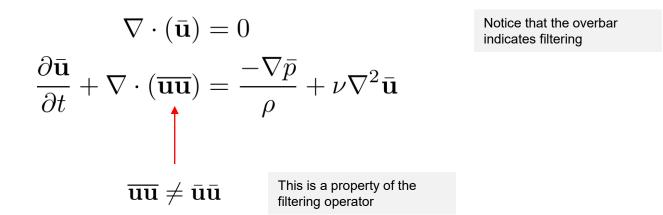
- As for the incompressible RANS/URANS equations, in LES the starting point are the exact incompressible Navier-Stokes equations or NSE.
- Using vector notation, the NSE can be written as follows,

$$\nabla \cdot (\mathbf{u}) = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

• Using index notation, the NSE can be written as follows,

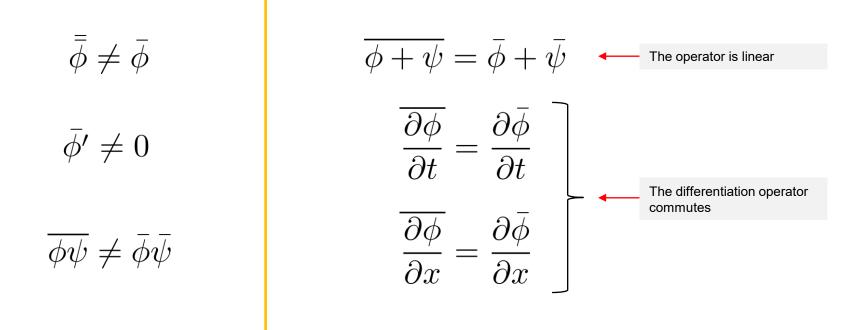
$$\frac{\partial u_i}{\partial x_i} = 0$$
$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j}$$

- Let us derive the filtered Navier-Stokes equations or FNS.
- The first step is to apply the filtering operator directly to the primitive variables.
- By doing so, we obtain the following set of equations,



- In this set of equations, we cannot solve the system for both $\, \bar{\mathbf{u}} \,$ and $\, \overline{\mathbf{uu}} \,$.
- Therefore, we need to manipulate this set of equations to get a new set of equations expressed only in function of $\,\bar{\mathbf{u}}$.
- Remember, the overbar indicates filtering.

• When deriving the filtered Navier-Stokes equations, we should be aware of the following filtering properties or rules.



- Remember, the overbar represents a spatial filter.
- Notice that the filtering properties in the left column are very different from the averaging rules used in RANS/URANS.

- To express the non-linear term \overline{uu} in function of \bar{u} , we can add and subtract the term $\bar{u}\bar{u}$ to the left-hand side of the filtered NSE equations.
- By doing so, we obtain the following set of equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}} + \bar{\mathbf{u}} \bar{\mathbf{u}} - \bar{\mathbf{u}} \bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}}$$

• After some algebra, we arrive to the exact filtered Navier-Stokes equations (FNS),

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

$$\uparrow$$

$$- (\overline{\mathbf{u}}\bar{\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}$$

• At the end of the day, the FNS equations are written as follows,

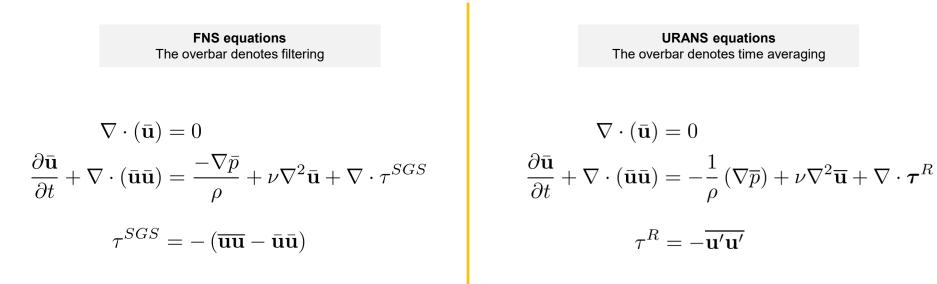
$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

• Where the sub-grid scale stress tensor τ^{SGS} is an apparent stress that arises from the filtering operation and is equivalent to,

$$au^{SGS} = -\left(\overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right) \qquad \text{or} \qquad au^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}}\bar{\mathbf{u}}$$

- This tensor represents the effect of filtered scales.
- As for the RANS/URANS equations, the sub-grid scale stress tensor au^{SGS} requires modeling.

Notice that the FNS and the URANS equations are very similar,

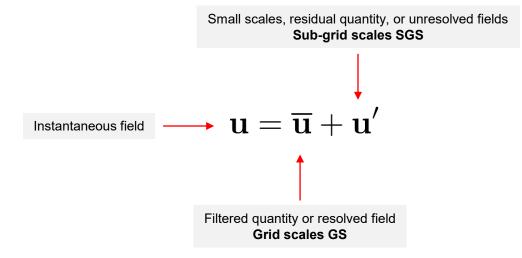


- The main differences are,
 - The FNS equations are filtered in space.
 - Whereas the URANS equations are time averaged.
 - Also, the apparent stress appearing in both the FNS and URANS equations are different.
 - Nevertheless, they both represent the smallest scales and they both need to be modelled.

- The sub-grid scale stress tensor au^{SGS} is given by,

 $\tau^{SGS} = -\left(\overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}}\right) \qquad \text{or} \qquad \tau^{SGS} = \overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}}$

- This term is analogous to the Reynolds stress tensor in the URANS/RANS equations.
- Let us expand this tensor by substituting the following expression (LES decomposition),



• After some algebra and by grouping some terms, we obtain the following equation (an expansion of the sub-grid scale tensor),

$$\tau^{SGS} = (\overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}}) + (\overline{\mathbf{u}}\mathbf{u'} + \overline{\mathbf{u'}\overline{\mathbf{u}}}) + \overline{\mathbf{u'}\mathbf{u'}}$$

- This decomposition is known as the triple decomposition.
- Notice that according to the filtering properties, the filtered sub-grid scale is not equal to zero, that is,

$$\bar{\phi'} \neq 0$$

 The triple decomposition is a little bit more complex than the definition of sub-grid scale stress tensor arising from the filtering operation, namely,

$$au^{SGS} = -\left(\overline{\mathbf{u}\mathbf{u}} - ar{\mathbf{u}}ar{\mathbf{u}}
ight)$$
 or $au^{SGS} = ar{\mathbf{u}}ar{\mathbf{u}} - ar{\mathbf{u}}ar{\mathbf{u}}$

- As it takes into account interactions between all scales, that is, resolved (grid scales) and unresolved (sub-grid scales).
- Remember, the overbar represents a spatial filter.

• The sub-grid scale stress tensor τ^{SGS} in the triple decomposition represents the effect of filtered scales and small scales, and can be written as,

$$\tau^{SGS} = \underbrace{(\overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}})}_{\mathbf{L}} + \underbrace{(\overline{\mathbf{u}}\mathbf{u'}}_{\mathbf{C}} + \overline{\mathbf{u'}\overline{\mathbf{u}}})}_{\mathbf{C}} + \underbrace{\overline{\mathbf{u'}\mathbf{u'}}}_{\mathbf{R}} = \mathbf{L} + \mathbf{C} + \mathbf{R}$$

- Where L is called the Leonard stresses, C is the called the cross-stress term, and R is called the sub-grid scale Reynolds stress (equivalent to the Reynolds stress tensor).
- The Leonard stresses (L) involves only the resolved quantities, and therefore it can be computed.
- The cross-term stresses (C) and SGS Reynolds stresses (R), involve unresolved scales and must be modeled.
- The cross-term stress represents the interaction of resolved and unresolved scales, whereas the SGS Reynolds stress represents the interaction of unresolved scales.
- It is important to mention that this decomposition is not unique.
- In this case we used the triple decomposition (or Leonard decomposition), which is often used with scale similarity models.
- At this point, the problem is how to model the cross-term stress and the sub-grid scale Reynolds stress.

 At the end of the day, these are the exact incompressible filtered Navier-Stokes equations to be used with LES models,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

• Where we can use any of the following definitions of the sub-grid scale stress tensor $\, au^{SGS}$,

$$\tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}}\bar{\mathbf{u}}$$

$$\tau^{SGS} = (\overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}}) + (\overline{\mathbf{u}}\mathbf{u'} + \overline{\mathbf{u'}\overline{\mathbf{u}}}) + \overline{\mathbf{u'u'}}$$

• At this point, we need to introduce models to approximate τ^{SGS} (similar to RANS).

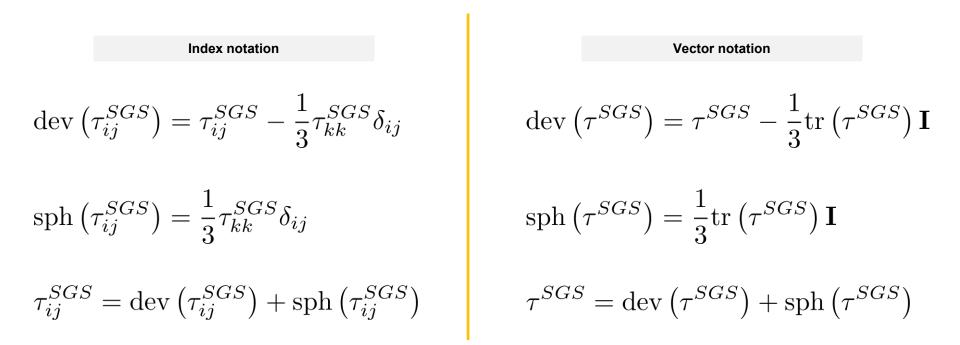
- From this point on, let us assume that we want to model the whole sub-grid scale stress tensor τ^{SGS} , such as,

$$\tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}}\bar{\mathbf{u}} = -2\nu_{SGS}\overline{\mathbf{S}}$$

$$\tau^{SGS} = (\overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}}) + (\overline{\mathbf{u}}\overline{\mathbf{u}'} + \overline{\mathbf{u}'}\overline{\mathbf{u}}) + \overline{\mathbf{u}'}\overline{\mathbf{u}'} = -2\nu_{SGS}\overline{\mathbf{S}}$$

- Notice that we are using the Boussinesq hypothesis, and that we are lumping all terms into this approximation.
- That is, we are not modeling individual effects (as it might be required for the triple decomposition).

• The sub-grid scale stress tensor τ^{SGS} can be decomposed into a deviatoric part (anisotropic part) and a spherical part (isotropic part), as follows,



- This procedure is known as additive decomposition of a second rank tensor.
- Notice that the deviatoric part (dev) is traceless.
- This kind of decomposition is often used in continuous mechanics.

• Using the additive decomposition of the sub-grid scale stress tensor, the exact FNS equations can be rewritten as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-1}{\rho}\nabla \overline{P} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \operatorname{dev}\left(\tau^{SGS}\right)$$

• Where,

$$\operatorname{dev}\left(\tau^{SGS}\right) = \tau^{SGS} - \frac{1}{3}\operatorname{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

$$\overline{P} = \overline{p} + \operatorname{sph}\left(\tau^{SGS}\right) = \overline{p} + \frac{1}{3}\operatorname{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

 In the previous equations, the deviatoric part of the sub-grid scale stress tensor is often modeled using the Boussinesq hypothesis,

$$\operatorname{dev}\left(\tau^{SGS}\right) = -2\nu_{SGS}\overline{\mathbf{S}}$$

Expanding and rearranging this equation, we obtain the following relationship,

$$\tau^{SGS} = -2\nu_{SGS}\overline{\mathbf{S}} + \frac{1}{3}\mathrm{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

- The second term in the right-hand side perform the same function as the equivalent term added in the RANS/URANS equation $(2/3\rho k I)$.
- It guarantees that the trace of the right-hand side is equal to the trace of the left-hand side.
- For an incompressible flow, the trace of the strain rate is zero; therefore, the identity holds.

- Notice that the spherical part of the sub-grid scale stress tensor $\, au^{SGS}$,

$$\operatorname{sph}\left(\tau^{SGS}\right) = \frac{1}{3}\operatorname{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

• Can be written as,

$$\operatorname{sph}\left(\tau^{SGS}\right) = \frac{1}{3}\operatorname{tr}\left(\tau^{SGS}\right)\mathbf{I} = \frac{2}{3}k^{SGS}\mathbf{I}$$

• Where,

$$\operatorname{tr}\left(\tau^{SGS}\right) = 2k^{SGS}$$

- Usually, the sub-grid scale kinetic energy k^{SGS} is omitted because there is no model for it (same for the deviatoric part of the sub-grid scale stress tensor).
- Therefore, its contribution is usually absorbed into the pressure (as we will show in the next slide).

 In the previous equations, the deviatoric part of the sub-grid scale stress tensor is often modeled using the Boussinesq hypothesis,

$$\operatorname{dev}\left(\tau^{SGS}\right) = -2\nu_{SGS}\overline{\mathbf{S}}$$

- The spherical part of the sub-grid scale stress tensor can be added to the filtered pressure $ar{p}$.
- Forming in this way the modified filtered pressure \overline{P} .

$$\overline{P} = \overline{p} + \operatorname{sph}\left(\tau^{SGS}\right) = \overline{p} + \frac{1}{3}\operatorname{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

Notice that we need to multiply the second term by the density, so we get the same pressure units

- The isotropic contribution (or the spherical part of the additive decomposition) in the modified filtered pressure \overline{P} can be resolved, modeled, or neglected [1,2].
- By using the additive decomposition of the sub-grid scale stress tensor τ^{SGS} , and substituting the Bousinessq hypothesis in the exact incompressible FNS equations, we can derive the solvable incompressible FNS equations.

 ^[1] R. Rogallo, P. Moin. Numerical simulation of turbulent flows. Annu. Rev. Fluid Mech., 16:99-137, 1984.
 [2] A Vreman. Direct and large eddy simulation of the compressible turbulent mixing layer. PhD Thesis. University of Twente Enschede, Dept. of Applied Mathematics, 1995.

• The incompressible solvable FNS equations can be written as,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\nabla \overline{P} + \nabla \cdot \left[\left(\nu + \nu^{SGS} \right) \nabla \bar{\mathbf{u}} \right]$$

• Where,

We need to multiply this term by the density, so we get the same pressure units

$$\overline{P} = \overline{p} + \operatorname{sph}\left(\tau^{SGS}\right) = \overline{p} + \frac{1}{3}\operatorname{tr}\left(\tau^{SGS}\right)\mathbf{I}$$

• At this point and similar to RANS/URANS turbulence modeling, the problem reduces to finding closure relations for the sub-grid scale viscosity ν^{SGS} .

- It is difficult to associate the sub-grid scale stress tensor τ^{SGS} directly with a physical process involving fluid motion because it is based on filtering rather than averaging.
- It is thus not surprising that the sub-grid scale stress tensor τ^{SGS} tends to be modeled formally without a detailed physical picture in mind.
- In fact, sub-grid scales models often use the Boussinesg hypothesis and the gradient diffusion hypothesis to model the sub-grid scale stress tensor τ^{SGS} and related terms.
- But instead of using the average velocities, the filtered velocities are used,

$$\tau^{SGS} - \frac{1}{3} \operatorname{tr} \left(\tau^{SGS} \right) \mathbf{I} = -2\nu_{SGS} \overline{\mathbf{S}}$$

• Where,

$$\overline{\mathbf{S}} = \frac{1}{2} \left(\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^{\mathrm{T}} \right) \qquad \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remember, in the context of LES simulations the overbar denotes filtering.

In the previous explanation, we assumed that the Leonard stress term L, the cross-stress term C, and the sub-grid scale Reynolds stress term R, appearing in the triple decomposition, they were all modeled using the Boussinesq hypothesis, as follows,

$$\tau^{SGS} = \underbrace{(\overline{\mathbf{u}}\overline{\mathbf{u}} - \overline{\mathbf{u}}\overline{\mathbf{u}})}_{\mathbf{L}} + \underbrace{(\overline{\mathbf{u}}\mathbf{u'}}_{\mathbf{C}} + \overline{\mathbf{u'}\overline{\mathbf{u}}})}_{\mathbf{C}} + \underbrace{\overline{\mathbf{u'}\mathbf{u'}}}_{\mathbf{R}} = -2\nu_{SGS}\overline{\mathbf{S}}$$

- It is also possible to only model the sub-grid scale Reynolds stress term R, and use approximate forms of the Leonard stress term L [1,2,3,4,5] and the cross-stress term C [3,4,5].
- This approach is known as scale similarity models [6,7], and it can account for backscatter effects.
- It is important to mention that different formulations of the approximate forms of the Leonard stress term L and the cross-stress term C exist depending of the particular filtering function used.

^[1] A. Leonard. Energy cascade in large eddy simulations of turbulent fluid flow. Advances in Geophysics, 18, 237, 1974.

^[2] N. Mansour, P. Moin, W. Reynolds, J. Ferziger. Improved Methods for Large Eddy Simulations of Turbulence. In: Turbulent Shear Flows I. Springer, 1979.

^[3] R. Clark, J. Ferziger, W. Reynolds. Evaluation of subgrid scale models using an accurately simulated turbulent flow. Journal of Fluid Mechanics, 91, 1, 1979.

^[4] R. Peyret, E. Krause. Advanced Turbulent Flow Computations. CISM Courses and Lecture No.395, International Centre for Mechanical Sciences, Vienna, 2000.

^[5] S. Shaanan, J. Ferziger, W. Reynolds. Numerical simulation of turbulence in the presence of shear. Report No. TF-6, Dept. Mech. Eng., Stanford University, 1975.

^[6] M. Germano. A proposal for a redefinition of the turbulent stresses in the filtered Navier-Stokes equations. Phys. Fluids, 29(7). 1986.

^[7] J. Bardina, J. Ferziger, W. Reynolds. Improved subgrid scale models for large eddy simulations. AIAA Paper 80-1357. 1980.