

- When deriving the filtered Navier-Stokes equations, we should be aware of the following filtering properties or rules.

$$\bar{\bar{\phi}} \neq \bar{\phi}$$

$$\bar{\phi'} \neq 0$$

$$\overline{\phi \psi} \neq \bar{\phi} \bar{\psi}$$

$$\overline{\phi + \psi} = \bar{\phi} + \bar{\psi}$$

The operator is linear

$$\overline{\frac{\partial \phi}{\partial t}} = \frac{\partial \bar{\phi}}{\partial t}$$

$$\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \bar{\phi}}{\partial x}$$

The differentiation operator commutes

- Remember, the overbar represents a spatial filter.
- Notice that the filtering properties in the left column are very different from the averaging rules used in RANS/URANS.

# LES equations – Filtered Navier-Stokes equations

- To express the non-linear term $\overline{\mathbf{u}\mathbf{u}}$ in function of $\bar{\mathbf{u}}$, we can add and subtract the term $\bar{\mathbf{u}}\bar{\mathbf{u}}$ to the left-hand side of the filtered NSE equations.
- By doing so, we obtain the following set of equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}} + \bar{\mathbf{u}}\bar{\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}}$$

- After some algebra, we arrive to the exact filtered Navier-Stokes equations (FNS),

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

$$-(\overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}})$$

# LES equations – Filtered Navier-Stokes equations

- At the end of the day, the FNS equations are written as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

- Where the sub-grid scale stress tensor $\tau^{SGS}$ is an apparent stress that arises from the filtering operation and is equivalent to,

$$\tau^{SGS} = -\left(\overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right) \qquad \text{or} \qquad \tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}}$$

- This tensor represents the effect of filtered scales.
- As for the RANS/URANS equations, the sub-grid scale stress tensor $\tau^{SGS}$ requires modeling.

# LES equations – Filtered Navier-Stokes equations

- Notice that the FNS and the URANS equations are very similar,

**FNS equations**
The overbar denotes filtering

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

$$\tau^{SGS} = -\left(\overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right)$$

**URANS equations**
The overbar denotes time averaging

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \overline{p}) + \nu \nabla^2 \overline{\mathbf{u}} + \nabla \cdot \boldsymbol{\tau}^R$$

$$\tau^R = -\overline{\mathbf{u}'\mathbf{u}'}$$

- The main differences are,
  - The FNS equations are filtered in space.
  - Whereas the URANS equations are time averaged.
  - Also, the apparent stress appearing in both the FNS and URANS equations are different.
  - Nevertheless, they both represent the smallest scales and they both need to be modelled.

# LES equations – Filtered Navier-Stokes equations

- The sub-grid scale stress tensor $\tau^{SGS}$ is given by,

$$\tau^{SGS} = -\left(\overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right) \quad \text{or} \quad \tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}}$$

- This term is analogous to the Reynolds stress tensor in the URANS/RANS equations.
- Let us expand this tensor by substituting the following expression (LES decomposition),

Small scales, residual quantity, or unresolved fields
**Sub-grid scales SGS**

Instantaneous field →

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$$

Filtered quantity or resolved field
**Grid scales GS**

# LES equations – Filtered Navier-Stokes equations

- After some algebra and by grouping some terms, we obtain the following equation (an expansion of the sub-grid scale tensor),

$$\tau^{SGS} = (\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}) + \left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right) + \overline{\mathbf{u}'\mathbf{u}'}$$

- This decomposition is known as the triple decomposition.
- Notice that according to the filtering properties, the filtered sub-grid scale is not equal to zero, that is,

$$\bar{\phi}' \neq 0$$

- The triple decomposition is a little bit more complex than the definition of sub-grid scale stress tensor arising from the filtering operation, namely,

$$\tau^{SGS} = -\left(\overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right) \qquad \text{or} \qquad \tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}}$$

- As it takes into account interactions between all scales, that is, resolved (grid scales) and unresolved (sub-grid scales).
- Remember, the overbar represents a spatial filter.

# LES equations – Filtered Navier-Stokes equations

- The sub-grid scale stress tensor $\tau^{SGS}$ in the triple decomposition represents the effect of filtered scales and small scales, and can be written as,

$$\tau^{SGS} = \underbrace{\left(\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right)}_{\mathbf{L}} + \underbrace{\left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right)}_{\mathbf{C}} + \underbrace{\overline{\mathbf{u}'\mathbf{u}'}}_{\mathbf{R}} = \mathbf{L} + \mathbf{C} + \mathbf{R}$$

- Where **L** is called the Leonard stresses, **C** is the called the cross-stress term, and **R** is called the sub-grid scale Reynolds stress (equivalent to the Reynolds stress tensor).
- The Leonard stresses (**L**) involves only the resolved quantities, and therefore it can be computed.
- The cross-term stresses (**C**) and SGS Reynolds stresses (**R**), involve unresolved scales and must be modeled.
- The cross-term stress represents the interaction of resolved and unresolved scales, whereas the SGS Reynolds stress represents the interaction of unresolved scales.
- It is important to mention that this decomposition is not unique.
- In this case we used the triple decomposition (or Leonard decomposition), which is often used with scale similarity models.
- At this point, the problem is how to model the cross-term stress and the sub-grid scale Reynolds stress.

# LES equations – Filtered Navier-Stokes equations

- At the end of the day, these are the exact incompressible filtered Navier-Stokes equations to be used with LES models,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

- Where we can use any of the following definitions of the sub-grid scale stress tensor $\tau^{SGS}$,

$$\tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}}$$

$$\tau^{SGS} = (\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}) + \left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right) + \overline{\mathbf{u}'\mathbf{u}'}$$

- At this point, we need to introduce models to approximate $\tau^{SGS}$ (similar to RANS).

# LES equations – Filtered Navier-Stokes equations

- From this point on, let us assume that we want to model the whole sub-grid scale stress tensor $\tau^{SGS}$, such as,

$$\tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}} = -2\nu_{SGS}\overline{\mathbf{S}}$$

$$\tau^{SGS} = (\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}) + \left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right) + \overline{\mathbf{u}'\mathbf{u}'} = -2\nu_{SGS}\overline{\mathbf{S}}$$

- Notice that we are using the Boussinesq hypothesis, and that we are lumping all terms into this approximation.
- That is, we are not modeling individual effects (as it might be required for the triple decomposition).

# LES equations – Filtered Navier-Stokes equations

- The sub-grid scale stress tensor $\tau^{SGS}$ can be decomposed into a deviatoric part (anisotropic part) and a spherical part (isotropic part), as follows,

| Index notation | Vector notation |
|---|---|
| $\text{dev}\left(\tau_{ij}^{SGS}\right) = \tau_{ij}^{SGS} - \frac{1}{3}\tau_{kk}^{SGS}\delta_{ij}$ | $\text{dev}\left(\tau^{SGS}\right) = \tau^{SGS} - \frac{1}{3}\text{tr}\left(\tau^{SGS}\right)\mathbf{I}$ |
| $\text{sph}\left(\tau_{ij}^{SGS}\right) = \frac{1}{3}\tau_{kk}^{SGS}\delta_{ij}$ | $\text{sph}\left(\tau^{SGS}\right) = \frac{1}{3}\text{tr}\left(\tau^{SGS}\right)\mathbf{I}$ |
| $\tau_{ij}^{SGS} = \text{dev}\left(\tau_{ij}^{SGS}\right) + \text{sph}\left(\tau_{ij}^{SGS}\right)$ | $\tau^{SGS} = \text{dev}\left(\tau^{SGS}\right) + \text{sph}\left(\tau^{SGS}\right)$ |

- This procedure is known as additive decomposition of a second rank tensor.
- Notice that the deviatoric part (dev) is traceless.
- This kind of decomposition is often used in continuous mechanics.

# LES equations – Filtered Navier-Stokes equations

- Using the additive decomposition of the sub-grid scale stress tensor, the exact FNS equations can be rewritten as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-1}{\rho}\nabla \overline{P} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \mathrm{dev}\left(\tau^{SGS}\right)$$

- Where,

$$\mathrm{dev}\left(\tau^{SGS}\right) = \tau^{SGS} - \frac{1}{3}\mathrm{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

$$\overline{P} = \bar{p} + \mathrm{sph}\left(\tau^{SGS}\right) = \bar{p} + \frac{1}{3}\mathrm{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

# LES equations – Filtered Navier-Stokes equations

- In the previous equations, the deviatoric part of the sub-grid scale stress tensor is often modeled using the Boussinesq hypothesis,

$$\text{dev}\left(\tau^{SGS}\right) = -2\nu_{SGS}\overline{\mathbf{S}}$$

- Expanding and rearranging this equation, we obtain the following relationship,

$$\tau^{SGS} = -2\nu_{SGS}\overline{\mathbf{S}} + \frac{1}{3}\text{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

- The second term in the right-hand side perform the same function as the equivalent term added in the RANS/URANS equation ( $2/3\rho k\mathbf{I}$ ).
- It guarantees that the trace of the right-hand side is equal to the trace of the left-hand side.
- For an incompressible flow, the trace of the strain rate is zero; therefore, the identity holds.

# LES equations – Filtered Navier-Stokes equations

- Notice that the spherical part of the sub-grid scale stress tensor $\tau^{SGS}$ ,

$$\text{sph}\left(\tau^{SGS}\right) = \frac{1}{3}\text{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

- Can be written as,

$$\text{sph}\left(\tau^{SGS}\right) = \frac{1}{3}\text{tr}\left(\tau^{SGS}\right)\mathbf{I} = \frac{2}{3}k^{SGS}\mathbf{I}$$

- Where,

$$\text{tr}\left(\tau^{SGS}\right) = 2k^{SGS}$$

- Usually, the sub-grid scale kinetic energy $k^{SGS}$ is omitted because there is no model for it (same for the deviatoric part of the sub-grid scale stress tensor).
- Therefore, its contribution is usually absorbed into the pressure (as we will show in the next slide).

# LES equations – Filtered Navier-Stokes equations

- In the previous equations, the deviatoric part of the sub-grid scale stress tensor is often modeled using the Boussinesq hypothesis,

$$\text{dev}\left(\tau^{SGS}\right) = -2\nu_{SGS}\overline{\mathbf{S}}$$

- The spherical part of the sub-grid scale stress tensor can be added to the filtered pressure $\bar{p}$.
- Forming in this way the modified filtered pressure $\overline{P}$.

$$\overline{P} = \bar{p} + \text{sph}\left(\tau^{SGS}\right) = \bar{p} + \frac{1}{3}\text{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

Notice that we need to multiply the second term by the density, so we get the same pressure units

- The isotropic contribution (or the spherical part of the additive decomposition) in the modified filtered pressure $\overline{P}$ can be resolved, modeled, or neglected [1,2].
- By using the additive decomposition of the sub-grid scale stress tensor $\tau^{SGS}$, and substituting the Bousinessq hypothesis in the exact incompressible FNS equations, we can derive the solvable incompressible FNS equations.

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[2] A Vreman. Direct and large eddy simulation of the compressible turbulent mixing layer. PhD Thesis. University of Twente Enschede, Dept. of Applied Mathematics, 1995.

# LES equations – Filtered Navier-Stokes equations

- The incompressible solvable FNS equations can be written as,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\nabla \overline{P} + \nabla \cdot \left[ \left( \nu + \nu^{SGS} \right) \nabla \bar{\mathbf{u}} \right]$$

- Where,

We need to multiply this term by the density, so we get the same pressure units

$$\overline{P} = \bar{p} + \mathrm{sph}\left(\tau^{SGS}\right) = \bar{p} + \frac{1}{3}\mathrm{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

- At this point and similar to RANS/URANS turbulence modeling, the problem reduces to finding closure relations for the sub-grid scale viscosity $\nu^{SGS}$.

# LES equations – Filtered Navier-Stokes equations

- It is difficult to associate the sub-grid scale stress tensor $\tau^{SGS}$ directly with a physical process involving fluid motion because it is based on filtering rather than averaging.
- It is thus not surprising that the sub-grid scale stress tensor $\tau^{SGS}$ tends to be modeled formally without a detailed physical picture in mind.
- In fact, sub-grid scales models often use the Boussinesq hypothesis and the gradient diffusion hypothesis to model the sub-grid scale stress tensor $\tau^{SGS}$ and related terms.
- But instead of using the average velocities, the filtered velocities are used,

$$\tau^{SGS} - \frac{1}{3}\mathrm{tr}\left(\tau^{SGS}\right)\mathbf{I} = -2\nu_{SGS}\overline{\mathbf{S}}$$

- Where,

$$\overline{\mathbf{S}} = \frac{1}{2}\left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^{\mathrm{T}}\right) \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Remember, in the context of LES simulations the overbar denotes filtering.

# LES equations – Filtered Navier-Stokes equations

- In the previous explanation, we assumed that the Leonard stress term **L**, the cross-stress term **C**, and the sub-grid scale Reynolds stress term **R**, appearing in the triple decomposition, they were all modeled using the Boussinesq hypothesis, as follows,

$$\tau^{SGS} = \underbrace{\left(\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right)}_{\mathbf{L}} + \underbrace{\left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right)}_{\mathbf{C}} + \underbrace{\overline{\mathbf{u}'\mathbf{u}'}}_{\mathbf{R}} = -2\nu_{SGS}\overline{\mathbf{S}}$$

- It is also possible to only model the sub-grid scale Reynolds stress term **R**, and use approximate forms of the Leonard stress term **L** [1,2,3,4,5] and the cross-stress term **C** [3,4,5].
- This approach is known as scale similarity models [6,7], and it can account for backscatter effects.
- It is important to mention that different formulations of the approximate forms of the Leonard stress term **L** and the cross-stress term **C** exist depending of the particular filtering function used.

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