Roadmap to Lecture 6

Appendix 1

1. **More turbulence models**

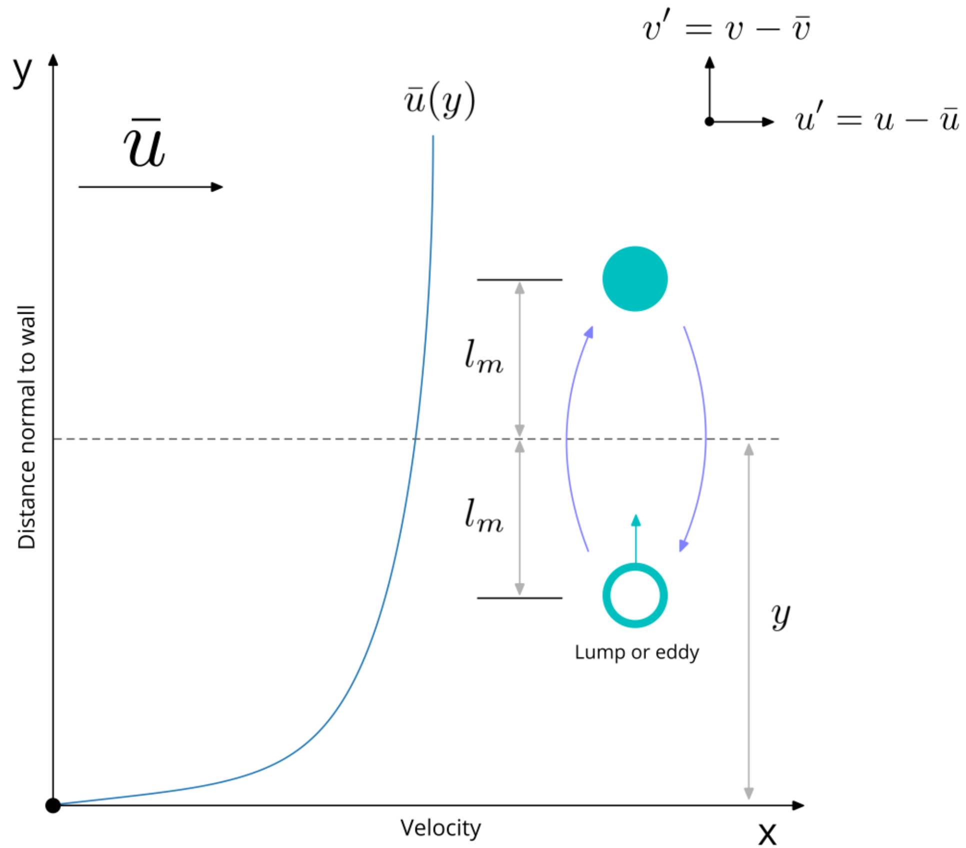
Algebraic models – Review of the Prandtl mixture length model

More turbulence models – The Prandtl mixture length model

- Prandtl in 1925 [1,2] introduced the concept of mixing length l_m theory that closely relates to eddy viscosity concept and formed the basis for all turbulent modeling efforts.
- The Prandtl mixing length is defined as the average distance travelled by a lump of fluid (or an eddy) in the normal direction across the flow, as illustrated in the figure.
- This concept is similar to the mean free path used in the kinetic theory of gases.
- In the 2D pure shear boundary layer shown in the figure, a fluid lump travels a distance of $2l_m$, exchanging momentum, and this results in a turbulent shear stress.
- The velocities at the two locations $(y + l_m)$ and $(y - l_m)$ are given as

$$\bar{u}(y+l_m) \approx \bar{u}(y) + l_m \frac{\partial \bar{u}}{\partial y}$$

$$\bar{u}(y-l_m) \approx \bar{u}(y) - l_m \frac{\partial \bar{u}}{\partial y}$$



[1] L. Prandtl. Report on Investigation of Developed Turbulence. NACA-TM-1231 (translation from German version), 1925.
[2] P. Bradshaw. Possible origin of Prandtl's mixing-length theory. Nature. 249 (6): 135–136, 1974.

More turbulence models – The Prandtl mixture length model

- Where the velocity fluctuation u' (in any of the locations $\pm l_m$) is given by,

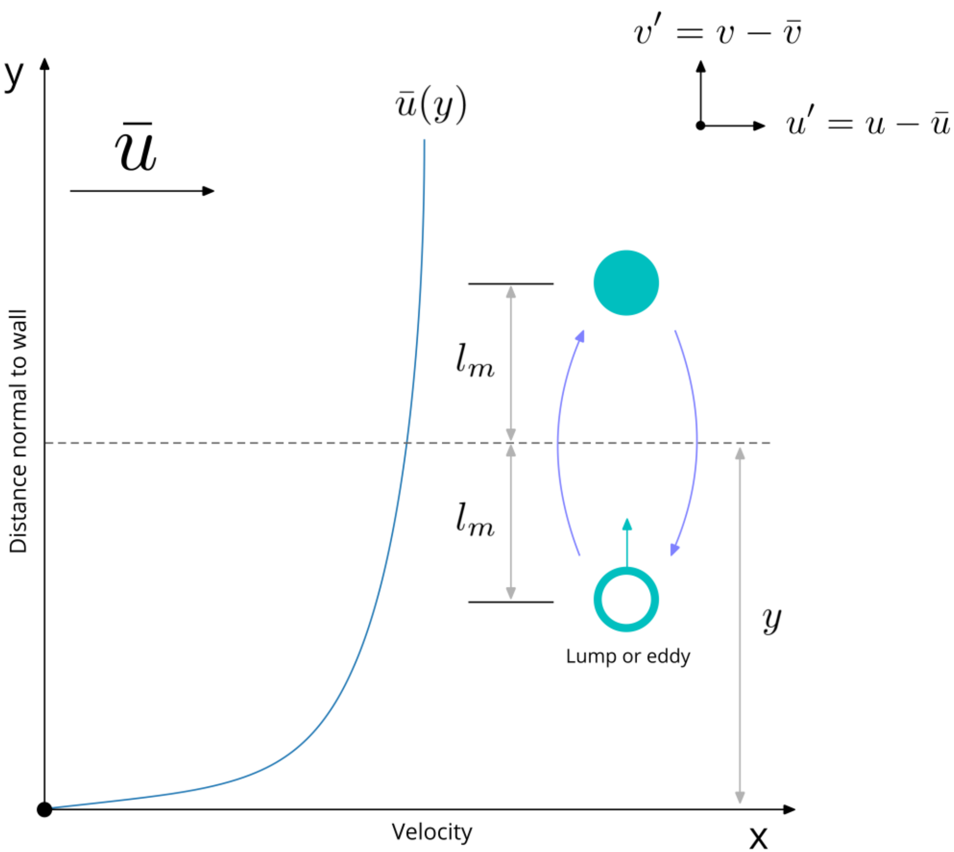
$$u' \approx l_m \frac{\partial \bar{u}}{\partial y}$$

- The turbulent velocity fluctuation, u', is proportional to the mean of the velocities u $(y + l_m)$ and u $(y - l_m)$ at locations $(y + l_m)$ and $(y - l_m)$ across the mixing plane.
- Here, the assumption is that a lump of fluid from a certain layer is displaced over a distance in the transverse direction.
- Then, the difference in velocity of the fluid lump (eddy) will differ from its surrounds by an amount.

$$l_m \frac{\partial \bar{u}}{\partial y}$$

- Or,

$$\bar{u} = \bar{u}(y) + u'$$



More turbulence models – The Prandtl mixture length model

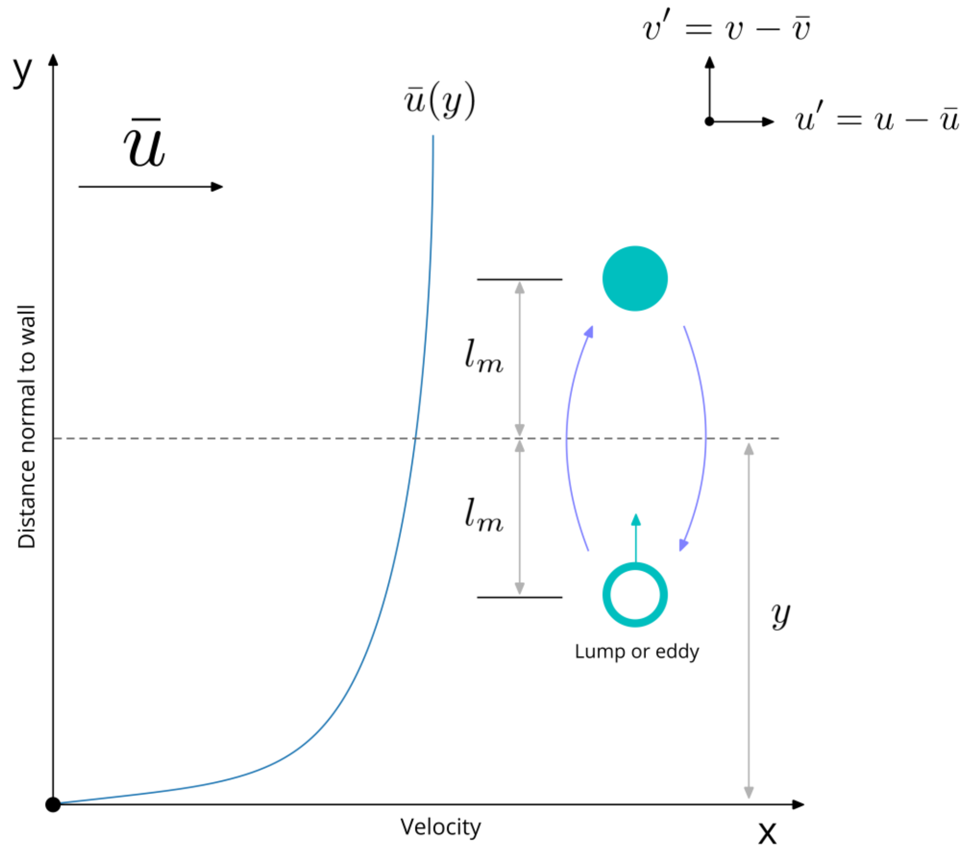
- Based on this concept and assuming v' of the same order as u', the turbulent stress and turbulent eddy viscosity are expressed in terms of **Prandtl mixing length hypothesis** in the following manner,

$$\tau_t = -\rho\overline{u'v'} = \mu_t \frac{\partial u}{\partial y} = \rho l_m^2 \left(\frac{\partial u}{\partial y}\right)^2$$

- Where μ_t is the turbulent eddy viscosity and is defined as,

$$\mu_t = \rho l_m^2 \left|\frac{\partial u}{\partial y}\right|$$

- And l_m is the turbulent mixing length, yet to be specify.
- This quantity conceptually defines the characteristic distance a fluid parcel is transported across the flow by turbulent fluctuations before becoming mixed with the surrounding fluid (hence taking on the flow properties at the new level).



More turbulence models – The Prandtl mixture length model

- Prandtl proposed that along with the strong variation of turbulent eddy viscosity, μ_t, within the boundary layer, the mixing length, l_m, also varies throughout the boundary layer following the relation,

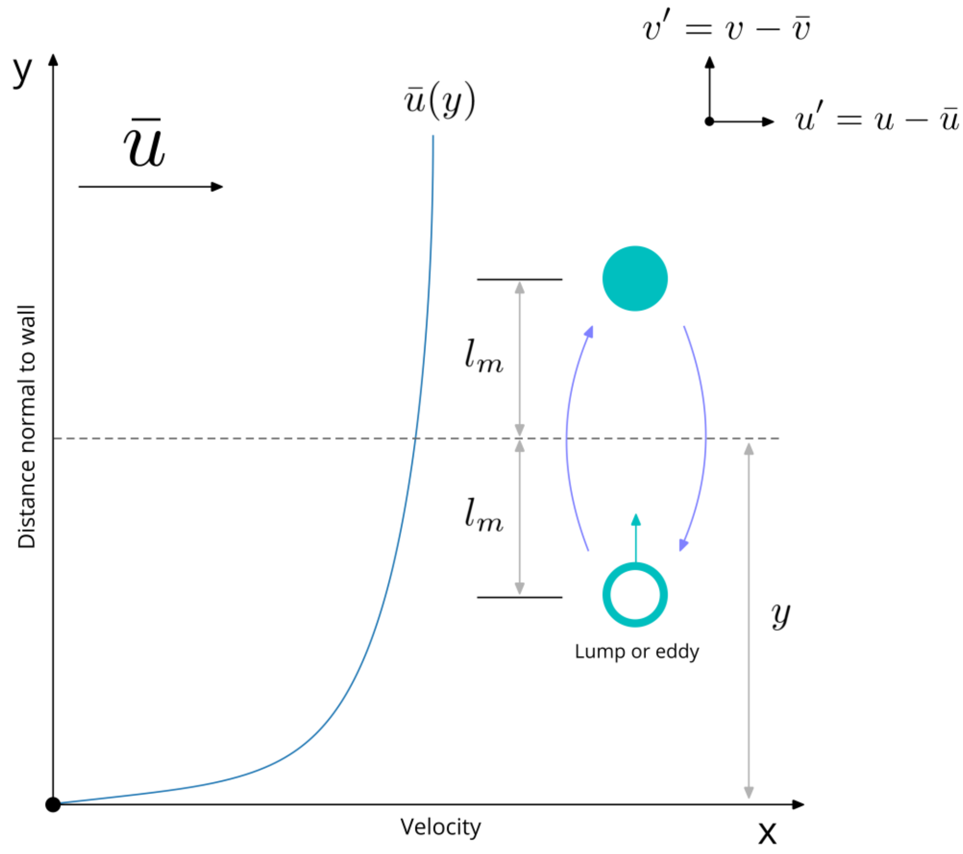
$$l_m \approx y \qquad \text{or} \qquad l_m \approx Ky$$

- Where K is a constant of proportionality.
- Substituting this expression into Prandtl mixing length equation, the turbulent eddy viscosity is expressed as,

$$\tau_t = -\rho\overline{u'v'} = \rho K^2 y^2 \left(\frac{\partial u}{\partial y}\right)^2$$

- Further assuming that in the near-wall region,

$$\tau_t = \tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$



More turbulence models – The Prandtl mixture length model

- After substitution and rearranging, we obtain the following expression,

$$\tau_w = \rho K^2 y^2 \left(\frac{\partial u}{\partial y}\right)^2$$

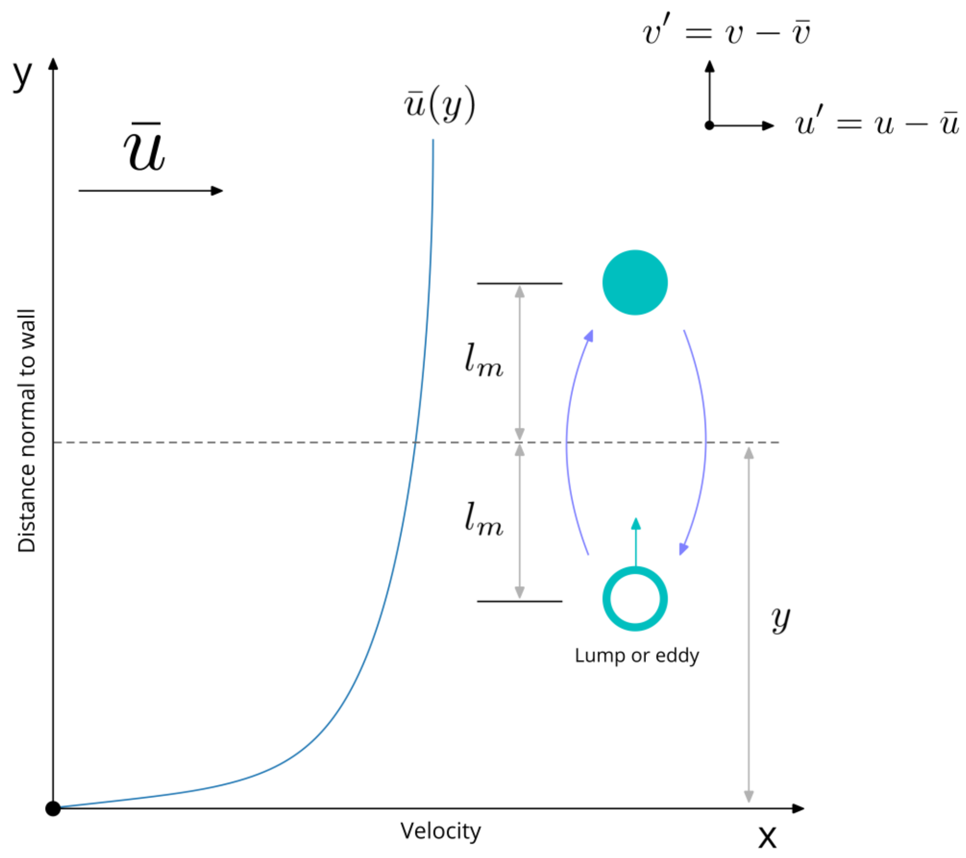
- Integration of this expression yields to the following velocity distribution profile,

$$u = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\ln y + E$$

- Where E is the constant of integration and τ_w is equal to,

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

- The equation of the velocity profile distribution matched well experimental data in the near-wall region of the turbulent boundary layer, except in the so-called laminar sublayer region close to the wall where viscous shear is dominant.



More turbulence models – The Prandtl mixture length model

- As we have done so far, many of the previous derivations are based on dimensional analysis.

$$\nu \sim \text{Velocity scale} \times \text{Length scale}$$

$$\text{Length scale} = l_m$$

$$\text{Velocity scale} \sim l_m \frac{\partial u}{\partial y}$$

- The velocity scale represent the velocity fluctuation u'.
- Using the previous scales, the eddy viscosity is equal to,

$$\mu_t = \rho l_m^2 \left| \frac{\partial u}{\partial y} \right|$$

- The turbulent stress and total stress can be computed as follows,

Based on the Boussinesq hypothesis

$$\tau_t = -\rho \overline{u'v'} = \mu_t \frac{\partial u}{\partial y}$$

Turbulent stress

$$\tau_{total} = \mu \frac{\partial u}{\partial y} + \mu_t \frac{\partial u}{\partial y} = \mu \frac{\partial u}{\partial y} + \rho l_m^2 \left(\frac{\partial u}{\partial y} \right)^2$$

Laminar stress

More turbulence models – The Prandtl mixture length model

- The Prandtl mixture length model is very simple and elegant.
- But its range of applicability is very limited.
- It is only accurate in 2D pure shear boundary layers of fully developed flows in flat surfaces.
- Also, as previously stated, it is only accurate in the log-layer of the boundary layer.
- More elaborate forms of l_m boundary layers have been proposed.
- For example, Van Driest proposed [1],

$$l_m = \kappa y \left[1 - e^{-y^+/A^+} \right]$$

l_m is use as the length scale to compute the turbulent viscosity

$$A^+ = 26$$

This coefficient depends on the pressure gradient

- This function is commonly used in mixing length models (or zero equation models) to improve their predicting capabilities.
- This model is also classified as an incomplete model because it requires the definition of $l_m(x)$, and the specification of l_m inevitably depends on the geometry.

[1] E. van Driest. On Turbulent Flow Near a Wall. J. Aero. Sci., 23, pp. 1007-1011, 1956.

More turbulence models – The Prandtl mixture length model

- For example, in the zero equation Cebeci-Smith model [1], the turbulent viscosity close to the walls is computed as follows,

$$\nu_t = l_m^2 \left(\frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} \right)^{1/2}$$

$$l_m = \kappa y \left[1 - e^{-y^+/A^+} \right]$$

- In the zero equation Baldwin-Lomax model [2], the turbulent viscosity close to the walls is computed as follows,

$$\nu_t = l_m^2 \left(\bar{\Omega} \bar{\Omega} \right)^{1/2}$$

$$l_m = \kappa y \left[1 - e^{-y^+/A^+} \right]$$

References:
[1] T. Cebeci, A.M.O. Smith. Analysis of Turbulent Boundary Layers. Academic Press, 1974.
[2] B. Baldwin, H. Lomax. Thin-layer Approximation and Algebraic Model for Separated Turbulent Flows. AIAA Paper No. 78-257, 1978.

More turbulence models – The Prandtl mixture length model

- The mixing length idea is also used in some LES models. For example, in the Smagorisky model [3], the turbulent viscosity close to the walls is computed as follows,

$$\nu_t = l_m^2 \left(2\overline{S}_{ij}\overline{S}_{ij}\right)^{1/2}$$

$$l_m = C_s \Delta$$

A constant times the grid spacing

- As the length scale is defined in an analytical way in this turbulence model, the user only needs to define the initial conditions and freestream boundary conditions values.
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
    - https://turbmodels.larc.nasa.gov/

References:
[1] J. Smagorinsky. General circulation experiments with primitive equations. I. The basic experiment, Monthly Weather Review, Vol. 91, pp. 99-164, 1963.

RNG $k - \epsilon$

More turbulence models – The RNG $k-\epsilon$ model

- This is a variant of the standard $k-\epsilon$ turbulence model.
- It is based on the Boussinesq hypothesis (linear eddy viscosity model or EVM).
- The development of this model can be attributed to Yakhot and Orszag [1,2].
- This model was developed using a statistical technique called renormalization group theory.
- Generally speaking, this model is more accurate and reliable for a wider class of flows than the standard $k-\epsilon$ model.
- In this model, the transport equations of the TKE and dissipation rate are the same as for the standard $k-\epsilon$ turbulence model.
- The main difference arises from the definition of the closure constants, where strain dependent terms are added in order to reduce the length scale at high strain rates.
- The RNG model improves the accuracy for rapidly strained and swirling flows.

References:
[1] V. Yakhot, S. Orszag. Renormalization group analysis of turbulence: 1. Basic theory. Journal of Scientific Computing. Vol. 1, pp. 3-51. 1986.
[2] V. Yakhot, S.A. Orszag, S. Thangam, and C.G. Speziale. Development of Turbulence Models for Shear Flows by a Double Expansion technique. Physics of Fluids A Fluid Dynamics, 4(7), 1992.

More turbulence models – The RNG $k - \epsilon$ model

- In this model, the transport equations of the TKE and dissipation rate are defined as follows,

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'} \right)$$

- The kinematic eddy viscosity is computed as follows,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- This model uses the following auxiliary relationships,

$$\omega = \frac{\epsilon}{C_\mu k} \qquad l = \frac{C_\mu k^{3/2}}{\epsilon}$$

- The closure coefficients in this model are defined as follows,

$$C_{\epsilon 2} = \tilde{C}_{\epsilon 2} + \frac{C_\mu \lambda^3 \left(1 - \dfrac{\lambda}{\lambda_0}\right)}{1 + \beta\lambda^3}$$

$$\lambda = \frac{k}{\epsilon}\sqrt{2S_{ij}S_{ij}}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

Index notation

$$\lambda = \frac{k}{\epsilon}\sqrt{2\mathbf{S}:\mathbf{S}}$$

$$\mathbf{S} = \frac{1}{2}\left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right)$$

Vector notation

$$C_{\epsilon 1} = 1.42 \qquad \tilde{C}_{\epsilon 1} = 1.68 \qquad C_\mu = 0.85$$

$$\sigma_k = 0.72 \qquad \sigma_\epsilon = 0.72 \qquad \beta = 0.012 \qquad \lambda_0 = 4.38$$

More turbulence models – The RNG $k-\epsilon$ model

- This model is based on the standard $k-\epsilon$ model; therefore, it is wall modelling.
- However, its range of applicability can be extended to lowRE cases by adding blending functions [1,2,3,4].
- The initial conditions and the free-stream boundary conditions are computed in the same way as for the highRE and lowRE $k-\epsilon$ models
- The wall boundary conditions are set in the same way as for the highRE and lowRE $k-\epsilon$ models.
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
  - https://turbmodels.larc.nasa.gov/

References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Realizable $k - \epsilon$

More turbulence models – The Realizable $k-\epsilon$ model

- This is a variant of the standard $k-\epsilon$ turbulence model.
- We will review the model developed by Shih et. al [1,2,3], based on the Boussinesq hypothesis (linear eddy viscosity model or EVM).
- This model differs from the standard and RNG $k-\epsilon$ versions in two different ways,
  - The coefficient C_μ in the eddy viscosity definition is no longer constant. The model uses a different eddy-viscosity formulation which is based on several realizability constraints for the turbulent Reynolds stresses.
  - A modified transport equation for the dissipation rate has been derived from an exact equation for the transport of the mean-square vorticity fluctuation.
- Generally speaking, this model is more accurate and reliable for a wider class of flows than the standard $k-\epsilon$ model.
- This model provides superior performance for flows involving strong streamline curvature, rotation, vortices, boundary layers under strong adverse pressure gradients, separation, and recirculation.
- One limitation of this model is that it produces nonphysical turbulent viscosities in situations when the computational domain contains both rotating and stationary fluid zones like in the use of multiple reference frames or rotating sliding meshes.

References:
[1] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New k-epsilon Eddy-Viscosity Model for High Reynolds Number Turbulent Flows. Computers Fluids. 24(3). 227–238. 1995.
[2] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New k-epsilon Eddy-Viscosity Model for High Reynolds Number Turbulent Flows – Model Development and Validation. NASA TM 106721. 1994.
[3] T. Shih, J. Zhu. A New Reynolds Stress Algebraic Equation Model. NASA TM 106644. 1994.

More turbulence models – The Realizable $k - \epsilon$ model

- The term realizable means that the model satisfies certain mathematical constraints on the Reynolds stresses, consistent with the physics of turbulent flows. Neither the standard nor the RNG $k - \epsilon$ models are realizable.

$$\overline{u'^2}, \overline{v'^2}, \overline{w'^2} \geq 0$$

Normal Reynolds stresses are always positive

$$\frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}}, \frac{\overline{u'w'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{w'^2}}}, \frac{\overline{v'w'}}{\sqrt{\overline{v'^2}}\sqrt{\overline{w'^2}}} < 1$$

Schwartz inequality

- The most straightforward way to ensure the realizability, that is, positivity of normal stresses and Schwarz inequality for shear stresses, is to make C_μ variable by sensitizing it to the mean flow (mean deformation) and the turbulent quantities.
- Several studies have shown that the realizable $k - \epsilon$ model provides the best performance of all the $k - \epsilon$ model versions for several validations of separated flows and flows with complex secondary flow features.

More turbulence models – The Realizable $k-\epsilon$ model

- In this model, the transport equations of the TKE and dissipation rate are defined as follows,

$$\frac{\partial k}{\partial t} + \frac{\partial \left(k \bar{u}_j \right)}{\partial x_j} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$

Note:

$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'} \right)$

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial \left(\epsilon \bar{u}_j \right)}{\partial x_j} = C_1 S \epsilon - C_2 \frac{\epsilon^2}{k + \sqrt{\nu \epsilon}} + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right]$$

- The eddy viscosity is computed as follows,

$$\mu_t = \rho C_\mu \frac{k^2}{\epsilon}$$

- Here the coefficient C_μ is no longer constant.
- Note that in the transport equations of the turbulent quantities we included the effect of transport by molecular viscosity, which is not included in the original references [1,2].

References:
[1] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New k-epsilon Eddy-Viscosity Model for High Reynolds Number Turbulent Flows. Computers Fluids. 24(3). 227–238. 1995.
[2] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New k-epsilon Eddy-Viscosity Model for High Reynolds Number Turbulent Flows – Model Development and Validation. NASA TM 106721. 1994.

More turbulence models – The Realizable $k - \epsilon$ model

- In this model, C_μ is computed as follows,

$$C_\mu = \frac{1}{A_0 + A_s U^{(*)} \frac{k}{\epsilon}}$$

- Where the variable $U^{(*)}$ is defined as follows,

$$U^{(*)} = \sqrt{S_{ij}S_{ij} + \tilde{\Omega}_{ij}\tilde{\Omega}_{ij}}$$

$$\tilde{\Omega}_{ij} = \Omega_{ij} - 2\epsilon_{ijk}\omega_k$$

This extra rotation term many times is not included as is not compatible multiple reference frames

$$\Omega_{ij} = \overline{\Omega_{ij}} - \epsilon_{ijk}\omega_k$$

Mean rate-of-rotation tensor viewed in a moving reference frame with the angular velocity ω_k

- The parameter A_s is defined as follows,

$$A_S = \sqrt{6}\cos\varphi$$

$$\varphi = \frac{1}{3}\cos^{-1}\left(\sqrt{6}W\right)$$

More turbulence models – The Realizable $k - \epsilon$ model

- The closure coefficient C_1 us defined as follows,

$$C_1 = \max\left[0.43, \frac{\eta}{\eta + 5}\right] \qquad \eta = S\frac{k}{\epsilon} \qquad S = \sqrt{2S_{ij}S_{ij}}$$

- Finally, this model uses the following support relations and closure coefficients,

$$W = \frac{S_{ij}S_{jk}S_{ki}}{\tilde{S}^3} \qquad \tilde{S} = \sqrt{S_{ij}S_{ij}}$$

$$A_0 = 4.04 \qquad C_2 = 1.9 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.2$$

More turbulence models – The Realizable $k - \epsilon$ model

- This model is based on the standard $k - \epsilon$ model; therefore, it is wall modelling.
- However, as the coefficient C_μ is variable, this model can also be used with wall resolving meshes (with some limitations of course).
- However, its range of applicability can be extended to lowRE cases by adding blending functions [1,2,3,4].
- The initial conditions and the free-stream boundary conditions are computed in the same way as for the highRE and lowRE $k - \epsilon$ models
- The wall boundary conditions are set in the same way as for the highRE and lowRE $k - \epsilon$ models.
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
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[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

The Menter $k-\omega$ SST

More turbulence models – The Menter $k - \omega$ SST model

- The development of the $k - \omega$ SST turbulence model can be attributed to Menter [1,2].
- This method is an improvement of the BSL $k - \omega$ model, also developed by Menter [1,2].
- It is worth mentioning that the BSL and SST models are almost identical. Only one constant and the expression for turbulent eddy viscosity are different.
- These models address some of the deficiencies of the Wilcox 1988 [2] and the Wilcox 1998 [3] $k - \omega$ turbulence models.
- Namely, overly sensitive to the free-stream boundary conditions and erratic performance when dealing with strong adverse pressure gradient (actually, every single turbulence model struggles to deal with this situation).
- This method can be seen as an hybrid method, where the good properties of the standard $k - \epsilon$ model (away from the walls), are merged with the good properties of the Wilcox $k - \omega$ model (close to the walls), with a blending function to compute the asymptotic turbulent behavior between both models.
- Thus, the shortcoming of a model are compensated by the improved behavior of the other model and vice versa.
- And as an extra bonus, there is no need to use damping functions near the walls or difficulties when setting the wall boundary conditions.

References:
[1] F. Menter. Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications. **AIAA Journal**, Vol. 32, No. 8, 1994, 1598-1605, 1994
[2] F. Menter. Improved Two-Equation k-omega Turbulence Models for Aerodynamic Flows. NASA TM-103975, 1992.
[3] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[4] D. C. Wilcox. Turbulence Modeling for CFD. Second edition, DCW Industries, 1998.

- The solvable equations of the $k-\omega$ SST turbulence model [1,2] are the following ones,

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\mathbf{u}\omega) = \frac{\alpha}{\nu_t} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\omega} \right) \nabla \omega \right] + 2\left(1 - F_1\right) \frac{\sigma_{\omega 2}}{\omega} \nabla k \cdot \nabla \omega$$

$$\mu_t = \rho \frac{a_1 k}{\max\left(a_1 \omega, F_2 \Omega\right)}$$

$$\Omega = \sqrt{2 W_{ij} W_{ij}}$$

Magnitude of the vorticity tensor

$$W_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

Anti-symmetric part of the velocity gradient (vorticity tensor)

- This model features several closure coefficients, blending functions, and auxiliary relations.
- Many of the coefficients used in this model are computed by a blend function between the respective constants of the $k-\epsilon$ and $k-\omega$ models.

References:
[1] F. Menter. Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications. AIAA Journal, Vol. 32, No. 8, 1994, 1598-1605, 1994.
[2] F. Menter. Improved Two-Equation k-omega Turbulence Models for Aerodynamic Flows. NASA TM-103975, 1992..

More turbulence models – The Menter $k-\omega$ SST model

- Letting ϕ denote any one of the parameters $\alpha, \sigma_k, \sigma_\omega, \beta, \beta^*, \mu_t$, then each of these parameters varies between a near-wall state ϕ_1 and a far-wall state ϕ_2 according to,

$$\phi = F_1\phi_1 + (1 - F_1)\phi_2$$

- Where F_1 and arg_1 are defined as follows,

$$F_1 = \tanh(\arg_1^4) \qquad \arg_1 = \min\left[\max\left(\frac{\sqrt{k}}{\beta^*\omega d}, \frac{500\nu}{d^2\omega}\right), \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}d^2}\right]$$

- Notice that $0 \leq F_1 \leq 1$, and $arg_1 \geq 0$.
- In the equations, d is the distance from the wall. So, as d increases, the two expressions in the maximum of arg_1 become smaller as well the term that the maximum is being compared to.
- Thus, arg_1 diminishes with d, causing F_1 to approach zero and ϕ to approach the far-field value ϕ_2 (the coefficients of the $k-\epsilon$ turbulence model).
- The opposite behavior occurs as the wall is approached with,

$$\arg_1 \rightarrow \infty, \quad F_1 \rightarrow 1, \quad \phi \rightarrow \phi_1$$

- That is, we obtain the coefficients of the $k-\omega$ turbulence model.

More turbulence models – The Menter $k-\omega$ SST model

- The additional expressions and coefficients used in the formulation are,

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}, 10^{-20}\right)$$

$$F_2 = \tanh(\arg_2^2) \qquad \arg_2 = \max\left(\frac{2\sqrt{k}}{\beta^* \omega d}, \frac{500\nu}{d^2\omega}\right)$$

$$\alpha_1 = \frac{\beta_1}{\beta^*} - \frac{\sigma_{\omega 1}\kappa^2}{\sqrt{\beta^*}} \qquad \alpha_2 = \frac{\beta_2}{\beta^*} - \frac{\sigma_{\omega 2}\kappa^2}{\sqrt{\beta^*}}$$

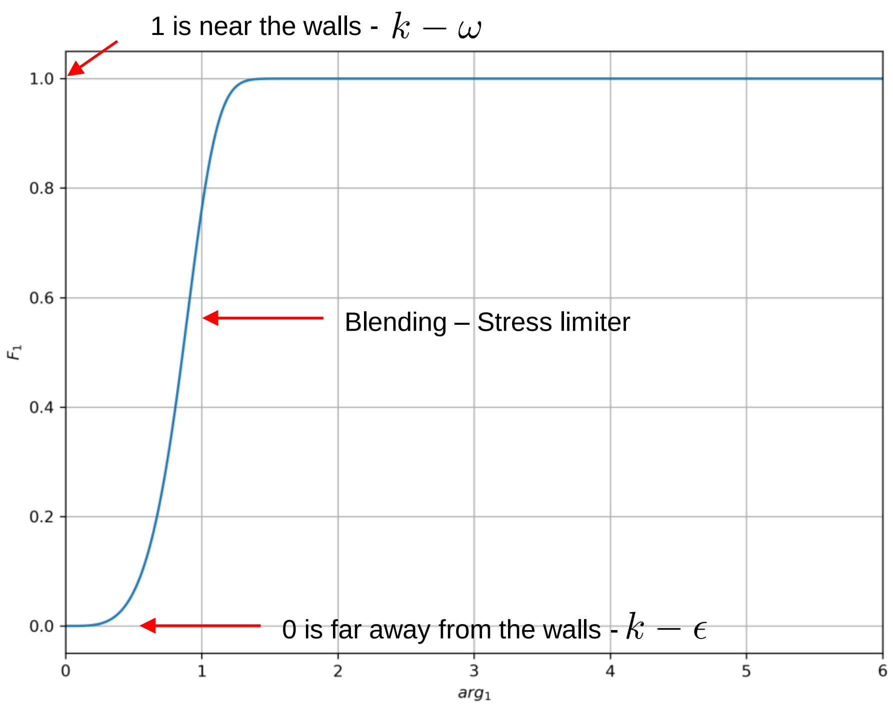
$$\sigma_{k1} = 0.85, \quad \sigma_{k2} = 1.0, \quad \sigma_{\omega 1} = 0.5, \quad \sigma_{\omega 2} = 0.856,$$

$$\beta_1 = 0.075, \quad \beta_2 = 0.0828, \quad \beta^* = 0.09, \quad \kappa = 0.41, \quad a_1 = 0.31$$

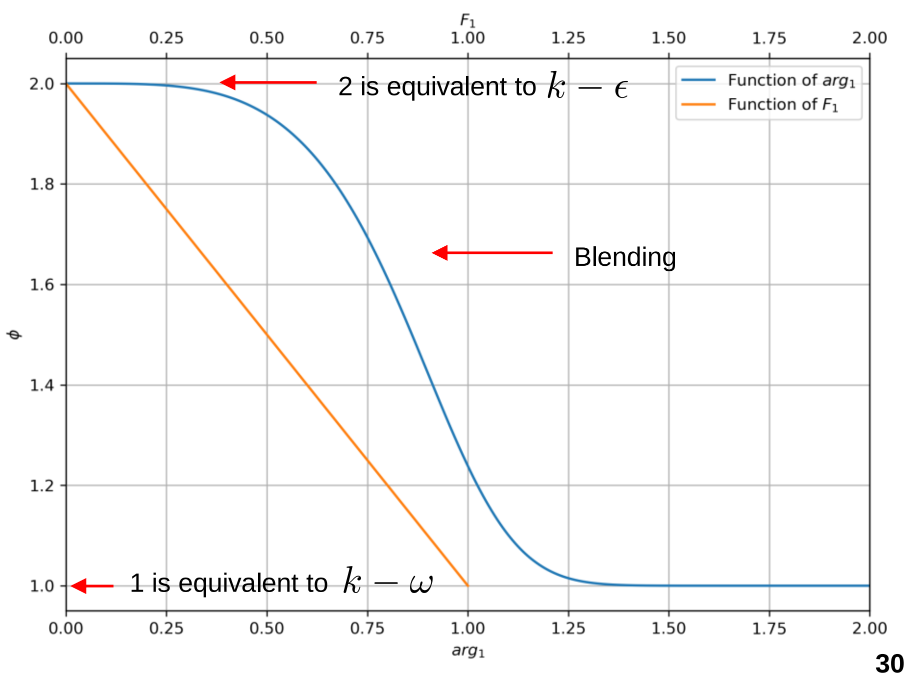
$\approx \sqrt{C_\mu}$ or $\approx \sqrt{\beta^*}$

More turbulence models – The Menter $k-\omega$ SST model

- Plot of the function F_1 as a function of arg_1.
- When $F_1 = 0$ the formulation is in the far-field conditions. That is, the formulation uses the $k-\epsilon$ model.
- And when $F_1 = 1$ the formulation is in the near wall region. That is, the formulation uses the $k-\omega$ model.
- The function F_1 represents a blending between the two turbulence models.



- Plot of the parameters ϕ and F_1 in function of arg_1.
- When F1 = 0 we use the parameters of the $k-\epsilon$ model.
- This function indicates how the parameters vary between a near-wall state and a far-wall state.
- The function ϕ blends the parameters of the two turbulence models.



More turbulence models – The Menter $k-\omega$ SST model

- The SST model and the Wilcox $k-\omega$ model share many common parameters.
- It is worth noting that the SST model conforms to the standard $k-\epsilon$ model away from walls.
- And close to the walls it uses a slightly modified version of the Wilcox 1988 $k-\omega$ model, which is y^+ insensitive.
- By using the following relationship,

$$\epsilon = \beta^* \omega k$$

- Taking its substantial derivative and substituting for the k and ω derivatives using the solvable equations of the SST model, gives a differential equation for ϵ of essentially the same form as the solvable equation of the dissipation rate ϵ of the generalized $k-\epsilon$ turbulence model.
- Substituting the far-field form of the constants into the expression yields an equation for the turbulent dissipation rate ϵ that conforms to that in the $k-\epsilon$ closure with only small differences.

More turbulence models – The Menter $k-\omega$ SST model

- The $k-\omega$ SST model is considered by many authors the most efficient and general RANS/URANS turbulence model.
- Therefore, it is strongly recommended to use this model.
- This model performs quite well for a wide variety of applications, to name a few,
  - Adverse pressure gradients.
  - Separated flows.
  - Turbulent heat transfer and mass transfer.
  - Transonic shock waves.
  - Aerospace applications.

- The initial conditions and the free-stream boundary conditions are computed in the same way as for the Wilcox 1988 $k-\omega$ turbulence model.

More turbulence models – The Menter $k-\omega$ SST model

- The wall boundary conditions for the turbulent variables can be computed as follows [1,2,3],

$$k = 0 \qquad \omega = \frac{6\nu}{\beta_0 y^2} \qquad \omega = 10\frac{6\nu}{\beta_0 y^2} \qquad \beta_0 \approx 0.075$$

Proposed by Wilcox [3,4]
Recommended to use a few cell center layers away from the wall

Proposed by Menter [3]
Recommended to use in the first cell center layer next to the wall

- In the ω wall boundary condition definitions, y is the distance normal to the wall. Also, the results are not sensitive to the factor 10 in the Menter formulation [3].
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
  - https://turbmodels.larc.nasa.gov/

References:
[1] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[2] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[3] F. Menter. Improved Two-Equation k-omega Turbulence Models for Aerodynamic Flows. NASA TM-103975, 1992.

Wilcox 2006 $k - \omega$

More turbulence models – The Wicox 2006 $k-\omega$

- The Menter $k-\omega$ SST model is very accurate, but at the same time the formulation is quite complex, as it uses many additional blending and limiter functions.
- Due to the complexity, the model contradicts the ideal development of turbulence models,

> *"An ideal model should introduce the minimum amount of complexity while capturing the essence of the relevant physics."*
>
> D. C. Wilcox

- To offer a similar level of accuracy and reliability to that of the Menter SST model and without the extra complexity, Wilcox further improved the 1988 $k-\omega$ [1] and $k-\omega$ 1998 [2] turbulence models.
- These developments resulted in the Wilcox 2006 $k-\omega$ turbulence model [3,4], which solved many deficiencies of previous versions of the Wilcox turbulence models.

References:
[1] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[2] D. C. Wilcox. Turbulence Modeling for CFD. Second edition, DCW Industries, 1998.
[3] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[4] D. C. Wilcox. Formulation of the k-omega turbulence model revisited. AIAA Journal 46(11), pp. 2823-2838, 2006.

More turbulence models – The Wicox 2006 $k-\omega$

- The most important differences between the Wilcox 2006 model and earlier versions created by Wilcox are the addition of a cross-diffusion term and a built-in stress limiter.
  - The cross-diffusion term was included to minimize the sensitivity of the model to free-stream boundary conditions in shear free regions [1,2].
  - The stress limiter was added to improved the modeling capabilities when dealing with flow separation, incompressible free shear flows, and transonic/supersonic flows [1,3].

- The Wilcox 2006 $k-\omega$ turbulence model has the same predicting capabilities as the Menter $k-\omega$ SST model, with the added bonus that it does need to use several complex blending functions [1,4].
- This model is a great contender to the Menter $k-\omega$ SST.
- However, is not implemented in Ansys Fluent.
- Nevertheless, it can be implemented in Ansys Fluent using UDFs.

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] C. Speziale, R. Abid, E. Anderson. A Crtical Evaluation of Two-Equation Models for Near Wall Turbulence. AIAA Paper 90-1481, 1990.
[3] F. Menter. Improved Two-Equation k-omega Turbulence Models for Aerodynamic Flows. NASA TM-103975, 1992.
[4] S. Rodriguez. Applied Computational Fluid Dynamics and Turbulence Modeling. Springer, 2019.

More turbulence models – The Wicox 2006 $k-\omega$

- The solvable equations of the Wilcox 2006 $k-\omega$ turbulence model are the following ones,

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \sigma^* \frac{k}{\omega} \right) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \sigma \frac{k}{\omega} \right) \nabla \omega \right] + \underbrace{\frac{\sigma_d}{\omega} \nabla k \cdot \nabla \omega}_{\text{Cross diffusion term}}$$

- With the following closure relation to compute the kinematic eddy viscosity,

$$\nu_t = \frac{k}{\tilde{\omega}} \qquad \tilde{\omega} = \underbrace{\max \left[\omega, C_{lim} \sqrt{\frac{2\mathbf{S} : \mathbf{S}}{\beta^*}} \right]}_{\text{Stress limiter}} \qquad C_{lim} = \frac{7}{8}$$

- The stress limiter is activated when the magnitude of the strain rate is too large, which happens with separated flows, high Mach numbers flows, and in the presence of shock waves.

More turbulence models – The Wicox 2006 $k - \omega$

- The cross-diffusion term was included to minimize sensitives to free-stream boundary conditions.
- The cross-diffusion coefficient is defined as follows,

$$\sigma_d = \begin{cases} 0, & \nabla k \cdot \nabla \omega \leq 0 \quad \text{Near the walls} \\ \sigma_{d0}, & \nabla k \cdot \nabla \omega > 0 \quad \text{Free shear} \end{cases}$$

- The near wall and shear free behaviors are controlled via the product of the gradients of TKE and specific dissipation rate.
- The coefficient is zero near the walls, so cross-diffusion is suppressed.
- Far from the walls (shear free), the coefficient is active, so it acts as a source term in the transport equation of the specific dissipation rate.
- That is, it increases the value of ω.

More turbulence models – The Wicox 2006 $k-\omega$

- The closure coefficients in this model are defined as follows,

$$\alpha=\frac{13}{25},\quad \beta=\beta_0 f_\beta,\quad \beta^*=\frac{9}{100},\quad \sigma=\frac{1}{2},\quad \sigma^*=\frac{3}{5},\quad \sigma_{d0}=\frac{1}{8}$$

Matrix (or second rank tensor) multiplication*

$$\beta_0=0.0708,\quad f_\beta=\frac{1+85\chi_\omega}{1+100\chi_\omega},\quad \chi_\omega=\left|\frac{\Omega_{ij}\Omega_{jk}S_{ki}}{(\beta^*\omega)^3}\right|$$

This is the magnitude of the second rank tensor resulting from this matrix multiplication (the tensor has dimensions i by i) **

Strain rate tensor

$$S_{ij}=\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\qquad \mathbf{S}=\frac{1}{2}\left(\nabla\bar{\mathbf{u}}+\nabla\bar{\mathbf{u}}^T\right)$$

Vorticity tensor (or spin tensor)

$$\Omega_{ij}=\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}-\frac{\partial\bar{u}_j}{\partial x_i}\right)\qquad \Omega=\frac{1}{2}\left(\nabla\bar{\mathbf{u}}-\nabla\bar{\mathbf{u}}^T\right)$$

* Using index and vector notation, matrix multiplication is written as follows.

$$\mathbf{C}=C_{ij}=A_{ik}B_{kj}=\mathbf{A}\cdot\mathbf{B}$$

** The magnitude of a second rank tensor is defined as follows,

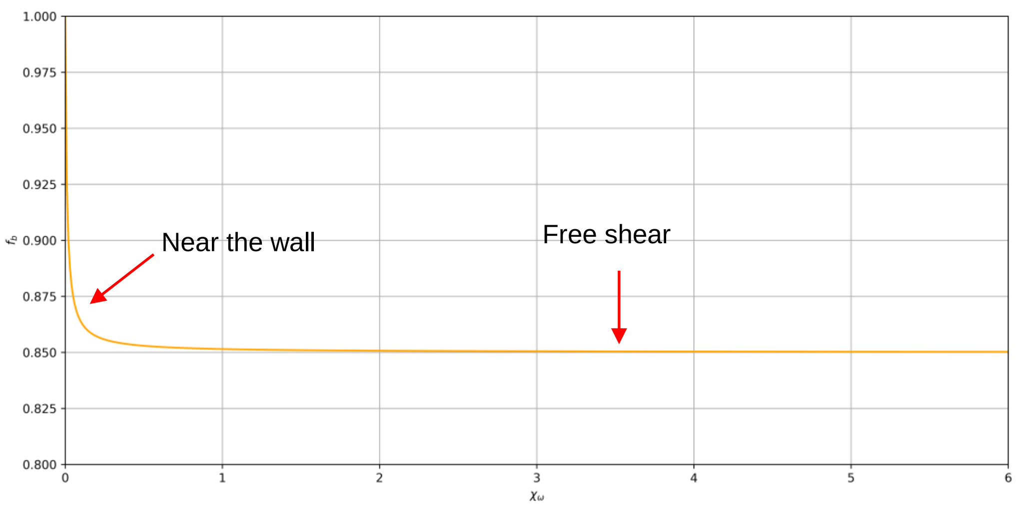
$$|\mathbf{C}|=|C_{ij}|=(2C_{ij}C_{ij})^{1/2}=(2\mathbf{C}:\mathbf{C})^{1/2}$$

More turbulence models – The Wicox 2006 $k - \omega$

- The blending function f_β, ranges from a minimum of 0.85 as χ_ω approaches large values for free shear, up to a peak value of 1.0 as χ_ω approaches 0 near the wall.
- In this context, χ_ω is the so-called non-dimensional vortex stretching.
- Recall that f_β and χ_ω are defined as,

$$f_\beta = \frac{1 + 85\chi_\omega}{1 + 100\chi_\omega} \qquad\qquad \chi_\omega = \left| \frac{\Omega_{ij}\Omega_{jk}S_{ki}}{(\beta^*\omega)^3} \right|$$

- Notice the introduction of the vorticity tensor Ω in this model, which is an ideal function for quantifying rotational flows. Turbulent coherent structures involve 3D sheets that folds with spiral like curvature; therefore, using the spin tensor is intuitive.
- Note that for χ_ω is zero for 2D flows, as there is no vortex stretching.



More turbulence models – The Wicox 2006 $k - \omega$

- Finally, this model uses the following auxiliary relationships,

$$l = \frac{k^{0.5}}{\omega}$$

$$\epsilon = \beta^* \omega k$$

$$\beta^* = \frac{9}{100}$$

- The initial conditions and the free-stream boundary conditions are computed in the same way as for the Menter SST $k - \omega$ turbulence models (or any $k - \omega$ turbulence model).

More turbulence models – The Wicox 2006 $k-\omega$

- The wall boundary conditions for the turbulent variables can be computed as follows [1,2,3],

$$k = 0 \qquad \omega = \frac{6\nu}{\beta_0 y^2} \qquad \omega = 10\frac{6\nu}{\beta_0 y^2} \qquad \beta_0 \approx 0.075$$

Proposed by Wilcox [3,4]
Recommended to use a few cell center layers away from the wall

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- In the ω wall boundary condition definitions, y is the distance normal to the wall. Also, the results are not sensitive to the factor 10 in the Menter formulation [3].
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
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[1] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[2] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[3] F. Menter. Improved Two-Equation k-omega Turbulence Models for Aerodynamic Flows. NASA TM-103975, 1992.

Compressible Wilcox 2006 $k-\omega$

More turbulence models – The compressible Wicox 2006 $k-\omega$

- To introduce the compressible Wilcox 2006 $k-\omega$ turbulence model [1,2], we need to introduce first the compressible averaged Navier-Stokes equations.
- This new set of equations is very similar to the incompressible RANS equations.
- But when dealing with compressible flows we use Favre average [1,2,3,4] instead of Reynolds average.
- The Favre average is very similar to the Reynolds average, but it is mass weighted in order to simplify the density fluctuations that arises from the Favre decomposition (similar to the Reynolds decomposition).
- As for the Reynolds decomposition, there a few rules of Favre averaging that we should be aware of.
- In our notation, the overbar denotes Reynolds averaging and the tilde denotes Favre averaging.

Favre averaging → $\tilde{\phi}$

$$\tilde{\phi} = \frac{\overline{\rho\phi}}{\bar{\rho}}$$

Reynolds averaging → $\overline{\rho\phi}$, $\bar{\rho}$

- We will review the Favre averaging in another lecture.

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.
3[] R. Pletcher, J. Tannehill, D. Anderson. Computational Fluid Mechanics and Heat Transfer. Third Edition. CRC Press, 2013.
[3] A. Favre. Equations des Gaz Turbulents Compressibles. Journal de Mecanique. 1965.

More turbulence models – The compressible Wicox 2006 $k-\omega$

- The Favre averaged compressible Navier-Stokes equations or FANS [1,2], read as follows,

$$\frac{\partial \bar{\rho}}{\partial t}+\frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\right)=0$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_i\right)+\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{u}_i\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\bar{t}_{ij}+\bar{\rho}\tau_{ij}\right]+S_u$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e}+\frac{\tilde{u}_i\tilde{u}_i}{2}+k\right)\right]+\frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h}+\frac{\tilde{u}_i\tilde{u}_i}{2}+k\right)\right]=\frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\mathrm{Pr}}+\frac{\mu_T}{\mathrm{Pr}_T}\right)\frac{\partial \tilde{h}}{\partial x_j}+\left(\mu+\sigma^*\frac{\bar{\rho}k}{\omega}\right)\frac{\partial k}{\partial x_j}\right]$$
$$+\frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\bar{t}_{ij}+\bar{\rho}\tau_{ij}\right)\right]+S_e$$

- Where we use the perfect gas law to relate pressure, density and temperature,

$$P=\bar{\rho}R\tilde{T}$$

- With the following thermodynamics relationships,

$$\tilde{e}=c_v\tilde{T} \qquad \tilde{h}=c_p\tilde{T}$$

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.

More turbulence models – The compressible Wicox 2006 $k-\omega$

- The Favre averaged compressible Navier-Stokes equations or FANS [1,2], read as follows,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{u}_i\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right] + S_u$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_T}{\mathrm{Pr}_T}\right)\frac{\partial \tilde{h}}{\partial x_j} + \left(\mu + \sigma^* \frac{\bar{\rho}k}{\omega}\right)\frac{\partial k}{\partial x_j}\right] + \frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right)\right] + S_e$$

- The molecular and Reynolds stresses can be written as follows,

$$\bar{t}_{ij} = 2\mu\bar{S}_{ij} \qquad \bar{\rho}\tau_{ij} = 2\mu_T\bar{S}_{ij} - \frac{2}{3}\bar{\rho}k\delta_{ij} \qquad \bar{S}_{ij} = S_{ij} - \frac{1}{3}\frac{\partial \tilde{u}_k}{\partial x_k}\delta_{ij}$$

- Note that in our notation the Reynolds stress is defined as follows,

$$\bar{\rho}\tau_{ij} = -\overline{\rho u_i'' u_j''}$$

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.

More turbulence models – The compressible Wicox 2006 $k - \omega$

- The Favre averaged compressible Navier-Stokes equations or FANS [1,2], read as follows,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\tilde{u}_i) = 0$$

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j\tilde{u}_i) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right] + S_u$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_T}{\mathrm{Pr}_T}\right)\frac{\partial \tilde{h}}{\partial x_j} + \left(\mu + \sigma^* \frac{\bar{\rho}k}{\omega}\right)\frac{\partial k}{\partial x_j}\right] + \frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right)\right] + S_e$$

- In this FANS set of equations we used the following closure relations for the turbulent heat flux vector and the molecular diffusion and turbulent transport terms,

$$q_{T_j} = \overline{\rho u_j'' h''} = -\frac{\mu_t c_p}{Pr_T}\frac{\partial \tilde{T}}{\partial x_j} = -\frac{\mu_t}{Pr_T}\frac{\partial \tilde{h}}{\partial x_j}$$

$$\overline{t_{ji}u_i''} - \rho u_j'' \frac{1}{2} u_i'' u_i'' = \left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}$$

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.

More turbulence models – The compressible Wicox 2006 $k - \omega$

- The Favre averaged compressible Navier-Stokes equations or FANS [1,2], read as follows,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{u}_i\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right] + S_u$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_T}{\mathrm{Pr}_T}\right)\frac{\partial \tilde{h}}{\partial x_j} + \left(\mu + \sigma^* \frac{\bar{\rho}k}{\omega}\right)\frac{\partial k}{\partial x_j}\right] + \frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right)\right] + S_e$$

- The previous FANS set of equations can be further simplified by introducing the following relations,

$$E = \tilde{e} + \frac{1}{2}\tilde{u}_i\tilde{u}_i + k \qquad H = \tilde{h} + \frac{1}{2}\tilde{u}_i\tilde{u}_i + k \qquad \bar{\rho}k = \frac{1}{2}\overline{\rho u_i'' u_i''}$$

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.

More turbulence models – The compressible Wicox 2006 $k-\omega$

- The Favre averaged compressible Navier-Stokes equations or FANS [1,2], read as follows,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{u}_i\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right] + S_u$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h} + \frac{\tilde{u}_i\tilde{u}_i}{2} + k\right)\right] = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_T}{\mathrm{Pr}_T}\right)\frac{\partial \tilde{h}}{\partial x_j} + \left(\mu + \sigma^* \frac{\bar{\rho}k}{\omega}\right)\frac{\partial k}{\partial x_j}\right]$$
$$+ \frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\bar{t}_{ij} + \bar{\rho}\tau_{ij}\right)\right] + S_e$$

- In our discussion, it is also necessary to relate the transported fluid properties, molecular viscosity μ and thermal conductivity k (do not confuse with TKE) to the thermodynamic variables.
- For example, we could use the following models,

$$\mu = \frac{C_1 \tilde{T}^{3/2}}{(T + C_2)} \qquad\qquad k = \frac{c_p \mu}{Pr}$$

Sutherland's formula with two coefficients

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.

- At this point, the transport equations of the compressible Wilcox 2006 $k-\omega$ turbulence model [1,2], are written as follows,

Note:

$$\bar{\rho}\tau_{ij} = -\overline{\rho u_i'' u_j''}$$

$$\frac{\partial}{\partial t}(\bar{\rho}k) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j k) = \bar{\rho}\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} - \beta^* \bar{\rho}k\omega + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma^* \frac{\bar{\rho}k}{\omega}\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial}{\partial t}(\bar{\rho}\omega) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j \omega) = \alpha\frac{\omega}{k}\bar{\rho}\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} - \beta\bar{\rho}\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma \frac{\bar{\rho}k}{\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \sigma_d \frac{\bar{\rho}}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

- With the following closure relation to compute the kinematic eddy viscosity,

$$\mu_T = \frac{\bar{\rho}k}{\tilde{\omega}} \qquad \tilde{\omega} = \max\left[\omega, C_{lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right] \qquad C_{lim} = \frac{7}{8}$$

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.

- The closure coefficients in this model are defined as follows,

$$\alpha = \frac{13}{25}, \quad \beta = \beta_0 f_\beta, \quad \beta^* = \frac{9}{100}, \quad \sigma = \frac{1}{2}, \quad \sigma^* = \frac{3}{5}, \quad \sigma_{d0} = \frac{1}{8}$$

$$\beta_0 = 0.0708, \quad Pr_T = \frac{8}{9}, \quad \sigma_d = \begin{cases} 0, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} \leq 0 \\ \sigma_{d0}, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} > 0 \end{cases}$$

- With the following closure functions,

$$f_\beta = \frac{1+85\chi_\omega}{1+100\chi_\omega}, \quad \chi_\omega = \left| \frac{\Omega_{ij}\Omega_{jk}\hat{S}_{ki}}{(\beta^*\omega)^3} \right|, \quad \hat{S}_{ki} = S_{ki} - \frac{1}{2}\frac{\partial \tilde{u}_m}{\partial x_m}\delta_{ki}$$

More turbulence models – The compressible Wicox 2006 $k-\omega$

- Except for some compressible corrections, additional terms, and the use of the Favre averaging method, the compressible $k-\omega$ Wilcox 2006 turbulence model [1,2] is exactly the same as the incompressible version.
- The free-stream and wall boundary conditions for the turbulent variables are set in the same way as for the incompressible version.
- Have in mind that there are many corrections and extra considerations when dealing with compressible high-speed flows (transonic, supersonic, hypersonic).
  - For example, a dilatation-dissipation correction [3,4] is often used to improve the predicting capabilities of turbulence models.
  - Sometimes, it might be necessary to adjust the closure coefficients according to the physics involved.
  - Very often, the compressibility effects corrections have been calibrated for a very limited number of free shear flow experiments and should be used with caution.
- In overall, this model shows similar performance to the compressible $k-\omega$ SST version.

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] K. Hoffmann, S. Chian. Computational Fluid Dynamics. Volume III. Fourth Edition. EES Books, 2000.
[3] S. Sarkar and L. Balakrishnan. Application of a Reynolds-Stress Turbulence Model to the Compressible Shear Layer. ICASE Report 90-18NASA CR 182002. 1990.
[4] D. C. Wilcox. Dilatation-Dissipation Corrections for Advanced Turbulence Models. AIAA Journal, Vol. 30 (11), 1992.

More turbulence models – The compressible Wicox 2006 $k - \omega$

- There are a few extra points worthy of mention regarding the closure equations [1].
    - The turbulent energy equation contains no special compressibility terms involving pressure work, diffusion or dilatation.
    - Although a dilatation-dissipation modification to the TKE equation improves compressibility mixing layer predictions, the same modification has a detrimental effect of shock-separated predictions.
    - Some researchers prefer the magnitude of the vorticity vector instead of the magnitude of the strain rate tensor in the stress-limiter modification. Using the magnitude of the vorticity with 0.95 instead of 7/8 is satisfactory for shock separated flow predictions up to Mach 3.
    - However, numerical experimentation with this model has shown that it has a detrimental effect on hypersonic shock-induced separation, some attached boundary layers and some free shear flows.

References:
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.

Wilcox Stress $-\ \omega$

More turbulence models – Wilcox Stress $-\ \omega$

- This model is part of the Reynolds Stress transport family of turbulence models or RSM.
- The RSM models are also known as second-order closure (SOC), second-moment closure (SMC), differential stress models (DSM), and stress-transport models (STM).
- In this model, Wilcox [1,2] proposed the use of the ω equation instead of the ϵ equation.
- Note that, by design, aside from the equation for the Reynolds-stress tensor replacing the stress-limiter, the underlying equations for k and ω are identical to those of the Wilcox 2006 $k-\omega$ turbulence model [1,3].
- All closure coefficients shared by the $k-\omega$ and the Stress $-\ \omega$ have the same values.
- The most significant difference between the LRR and the Stress $-\ \omega$ models is in the scale-determining equation. The LRR model uses the ϵ equation while the Stress $-\ \omega$ model uses the ω equation. All other differences are minor by comparison.
- This strongly suggests that the end accomplished by the LRR wall-reflection term $\Pi_{ij}^{(w)}$ may be to mitigate a shortcoming of the model equation for ϵ rather than to correctly represent the physics of the pressure echo process [1].
- When using this model, you should follow the same standard practices as the ones recommended for the Wilcox 2006 $k-\omega$ turbulence model.

References:

[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2006.

[2] D. C. Wilcox. Multiscale Model for Turbulent Flows. AIAA Journal, Vol. 26(11), 1988.

[3] D. C. Wilcox. Formulation of the k-omega turbulence model revisited. AIAA Journal 46(11), pp. 2823-2838, 2006.

- Let us write down the compressible equations of the Wilcox Stress $-\ \omega$ turbulence model.
- The Reynolds-Stress tensor can be written as follows,

$$\bar{\rho}\frac{\partial \tau_{ij}}{\partial t} + \bar{\rho}\tilde{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\bar{\rho}P_{ij} + \frac{2}{3}\beta^* \bar{\rho}\omega k \delta_{ij} - \bar{\rho}\Pi_{ij} + \frac{\partial}{\partial x_k}\left[(\mu + \sigma^* \mu_t)\frac{\partial \tau_{ij}}{\partial x_k}\right]$$

Note:

$$\bar{\rho}\tau_{ij} = -\overline{\rho u_i'' u_j''}$$

- The specific dissipation rate equation is given as follows,

$$\frac{\partial}{\partial t}(\bar{\rho}\omega) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j \omega) = \alpha \frac{\omega}{k}\bar{\rho}\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} - \beta \bar{\rho}\omega^2 + \sigma_d \frac{\bar{\rho}}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} + \frac{\partial}{\partial x_k}\left[(\mu + \sigma \mu_t)\frac{\partial \omega}{\partial x_k}\right]$$

- This model uses the following pressure-strain correlation,

$$\Pi_{ij} = \beta^* C_1 \omega \left(\tau_{ij} + \frac{2}{3}k\delta_{ij}\right) - \hat{\alpha}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - \hat{\beta}\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right) - \hat{\gamma}k\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)$$

Note:
If the density is constant, $\tilde{\phi} = \bar{\phi}$ and $\phi'' = \phi'$, and we recast the incompressible RANS equations.

More turbulence models – Wilcox Stress — ω

- With the following auxiliary relations,

$$\mu_t = \frac{\overline{\rho} k}{\omega}$$

$$P_{ij} = \tau_{im} \frac{\partial \tilde{u}_j}{\partial x_m} + \tau_{jm} \frac{\partial \tilde{u}_i}{\partial x_m} \qquad D_{ij} = \tau_{im} \frac{\partial \tilde{u}_m}{\partial x_j} + \tau_{jm} \frac{\partial \tilde{u}_m}{\partial x_i} \qquad P = \frac{1}{2} P_{kk}$$

- And the following closure coefficients,

$$\hat{\alpha} = \frac{8 + C_2}{11} \qquad \hat{\beta} = \frac{8C_2 - 2}{11} \qquad \hat{\gamma} = \frac{60C_2 - 4}{55}$$

$$C_1 = \frac{9}{5} \qquad C_2 = \frac{10}{19}$$

More turbulence models – Wilcox Stress $-\ \omega$

- And the following closure coefficients,

$$\alpha = \frac{13}{25}, \quad \beta = \beta_0 f_\beta, \quad \beta^* = \frac{9}{100}, \quad \sigma = \frac{1}{2}, \quad \sigma^* = \frac{3}{5}, \quad \sigma_{d0} = \frac{1}{8}$$

$$\beta_0 = 0.0708, \quad Pr_T = \frac{8}{9}, \quad \sigma_d = \begin{cases} 0, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} \leq 0 \\ \sigma_{d0}, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} > 0 \end{cases}$$

- With the following closure functions,

$$f_\beta = \frac{1 + 85\chi_\omega}{1 + 100\chi_\omega}, \quad \chi_\omega = \left| \frac{\Omega_{ij}\Omega_{jk}\hat{S}_{ki}}{(\beta^*\omega)^3} \right|, \quad \hat{S}_{ki} = S_{ki} - \frac{1}{2}\frac{\partial \tilde{u}_m}{\partial x_m}\delta_{ki}$$

ASM and EARSM

More turbulence models – ASM and EARSM

- Under certain assumptions the exact Reynolds stress equations can be reduced to a system of algebraic equations that require knowledge of the turbulent kinetic energy TKE and the turbulent dissipation rate ϵ.
- This class of models is known as algebraic stress models or ASM.
- ASM models reduce the closure problem to only solving two transport equations and a system of algebraic equations.
- This is significantly faster easier than solving the full RSM equations.
- The ASM equation include most of the models and assumptions that are used to solve the full RSM equations. Therefore, the complexity of the constitutive equations of the ASM models depends on the complexity of the closure approximations.
- ASM and EARSM (Explicit Algebraic Reynolds Stress Model) models are in between RSM and eddy viscosity models.
- They are characterized by less computational demands and (arguably) better accuracy compared to LEVM.
- ASM and EARSM models are non-nonlinear (anisotropic viscosity), so they do not suffer of the same deficiencies of LEVM.
- However, not necessary they give better results and sometimes they are difficult to make converge.

More turbulence models – ASM and EARSM

- The mathematical framework behind ASM and EARSM models is quite complex and will not address it here.
- There are many formulations, the interested reader can refer to the following publications [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15], to name a few.
- Hereafter, we will briefly address the derivation of the algebraic stress approximation.
- By introducing a few approximations, the RSM model can be reduced to a set of algebraic stress equations, which implicitly determines the Reynolds stresses (locally) as function of TKE, turbulent dissipation rate, and the mean velocity gradients.
- Because of the approximations involved, ASM models are less general and less accurate than RSM models

References:
[1] J. Lumley. Toward a Turbulent Constitutive Equation. Jounral of Fluid Mechanics, Vol. 41, 1970.
[2] W. Rodi. The Prediction of Free Turbulent Boundary Layers Using a Two-Equation Model of Turbulence. Ph.D thesis, Imperial College, 1972.
[3] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[4] W. Rodi. A New Algebraic Relation for Calculating Reynolds Stresses. ZAMM, Vol. 56, 1976.
[5] P. Saffman. Development of a Complete Model for the Calculation of Turbulent Shear Flows. Symposium of Turbulence and Dynamical Systems. Duke Univeristy, 1976.
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[7] T. Gatski, C. Speziale. On Explicit Algebraic Stress Models for Complex Turbulent Flows. Journal of Fluid Mechanics, Vol. 254, 1993.
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[12] T. Gatski, C. Rumsey. Linear and nonlinear eddy viscosity models. In Closure Strategies for Turbulent and Transitional Flows. Cambridge Univ. Press, 2002.
[13] S. Pope. Turbulent Flows. Cambridge University Press, 2010.
[14] P. Durbin. Some Recent Developments in Turbulence Closure Modeling. Annual Review of Fluid Mechanics, 2018.
[15] P. Durbin. Advanced Appraoches in Turbulence. Theory, MOdeling, Simulation, and Data Analysis for Turbulent Flows. Elsevier, 2022.

More turbulence models – ASM and EARSM

- One way to devise a constitutive equation is to reduce a Reynolds stress closure models to a set of algebraic equations.
- Let us assume the case of homogenous turbulence, in this case the Reynolds stress transport equation takes the following form,

$$\frac{D}{D_t}\overline{u_i'u_j'} = P_{ij} + \phi_{ij} - \frac{2}{3}\epsilon\delta_{ij}$$

- Where P_{ij} is the rate of production of Reynolds stress and ϕ_{ij} is the redistribution between components of the stress tensor.
- Redistribution incorporates anisotropy of dissipation, so only the isotropic part $\epsilon\delta_{ij}$ appears.
- One half of the trace of this equation is the turbulent kinetic energy equation; therefore, we obtain,

$$\frac{D}{D_t}k = P - \epsilon$$

More turbulence models – ASM and EARSM

- By using the equilibrium stress approximation,

$$\frac{D}{D_t}\frac{\overline{u_i'u_j'}}{k} = 0$$

- We obtain the following equation,

$$P_{ij} + \phi_{ij} - \frac{2}{3}\epsilon\delta_{ij} - (P - \epsilon)\frac{\overline{u_i'u_j'}}{k} = 0$$

- This is the set of equations solved in the ASM model.
- It is a system of equations for $\overline{u_i'u_j'}$ (or equivalently for the normalized anisotropy tensor b_{ij}) as an implicit function of the mean velocity gradient $\partial_i \bar{u}_j$.

More turbulence models – ASM and EARSM

- For example, if we derive the ASM model from the LRR RSM model [1], we obtain the following equation for the normalized anisotropy tensor b_{ij},

$$b_{ij} = \frac{\frac{1}{2}\left(1 - C_2\right)}{C_R - 1 + P/\epsilon} \frac{\left(P_{ij} - \frac{2}{3} P \delta_{ij}\right)}{\epsilon}$$

- Where the normalized anisotropy tensor b_{ij}, can be written as follows,

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\overline{u_i' u_j'}}{\overline{u_k' u_k'}} - \frac{1}{3}\delta_{ij}$$

- And where a_{ij} is the deviatoric anisotropic part of the Reynolds stress tensor,

$$a_{ij} = \overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij}$$

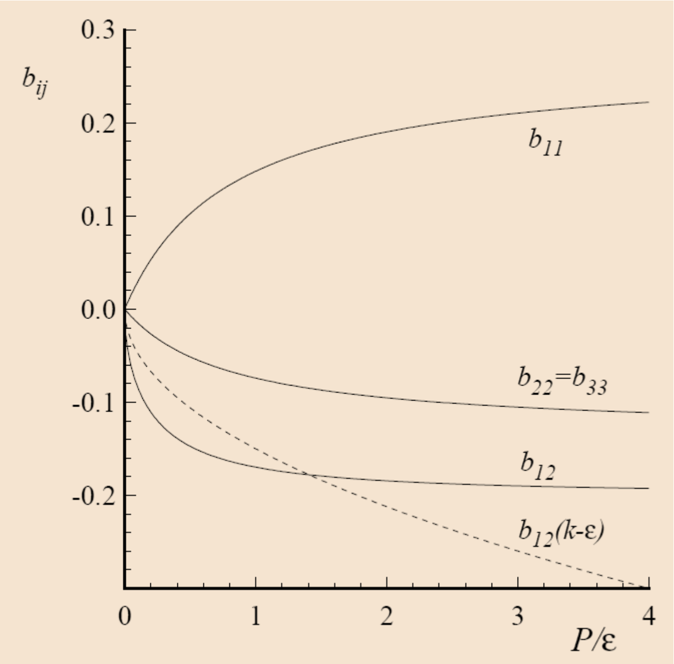
[1] B. E. Launder, G. J. Reece, W. Rodi. Progress in the Development of a Reynolds-Stress Turbulence Closure. 1975.

More turbulence models – ASM and EARSM

- An implication of the model is that the Reynolds stress anisotropy is directly proportional to the production anisotropy.

$$b_{ij} = \frac{\frac{1}{2}\left(1 - C_2\right)}{C_R - 1 + P/\epsilon} \frac{\left(P_{ij} - \frac{2}{3} P \delta_{ij}\right)}{\epsilon}$$

- For simple shear flows, the equation of b_{ij} can be solved to obtain the results shown in the figure below.



- In the figure [1], the anisotropies b_{ij} are plotted in function of P/ϵ according to the LRR ASM model.
- For large P/ϵ values, $|b_{12}|$ tends to asymptote

$$\sqrt{\frac{1}{6} C_2 \left(1 - C_2\right)} = \frac{1}{5}$$

- Whereas, the value given by the $k - \epsilon$ model continually increases and becomes non-realizable.

[1] S. Pope. Turbulent Flows. Cambridge University Press, 2010.

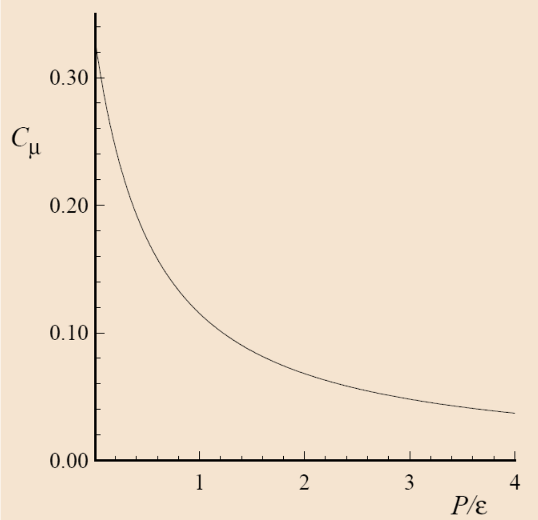
More turbulence models – ASM and EARSM

- By using the following relation for simple shear flows,

$$-\overline{uv} = \frac{C_\mu k^2}{\epsilon}\frac{\partial \bar{u}}{\partial y}$$

- We can obtain the following relation for the coefficient C_μ for the LRR ASM model,

$$C_\mu = \frac{\frac{2}{3}(1 - C_2)(C_R - 1 + C_2 P/\epsilon)}{(C_R - 1 + P/\epsilon)^2}$$



- In the figure [1], the value of C_μ a function of P/ϵ is plotted.
- The results correspond to the LRR ASM model.
- As can be seen, the value of C_μ decreases with increasing P/ϵ, corresponding to shear-thinning behavior.
- That is, C_μ decreases with increasing shearing,

$$\frac{\mathbf{S}k}{\epsilon}$$

[1] S. Pope. Turbulent Flows. Cambridge University Press, 2010.

More turbulence models – ASM and EARSM

- The ASM constitutive equations is an implicit equation for $\overline{u'_i u'_j}$ (or equivalently for the anisotropy tensor b_{ij}), *i.e.*, the Reynolds stresses appear both on the left and the right sides of the equation.

$$(P - \epsilon) \frac{\overline{u'_i u'_j}}{k} = P_{ij} + \phi_{ij} - \frac{2}{3} \epsilon \delta_{ij}$$

- It would be of course be advantageous to be able to get an explicit expression for the Reynolds stresses or the anisotropy tensor b_{ij}.

$$b_{ij} = \mathcal{B}_{ij} \left(\mathbf{S}, \Omega \right) = \sum_{1}^{10} G^n \mathcal{T}_{ij}^n$$

- Pope [1,2], maybe was the first one to derive an explicit expression for the ASM.
- He assumed that the Reynolds stress tensor can be expressed in function of the strain-rate tensor, **S**, and the vorticity tensor, Ω.
- Furthermore, he showed that the coefficients, G^n, in the expression of the anisotropy tensor b_{ij} can be a function of not more than five invariants $\mathcal{T}_{ij}^n$.

References:
[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.

More turbulence models – ASM and EARSM

- The exact explicit ASM or EARSM constitutive equations can be written as follows,

$$
\begin{aligned}
\overline{u_i'u_j'} - \frac{2}{3}k\delta_{ij} = a_{ij} = C_\mu k\mathbf{S} + C_1 k\left(\mathbf{S}\cdot\Omega - \Omega\cdot\mathbf{S}\right) + \\
C_2 k\left[\Omega^2 - \frac{1}{3}tr\left(\Omega^2\right)\mathbf{I}\right] + \\
C_3 k\left[\mathbf{S}\cdot\Omega^2 + \Omega^2\cdot\mathbf{S} - \frac{2}{3}tr\left(\Omega^2\cdot\mathbf{S}\right)\mathbf{I}\right] + \\
C_4 k\left(\Omega\cdot\mathbf{S}\cdot\Omega^2 - \Omega^2\cdot\mathbf{S}\cdot\Omega\right)
\end{aligned}
$$

- This corresponds to the three-dimensional expression of the Reynolds stresses.
- At this point, we need to find the closure coefficients, which can be constant or a function of a variable or combination of tensors.
- The derivation of the invariants is addressed in references [1,2,3].

References:

[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.

[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.

[3] M. Leschziner. Statistical Turbulence Modeling for Fluid Dynamics. Imperial College Press, 2016.

More turbulence models – ASM and EARSM

- The exact two-dimensional expression only retain the first two invariants.

$$\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij} = a_{ij} = C_\mu k \mathbf{S} + C_1 k \left(\mathbf{S} \cdot \Omega - \Omega \cdot \mathbf{S} \right)$$

- The exact two-dimensional expression only retain the first two invariants.
- At this point, we need to find the closure coefficients, which can be constant or a function of a variable or combination of tensors.
- The derivation of the invariants is addressed in references [1,2,3].
- Note that if $C_1 = 0$, $C_\mu = -C_\mu$, and $\mathbf{S} = \frac{k}{\epsilon} 2 S_{ij}$, we recover the linear $k - \epsilon$ turbulence viscosity formulation,

$$\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij} = -C_\mu \frac{k^2}{\epsilon} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

References:
[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.
[3] M. Leschziner. Statistical Turbulence Modeling for Fluid Dynamics. Imperial College Press, 2016.

More turbulence models – ASM and EARSM

- The model of Pope [1,2].
- This model is based on the RSM LRR model [3].
- The constitute equations of this model read as follows,

$$a_{ij} = -b_0 \left[b_1 S^*_{ij} + b_2 \left(a_{ik} S^*_{kj} + S^*_{ik} a_{jk} - \frac{2}{3} \delta_{ij} a_{kl} S^*_{lk} \right) - b_3 \left(a_{ik} \Omega^*_{kj} + \Omega^*_{ik} a_{jk} \right) \right]$$

$$b_0 = \left(\frac{P_k}{\epsilon} + C_1 - 1 \right)^{-1} \qquad b_1 = \frac{8}{15} \qquad b_2 = \frac{1}{11}(5 - 9C_2)$$

$$b_3 = \frac{1}{11}(7C_2 + 1) \qquad C_1 = 1.5 \qquad C_2 = 0.4$$

$$S^*_{ij} = \frac{k}{\epsilon} S_{ij} \qquad \Omega^*_{ij} = \frac{k}{\epsilon} \Omega_{ij}$$

- Note that the formulation shown is two-dimensional.

References:
[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.
[3] B. E. Launder, G. J. Reece, W. Rodi. Progress in the Development of a Reynolds-Stress Turbulence Closure. 1975.

More turbulence models – ASM and EARSM

- The model of Gatski and Speziale [1,2].
- This model is based on the RSM LRR model [3].
- The constitute equations of this model read as follows,

$$a_{ij} = -b_0 \left[b_1 S^*_{ij} + b_2 \left(a_{ik} S^*_{kj} + S^*_{ik} a_{jk} - \frac{2}{3} \delta_{ij} a_{kl} S^*_{lk} \right) - b_3 \left(a_{ik} \Omega^*_{kj} + \Omega^*_{ik} a_{jk} \right) \right]$$

$$b_0 = \left(1 - \frac{P_k}{\epsilon} - \frac{1}{2} C_1 \right)^{-1} \qquad b_1 = C_2 - \frac{4}{3} \qquad b_2 = \frac{1}{2} C_3 - 1 \qquad b_3 = 1 - \frac{1}{2} C_4$$

$$C_1 = 3.4 + 1.8 \frac{P_k}{\epsilon} \qquad C_2 = 0.8 - 0.65 \left(a_{kl} a_{kl} \right)^{1/2} \qquad C_3 = 1.25 \qquad C_4 = 0.4$$

$$S^*_{ij} = \frac{k}{\epsilon} S_{ij} \qquad \Omega^*_{ij} = \frac{k}{\epsilon} \Omega_{ij}$$

- Note that the formulation shown is two-dimensional.

References:
[1] T. Gatski, C. Speziale. On Explicit Algebraic Stress Models for Complex Turbulent Flows. Journal of Fluid Mechanics, Vol. 254, 1993.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.
[3] B. E. Launder, G. J. Reece, W. Rodi. Progress in the Development of a Reynolds-Stress Turbulence Closure. 1975.

More turbulence models – ASM and EARSM

- The mathematical formalism of ASM and EARSM models is the starting point to develop non-linear eddy viscosity models (NLEVM).
- NLEVM models not based on algebraic stress models have been also developed.
- References [1,2,3,4,5,6,7] addressed a few NLEVM not based on ASM/EARSM.
- In all models, the way in which non-linear stress-strain relationships are derived differs greatly but ultimately the derivation involves expansions with strain and vorticity tensors.
- In some formulations, in particular cubic formulations, the quadratic terms allow anisotropic to be modelled and the cubic terms the consequences of streamline curvature.
- These models also involve variable C_μ (or equivalent coefficient) formulations based on **S** and Ω , which helps avoid excessive turbulence prediction at stagnation points.
- In these models, the realizability conditions are always enforced.

References:
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More turbulence models – ASM and EARSM

- Despite being often marketed as a means of nearly getting Reynolds stress model performance at linear eddy viscosity model cost, practice shows this can be far from reality.
- Solutions obtained with these models can be nearly as expensive as performing a hybrid RANS-LES.
- Also, results can be worse can be worse than for EVM.
- The complexity of the constitutive equations of these models and therefore the accuracy, depends on the complexity of the closure approximations and alternative approximations.
- No need to mention that the mathematical formalism behind these models is quite complex.
- The EARSM derived by Pope [1] was a two-dimensional formulation. This formulation was later extended and refined by Gatski and Speziale [2] and by Jongen and Gatski [3] for 3-D flows.
- The EARSM models are popular for predicting aeronautical flows, in particular with the model by Wallin and Johansson [4].

References:
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[2] T. Gatski, C. Speziale. On Explicit Algebraic Stress Models for Complex Turbulent Flows. Journal of Fluid Mechanics, Vol. 254, 1993.
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