- Remember, before starting to use any turbulence model, it is strongly recommended to know its range of applicability and limitations.
- It is also important to know the recommend values for the boundary conditions and initial conditions.
- And any other information that may be useful when setting the simulation.
- Therefore, it is extremely recommended to read the original source of documentation of the model.
- This can be a paper or the help system of the CFD solver you are using.
- From now on, we will always mention the specific version of the turbulence model that we are going to use.
- We will give the main and some additional references.
- You do not do CFD and turbulence modeling without understanding the theoretical background.

- The order in which we are going to present the turbulence models does not reflect the accuracy, importance, number of equations, release date, type of approximations used, or efficiency of the models.
- It is an order that we think follows the derivation of the exact equations.

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
2. ~~Revisiting the closure problem~~
3. **Two equations models – The $k-\epsilon$ model**
4. Two equations models – The $k-\omega$ model
5. One equation model – The Spalart-Allmaras model
6. Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

Two equations models – The $k-\epsilon$ model

- This is maybe the most popular family of two-equation turbulence models.
- It is based on the Boussinesq hypothesis (linear eddy viscosity model or EVM).
- The initial development of this model can be attributed to Chou [1], circa 1945.
- Jones and Launder [2], Launder and Spalding [3], and Launder and Sharma [4] further developed and calibrated the model.
- They all contributed to what is generally referred to as the Standard $k-\epsilon$ turbulence model.
- This is the model that we are going to address hereafter.
- Have in mind that there are many variations of this model. Each one designed to add new capabilities and overcome the limitations of the standard $k-\epsilon$ turbulence model.
- The most notable limitation of the standard $k-\epsilon$ model is that it requires the use of wall functions.
- Variants of this model include the RNG $k-\epsilon$ model [5] and the Realizable $k-\epsilon$ model [6], to name a few.

References:
[1] P. Y. Chou. On Velocity Correlations and the Solutions of the Equations of Turbulent Fluctuation. Quarterly of Applied Mathematics. 1945.
[2] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[3] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.
[4] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.
[5] V. Yakhot, S. A. Orszag. Renormalization Group Analysis of Turbulence I Basic Theory. Journal of Scientific Computing. 1986.
[6] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New - Eddy-Viscosity Model for High Reynolds Number Turbulent Flows - Model Development and Validation. Computers Fluids. 1995.

Two equations models – The $k-\epsilon$ model

- It is called $k-\epsilon$ because it solves two additional equations for modeling the turbulent viscosity, namely, the turbulent kinetic energy k and the turbulence dissipation rate ϵ.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- With the following closure coefficients,

$$C_{\epsilon_1} = 1.44 \qquad C_{\epsilon_2} = 1.92 \qquad C_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.3$$

- And the following auxiliary relationships,

$$\omega = \frac{\epsilon}{C_\mu k} \qquad l = \frac{C_\mu k^{3/2}}{\epsilon}$$

Two equations models – The $k-\epsilon$ model

- The **solvable** closure equations of the standard $k-\epsilon$ turbulence model have been manipulated so there are no terms involving fluctuating quantities (*i.e.*, velocity and pressure), and double or triple correlations of the fluctuating quantities.
- Remember, the Reynolds stress tensor is modeled using the Boussinesq approximation.
- The turbulence dissipation rate ϵ is modeled using a second transport equation that we will derive later.

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \underbrace{\tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underbrace{\epsilon}_{\text{Dissipation}} + \underbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]}_{\text{Diffusion}}$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = \underbrace{C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underbrace{C_{\epsilon_2} \frac{\epsilon^2}{k}}_{\text{Dissipation}} + \underbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]}_{\text{Diffusion}}$$

Two equations models – The $k-\epsilon$ model

- The **exact** transport equation of the turbulent kinetic energy k was derived in the previous lectures.
- For convenience, we rewrite the **exact** TKE equation hereafter,

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon + \frac{\partial}{\partial x_j} \left[\nu \frac{\partial k}{\partial x_j} - \frac{1}{2} \overline{u_i' u_i' u_j'} - \frac{1}{\rho} \overline{p' u_j'} \right]$$

- At this point, let us focus our attention on the derivation of the **exact** transport equation for the turbulence dissipation rate ϵ.
- The derivation of the exact equation of the turbulence dissipation rate ϵ and the initial developments of the model can be traced back to the work of Davidov [1], Harlow et. al [2], and Hanjalic [3].

References:
[1] B. Davidov. On the Statistical Dynamics of an Incompressible Turbulent Fluid. Dokl. Akad. Nauk SSSR 136, 1961.
[2] F. Harlow, P. Nakayama. Transport of Turbulence Energy Decay Rate. University of California Report LA-3854, 1968.
[3] K. Hanjalic. Two Dimensional Asymmetric Turbulent Flow in Ducts. Ph. D thesis, University London, 1970.

Two equations models – The $k - \epsilon$ model

- The transport equation of the turbulence dissipation rate ϵ used in this model can be derived by taking the following moment of the NSE equations,

$$2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial}{\partial x_j} \left[\mathcal{N}\left(u_i \right) \right]} = 0$$

- Where the operator $\mathcal{N}(u_i)$ is equal to,

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k x_k}$$

- The **exact** turbulence dissipation rate transport equation is far more complicated than the turbulent kinetic energy equation.
- This equation contains several new unknown double and triple correlations of fluctuating velocity, pressure, and velocity gradients.
- There is a lot uncertainty related to this equation, and lot of authors agree that this is the largest source of error in the $k - \epsilon$ family of turbulence models.

Two equations models – The $k-\epsilon$ model

- There is a lot of algebra involved in the derivation of the **exact** turbulence dissipation rate transport equation.
- The final equation looks like this,

$$\underbrace{\frac{\partial \epsilon}{\partial t}}_{1} + \underbrace{\overline{u}_j \frac{\partial \epsilon}{\partial x_j}}_{2} = \underbrace{-2\nu \frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}}\right)}_{3} \underbrace{-2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j}\overline{u_k' \frac{\partial u_i'}{\partial x_j}}}_{4}$$

$$\underbrace{-2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}}_{5} \underbrace{-2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m}\frac{\partial^2 u_i'}{\partial x_k \partial x_m}}}_{6}$$

$$\underbrace{+\nu \frac{\partial \left(\frac{\partial \epsilon}{\partial x_j}\right)}{\partial x_j}}_{7} \underbrace{-\nu \frac{\partial \left(\overline{u_j' \frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)}{\partial x_j}}_{8} \underbrace{-2\frac{\nu}{\rho}\frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}}\right)}{\partial x_j}}_{9}$$

1. Transient rate of change term.
2. Convective term.
3. Production term that arises from the product of the gradients of the fluctuating and mean velocities.
4. Production term that generates additional dissipation based on the fluctuating and mean velocities.
5. Dissipation (destruction) associated with eddy velocity fluctuating gradients.
6. Dissipation (destruction) arising from eddy velocity fluctuating diffusion.
7. Viscous diffusion.
8. Diffusive turbulent transport resulting from the eddy velocity fluctuations.
9. Dissipation of turbulent transport arising from eddy pressure and fluctuating velocity gradients.

Two equations models – The $k - \epsilon$ model

- To derive the **solvable** transport equation of the turbulence dissipation rate, we need to use approximations in place of the terms that contain fluctuating quantities (velocity, pressure, and so on).
- The following approximations can be added to the **exact** turbulence dissipation rate transport equation in order to obtain the **solvable** transport equation.
- Note that the gradient diffusion hypothesis and the product rule is consistently used when deriving the turbulence dissipation rate.

$$\underbrace{-2\nu \frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} + \overline{\frac{\partial u'_k}{\partial x_i}\frac{\partial u'_k}{\partial x_j}}\right)}_{3} \underbrace{-2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j}\overline{u'_k \frac{\partial u'_i}{\partial x_j}}}_{4} \longrightarrow C_{\epsilon 1}\frac{\epsilon}{k}\tau_{ij}\frac{\partial \overline{u}_i}{\partial x_j}$$

Production

$$\underbrace{-2\nu \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_k}{\partial x_m}}}_{5} \underbrace{-2\nu^2 \overline{\frac{\partial^2 u'_i}{\partial x_k \partial x_m}\frac{\partial^2 u'_i}{\partial x_k \partial x_m}}}_{6} \longrightarrow -C_{\epsilon 2}\frac{\epsilon^2}{k}$$

Dissipation

$$\underbrace{\nu \frac{\partial \left(\frac{\partial \epsilon}{\partial x_j}\right)}{\partial x_j}}_{7} \underbrace{-\nu \frac{\partial \left(\overline{u'_j \frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}\right)}{\partial x_j}}_{8} \underbrace{-2\frac{\nu}{\rho}\frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}\right)}{\partial x_j}}_{9} \longrightarrow \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right]$$

Diffusion

Two equations models – The $k-\epsilon$ model

- By substituting the previous approximations in the **exact** turbulence dissipation rate transport equation, we obtain the **solvable** equation.
- It is not easy to elucidate the behavior of each term appearing in the **exact** turbulence dissipation rate transport equation.
- All the approximations added are based on DNS simulations, experimental data, analytical solutions, or engineering intuition.
- The **solvable** turbulence dissipation rate transport equation takes the following form,

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} = \underbrace{C_{\epsilon_1} \frac{\epsilon}{k} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Production}} - \underbrace{C_{\epsilon_2} \frac{\epsilon^2}{k}}_{\text{Dissipation}} + \underbrace{\frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right]}_{\text{Diffusion}}$$

Production

Dissipation

Diffusion

Two equations models – The $k - \epsilon$ model

- When using turbulence models, we also need to define boundary conditions at the walls.
- The standard $k - \epsilon$ turbulence model use wall functions.
- When using commercial solvers (Ansys Fluent in our case) you do not need to be concerned about the boundary conditions at the walls because this is done automatically by the solver and the wall treatment implementation.
- Have in mind that there are many wall function treatment implementations, each one having different capabilities and limitations. We will talk more about this later.
- As previously mentioned, the biggest deficiency of the wall treatment in the standard $k - \epsilon$ turbulence model is that it must be used with wall functions.
- If you use wall resolving meshes, you will get a solution, but it will deteriorate.
- The method is very sensitive to y^+ values.
- As a guideline, the y^+ value must be more than 50.
- We will study later the source and how to overcome this problem.

Two equations models – The $k - \epsilon$ model

- There is no common agreement on the best values of the wall boundary conditions.
- Everything depends on the version of the turbulence model and the specific wall treatment method used.
- Using the standard walls functions approach developed by Launder and Spalding [1] and Launder and Sharma [2], the recommended numerical values of the boundary conditions at the walls can be computed as follows,

$$\frac{\partial k}{\partial n} = 0 \qquad \epsilon_P = \frac{C_\mu^{3/4} k_P^{3/2}}{\kappa y_P}$$

Where the subscript P means cell center

- The free-stream values can be computed using the method introduced in Lecture 4.
- It is strongly recommended to not initialize these quantities with the same value or with values close to zero (in particular the turbulence dissipation rate).
- The boundary conditions for wall resolving meshes are different, we will study this scenario later.
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
  - https://turbmodels.larc.nasa.gov/

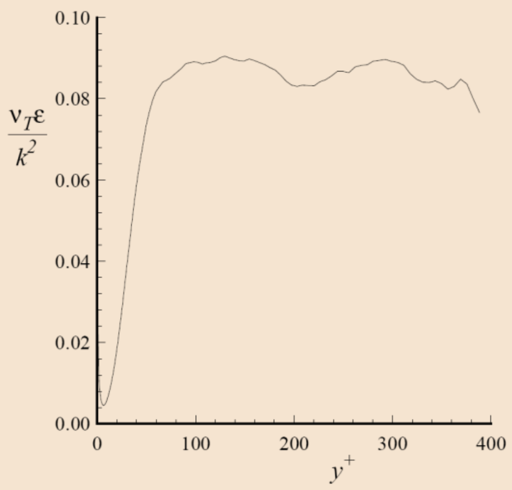
[1] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.
[2] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.

Two equations models – The $k-\epsilon$ model

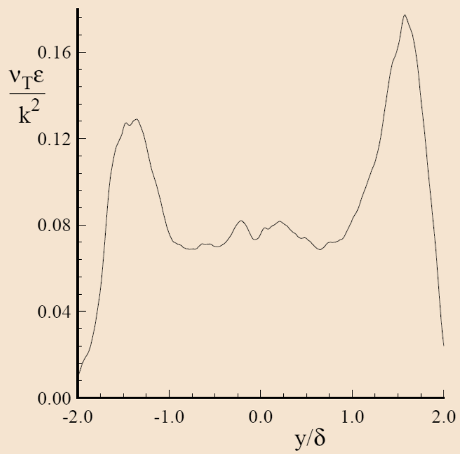
- The inaccuracies in the $k-\epsilon$ turbulence model stem from two sources, the turbulent viscosity computation and the turbulent dissipation rate equation.
- Recall that in this model the turbulent viscosity is computed as follows,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- Experimental observations and numerical simulations show the value of C_μ decreases as y^+ decreases below 50, as illustrated in the figures below [1].



J. Kim, P. Moin, R. Moser. Turbulence statistics in fully developed channel flow at low Reynolds number. 1987.



M. Rogers, R. Moser. Direct simulation of a self-similar turbulent mixing layer. 1994.

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Two equations models – The $k-\epsilon$ model

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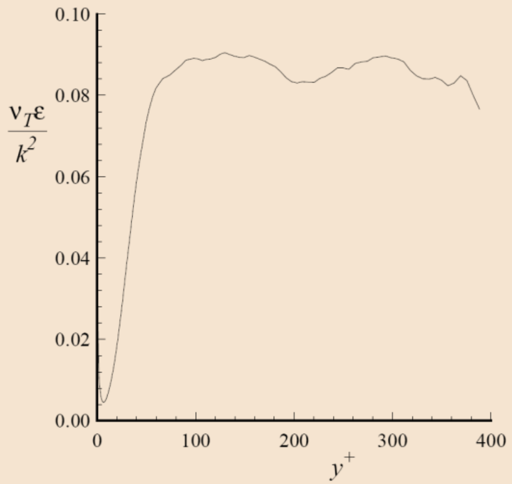
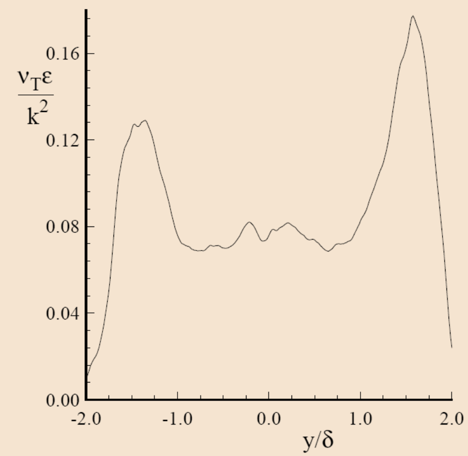


J. Kim, P. Moin, R. Moser. Turbulence statistics in fully developed channel flow at low Reynolds number. 1987.



M. Rogers, R. Moser. Direct simulation of a self-similar turbulent mixing layer. 1994.

- This behavior of C_μ suggests that in order to get the correct results of turbulence viscosity close to the walls we need to use a damping function.
- The goal of this damping function is to correct the turbulence viscosity, so it approaches to zero as we approach to the wall.
- We will study later a few modifications of the standard $k-\epsilon$ turbulence model to deal with wall resolving meshes.
- We will also talk about how the closure coefficients has been estimated.

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Two equations models – The $k - \epsilon$ model

- The inaccuracies related to the turbulent dissipation rate equation are a little bit more difficult to elucidate.
- The turbulent dissipation rate equation is a very complex one and the behavior of each term appearing in this equation is not well understood.
- And to make matters even worst, it is difficult to measure the budget of each term, numerically and experimentally.
- Generally speaking, the values of the coefficients in this equation represent a compromise.
- For any particular problem it is likely that the accuracy of the model calculations can be improved by adjusting the constants.
- Likely, the coefficients appearing in this equation show a similar behavior to that of the coefficient C_μ.
- As the reader might expect, corrections has been implemented in order to add a dependence on the y^+ value, turbulent Reynolds number, vorticity, strain rate, and so on.
- Later, we will study a variation of the standard $k - \epsilon$ turbulence model that aims at addressing this issue.

Two equations models – The $k - \epsilon$ model

- Another issue related to the turbulent quantities is the synchronization of these variables as we approach to the walls.
- Recall that the turbulent dissipation rate ϵ represents the dissipated turbulent kinetic energy per unit time, with the following base units,

$$\frac{m^2}{s^2}\frac{1}{s} = \frac{m^2}{s^3}$$

Base units of turbulent kinetic energy

- In the $k - \epsilon$ family of turbulence models, the turbulent kinetic energy k and the turbulent dissipation rate ϵ they are about the same order of magnitude.
- Therefore, they must go to zero at the correct rate in order to balance the equations and to avoid excessive production of turbulent viscosity production close to walls.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
2. ~~Revisiting the closure problem~~
3. ~~Two equations models – The $k-\epsilon$ model~~
4. **Two equations models – The $k-\omega$ model**
5. One equation model – The Spalart-Allmaras model
6. Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

Two equations models – The $k-\omega$ model

- This is another widely used and very popular family of two-equation turbulence models.
- It is based on the Boussinesq hypothesis (linear eddy viscosity model or EVM).
- The initial development of this model can be attributed to Kolmogorov [1], circa 1942. This was the first two-equation model of turbulence.
- The method was further developed and improved by Saffman [2], Launder and Spalding [3], Wilcox [4,5], Menter [6] and many more.
- There are many variations of this model. Hereafter, we will address the $k-\omega$ Wilcox 1988 model, which probably is the first formulation of the modern $k-\omega$ family of turbulence models.
- Each variation is designed to add new capabilities and overcome the limitations of the predecessor formulations.
- The most notable drawbacks of the $k-\omega$ Wilcox 1988 model are its limitation to resolve streamline curvature and its overly sensitivity to initial conditions.
- This family of models is y^+ insensitive.
- Variants of this model include the Wilcox 1998 $k-\omega$ [5], Wilcox 2006 $k-\omega$ [5], and Menter 2003 $k-\omega$ SST [6].

References:
[1] A. N. Kolmogorov. Equations of Turbulent Motion in an Incompressible Fluid. Physics. 1941.
[2] P. Saffman. A Model for Inhomogeneous Turbulent Flow. Proceedings of the Royal Society of London. 1970.
[3] B. E. Launder, D. B. Spalding. Mathematical Models of Turbulence. Academic Press. 1972.
[4] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[5] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[6] F. Menter, M. Kuntz, R. Langtry. Ten Years of Industrial Experience with the SST Turbulence Model. Turbulence, Heat and Mass Transfer. 2003.

Two equations models – The $k-\omega$ model

- It is called $k-\omega$ because it solves two additional equations for modeling the turbulent viscosity, namely, the turbulent kinetic energy k and the specific turbulence dissipation rate ω.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \sigma^* \nu_t \right) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \sigma \nu_t \right) \nabla \omega \right]$$

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'} \right)$$

- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{k}{\omega}$$

- Note that there is no dependence on the coefficient C_μ or any other coefficient to this matter.

- With the following closure coefficients,

$$\alpha = 5/9 \qquad \beta = 3/40 \qquad \beta^* = 9/100 \qquad \sigma = 1/2 \qquad \sigma^* = 1/2$$

- And auxiliary relationships,

$$\epsilon = \beta^* \omega k \qquad l = \frac{k^{1/2}}{\omega}$$

Two equations models – The $k-\omega$ model

- The closure equations of the Wilcox (1988) $k-\omega$ model have been manipulated so there are no terms including fluctuating quantities (*i.e.*, velocity and pressure), and double or triple correlations of the fluctuating quantities.
- Remember, the Reynolds stress tensor is modeled using the Boussinesq approximation.
- The specific turbulence dissipation rate is modeled using a second transport equation.

Note:

$$\tau^R = -\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

Dissipation

Production

Diffusion

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \sigma^* \nu_t \right) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \sigma \nu_t \right) \nabla \omega \right]$$

Production

Diffusion

Dissipation

Two equations models – The $k-\omega$ model

- In the Wilcox (1988) $k-\omega$ turbulence model, the production, dissipation, and diffusion terms of the specific turbulence dissipation rate ω are given by,

$$P^{\omega} = \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j} = \alpha \frac{\omega}{k} P^k$$

Note:

$$\tau_{ij}^R = -\left(\overline{u_i' u_j'}\right)$$

$$\epsilon^{\omega} = \beta \omega^2$$

$$D^{\omega} = \frac{\partial}{\partial x_j} \left[(\nu + \sigma \nu_T) \frac{\partial \omega}{\partial x_j} \right]$$

Two equations models – The $k - \omega$ model

- The transport equation for the specific turbulence dissipation rate ω can be derived from the transport equation of the turbulence dissipation rate ϵ.
- The model can be thought as the ratio of ϵ to k.
- To derive the transport equation of the turbulence dissipation rate ω, we can start by using the following relation,

$$\epsilon = \beta^* \omega k$$

- Then, by using the product rule we can obtain the material derivative of the specific turbulence dissipation rate ω.
- At this point, we can substitute the material derivative of the variables of the $k - \epsilon$ turbulence model into the material derivative of the specific turbulence dissipation rate ω (the material derivative we just obtained).
- By proceeding in this way, we can obtain the **exact** equations of ω.
- To derive the **solvable** equations of the Wilcox (1988) $k - \omega$ turbulence model, we just need to insert the approximations into the exact equations (Boussinesq hypothesis, gradient diffusion hypothesis, and so on).

Two equations models – The $k-\omega$ model

- By using the following equations, it is possible to derive an **exact** transport equation for the specific turbulence dissipation rate ω.

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\epsilon+\frac{\partial}{\partial x_j}\left[\nu\frac{\partial k}{\partial x_j}-\frac{1}{2}\overline{u_i'u_i'u_j'}-\frac{1}{\rho}\overline{p'u_j'}\right]$$

$$\frac{\partial \epsilon}{\partial t}+\overline{u}_j\frac{\partial \epsilon}{\partial x_j}=-2\nu\frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}+\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}}\right)-2\nu\frac{\partial^2\overline{u}_i}{\partial x_k\partial x_j}\overline{u_k'\frac{\partial u_i'}{\partial x_j}}-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}$$

$$-2\nu^2\overline{\frac{\partial^2 u_i'}{\partial x_k\partial x_m}\frac{\partial^2 u_i'}{\partial x_k\partial x_m}}+\nu\frac{\partial\left(\frac{\partial \epsilon}{\partial x_j}\right)}{\partial x_j}-\nu\frac{\partial\left(\overline{u_j'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)}{\partial x_j}-2\frac{\nu}{\rho}\frac{\partial\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}}\right)}{\partial x_j}$$

$$\epsilon=\beta^*\omega k$$

- The new **exact** transport equation for the specific turbulence dissipation rate ω can be derived from the turbulence dissipation rate equation ϵ; therefore, they share many similarities.

Two equations models – The $k-\omega$ model

- As for the turbulence dissipation rate equation ϵ, there is a lot of algebra involved.
- Hereafter, we will show the most important steps.
- By using the product rule, we can write $\epsilon = \beta^* \omega k$ as follows,

$$\frac{d\epsilon}{dt} = \beta^* k \frac{d\omega}{dt} + \beta^* \omega \frac{dk}{dt} \quad \Longrightarrow \quad \frac{d\omega}{dt} = \frac{1}{\beta^* k}\frac{d\epsilon}{dt} - \frac{\omega}{k}\frac{dk}{dt}$$

- Where d/dt is the material derivative (dependent of the mean velocity),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}$$

- By substituting the following relations into $d\omega/dt$,

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \overline{u}_j \frac{\partial \epsilon}{\partial x_j}$$

Exact transport equation of turbulence dissipation rate

$$\frac{dk}{dt} = \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j}$$

Exact transport equation of turbulence kinetic energy

- And doing a lot algebra, we obtain the **exact** equations of ω.

Two equations models – The $k - \omega$ model

- The final **exact** equation of the specific turbulence dissipation rate ω, can be written as follows,

> - Note that this equation is a combination of the equations for the turbulent kinetic energy and the turbulent dissipation rate.
> - Therefore, in order to determine the contribution of each term, just look at the starting equations and group the respective terms.

$$\frac{d\omega}{dt} = \frac{\omega}{k}\left[-\tau_{ij} - 2\nu\frac{\overline{u'_{i,k}u'_{j,k}} + \overline{u'_{k,i}u'_{k,j}}}{\beta^*\omega}\right]\frac{\partial \overline{u}_i}{\partial x_j}$$

$$-2\nu\frac{u'_k u'_{i,j}}{\beta^* k}\frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j}$$

$$-\left[2\nu\frac{\overline{u'_{i,k}u'_{i,m}u'_{k,m}} + \nu\overline{u'_{i,km}u'_{i,km}}}{\beta^*\omega} - \beta^*\omega^2\right]$$

$$+\frac{\partial}{\partial x_j}\left[\nu\frac{\partial \omega}{\partial x_j} - \nu\frac{\overline{u'_j u'_{i,m}u'_{i,m}}}{\beta^* k}\right] + \frac{1}{2}\omega\frac{\overline{u'_j u'_i u'_i}}{k} - 2\nu\frac{\overline{p'_{,m}u'_{j,m}}}{\beta^*\rho k} + \omega\frac{\overline{p'u'_j}}{\rho k}$$

$$+\frac{1}{k}\left[2\nu\frac{\partial k}{\partial x_j} - \frac{1}{2}\overline{u'_j u'_i u'_i} - \frac{1}{\rho}\overline{p'u'_j}\right]\frac{\partial \omega}{\partial x_j}$$

$$+\frac{1}{k^2}\left[\frac{\omega}{2}\overline{u'_j u'_i u'_i} - \frac{1}{\beta^*}\nu\overline{u'_j u'_{i,m}u'_{i,m}} + \frac{\omega}{\rho}\overline{p'u'_j} - \frac{2}{\beta^*}\frac{\nu}{\rho}\overline{p'_{,m}u'_{j,m}}\right]\frac{\partial k}{\partial x_j}$$

- As for the **exact** turbulence dissipation rate transport equation, it is not easy to elucidate the behavior of each term appearing in this equation.
- As this equation was derived from **exact** turbulence dissipation rate transport equation, we can use similar approximations.

Two equations models – The $k-\omega$ model

- When using turbulence models, we also need to define boundary conditions at the walls.
- The $k-\omega$ family of turbulence models are y^+ insensitive.
- When using commercial solvers (Ansys Fluent in our case) you do not need to be concerned about the boundary conditions at the walls because this is done automatically by the solver and the wall treatment implementation.
- Also, have in mind that there are many wall function treatment implementations, each one having different capabilities and limitations. We will talk more about this later.
- Unlike the standard $k-\epsilon$ model and some other models, the $k-\omega$ family of turbulence models can be integrated through the viscous sublayer without the need of damping functions.
- These models work by blending the viscous sublayer formulation and the logarithmic layer formulation based on the y^+.
- No need to mention that there many formulations in order to address the blending.

Two equations models – The $k-\omega$ model

- In this family of turbulence models, the community seems to have a better agreement when it comes to the wall boundary conditions.
- The wall boundary conditions for the turbulent variables can be computed as follows [1,2,3],

$$k = 0 \qquad \omega = \frac{6\nu}{\beta_0 y^2} \qquad \omega = 10\frac{6\nu}{\beta_0 y^2} \qquad \beta_0 \approx 0.075$$

Proposed by Wilcox [3,4]
Recommended to use a few cell center layers away from the wall

Proposed by Menter [3]
Recommended to use in the first cell center layer next to the wall

- In the ω wall boundary condition definitions, y is the distance normal to the wall. Also, the results are not sensitive to the factor 10 in the Menter formulation [3].
- The free-stream values can be computed using the method introduced in Lecture 4.
- It is strongly recommended to not initialize these quantities with the same value or with values close to zero (in particular the specific turbulence dissipation rate).
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
  - https://turbmodels.larc.nasa.gov/

[1] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[2] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[3] F. Menter. Improved Two-Equation k-omega Turbulence Models for Aerodynamic Flows. NASA TM-103975, 1992.
[4] D. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence models. AIAA Journal, vol. 26, pp. 1299-1310, 1988.

Two equations models – The $k-\omega$ model

- We previously mentioned that in the $k-\epsilon$ family of turbulence models, the turbulent kinetic energy k and the turbulent dissipation rate ϵ they must go to zero at the correct rate in order to balance the equations, and to avoid production of turbulent viscosity production close to walls.
- Instead, the $k-\omega$ family of turbulence models do not suffer of this problem as the turbulence specific dissipation rate ω is proportional to $\omega \propto y^{-2}$ as we approach to the walls [1,2].
- Therefore, the specific dissipation rate ω close to the walls is usually a large value.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \sigma^* \nu_t \right) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \sigma \nu_t \right) \nabla \omega \right]$$

$$\nu_t = \frac{k}{\omega} \qquad \omega = \frac{6\nu}{\beta_0 d^2} \qquad \beta_0 \approx 0.075 \qquad \epsilon = \beta^* \omega k$$

d is the distance to the first cell center normal to the wall (y)

- We can see the specific dissipation rate ω as the dissipation frequency, which is high (recall the turbulent energy spectrum).

[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Two equations models – The $k - \omega$ model

- Finally, notice that the coefficients in the $k - \omega$ turbulence model are different from those of the $k - \epsilon$ turbulence model (as expected).

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot [(\nu + \sigma^* \nu_t) \nabla k]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot [(\nu + \sigma \nu_t) \nabla \omega]$$

$$\nu_t = \frac{k}{\omega}$$

$$\alpha = 5/9 \qquad \beta = 3/40 \qquad \beta^* = 9/100 \qquad \sigma = 1/2 \qquad \sigma^* = 1/2$$

- In the $k - \omega$ turbulence model, the turbulent viscosity does not depend on a coefficient.
- So, there is no need to add complex damping functions in order to mimic an observed behavior.
- As described in detail by Wilcox [1], for boundary layer flows, the $k - \omega$ model is superior both in its treatment of the viscous near-wall region, and in its accounting for the effects of streamwise pressure gradients.
- Later, we will study a few variants of the Wilcox (1988) $k - \omega$ turbulence model, aiming at improving the predictions of turbulent flows.

[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.

Two equations models – The $k - \omega$ model

"The 1988 model is elegant because it captures the major elements of transport for both k and w, while having neither limiters not blending functions, and requires only a minimal number of closure coefficients. The model is very useful for low Re and near wall boundary layers."

S. Rodriguez [1]

"An ideal model should introduce the minimum amount of complexity while capturing the essence of the relevant physics."

D. C. Wilcox [2]

[1] S. Rodriguez. Applied Computational Fluid Dynamics and Turbulence Modeling. Springer, 2019.
[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
2. ~~Revisiting the closure problem~~
3. ~~Two equations models – The $k-\epsilon$ model~~
4. ~~Two equations models – The $k-\omega$ model~~
5. **One equation model – The Spalart-Allmaras model**
6. Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

One equation model – The Spalart-Allmaras model

- The Spalart-Allmaras model [1,2] is a one-equation model that solves a model transport equation for the modified turbulent kinematic viscosity (artificial variable).
- It is based on the Boussinesq hypothesis (linear eddy viscosity model or EVM).
- By far, this is the most popular and successful one-equation model.
- It also has been adopted as the foundation for DES models [3].
- The Spalart-Allmaras model was designed specifically for aerospace applications involving wall-bounded flows.
- In its original form, the Spalart-Allmaras model is a wall resolving method, requiring the use of fine meshes in order to resolve the viscous sublayer.
- Over the years this method has been improved. Each variation is designed to add new capabilities and overcome the limitations of the predecessor formulations.
- The most notable drawback is its limitation to deal with massive flow separation (like most of the turbulence models).
- Variants of this model include the addition of rotation/curvature corrections, trip terms, production limiters, strain adaptive formulations, wall roughness corrections, compressibility corrections, extension to y^+ insensitive treatment, and so on.
- Hereafter, we will address the model formulation described in reference [1] (which is probably the original formulation).

References:
[1] P. Spalart, S. Allmaras, A One-Equation Turbulence Model for Aerodynamic Flows, Recherche Aerospatiale, No. 1, pp. 5-21, 1994.
[2] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. AIAA Conference Paper AIAA-92-0439, 1992.
[3] M. Shur, P. R. Spalart, M. Strelets, A. Travin. Detached-Eddy Simulation of an Airfoil at High Angle of Attack. 1999.

One equation model – The Spalart-Allmaras model

- The Spalart-Allmaras model is based on the Boussinesq hypothesis.

$$\tau_{ij}^R = -\rho\overline{u_i'u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

- Where the circled term is generally ignored because information about the turbulent kinetic energy k is not readily available.
- Details regarding a nonlinear implementation that also includes an approximation for the term,

$$-\frac{2}{3}\rho k \delta_{ij}$$

can be found in references [1, 2, 3].

- It is worth noting that most of the one equation turbulence models (unless they are based on a transport equation for the turbulent variable k), do not provide information about the turbulent kinetic energy.

[1] M. Mani, D. Babcock, C. Winkler, P. Spalart, Predictions of a Supersonic Turbulent Flow in a Square Duct, AIAA Paper 2013-0860, January 2013.
[2] C. Rumsey, H. Lee, T. Pulliam, Reynolds-Averaged Navier-Stokes Computations of the NASA Juncture Flow Model Using FUN3D and OVERFLOW, AIAA Paper 2020-1304, 2020.
[3] C. Rumsey, J. Carlson, T. Pulliam, P. Spalart, Improvements to the Quadratic Constitutive Relation Based on NASA Juncture Flow Data, AIAA Journal, Vol. 58, No. 10, pp. 4374-4384, 2020.

One equation model – The Spalart-Allmaras model

- In the Spalart-Allmaras model (SA), a closed equation for the turbulent eddy viscosity is artificially created that fits well a range of experimental and empirical data.
- To accomplish this, the SA ν_T equation is built up term by term in a series of calibrations involving flows of increasing complexity.

$$\underbrace{\nabla_t \nu_T}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \nu_T}_{\text{Convection}} = \underbrace{P^{\nu_T}}_{\text{Production}} + \underbrace{\epsilon^{\nu_T}}_{\text{Dissipation}} + \underbrace{D^{\nu_T}}_{\text{Diffusion}} + \underbrace{S^{\nu_T}}_{\text{Source terms}}$$

- The resulting model has gone through a number of developmental iterations beyond its original form and has been widely tested for different external aerodynamics applications.
- Probably, this is the turbulence model that has undergone more modifications.
- It is beyond the scope of this discussion to delve the different calibration steps of each term and the choice of the closure coefficients.
- A good description of the background of the SA model is given in reference [1].
- For a description of different versions of the SA model, the interested reader should take a look at the following link,
    - https://turbmodels.larc.nasa.gov/spalart.html

[1] P.R. Spalart. Strategies for turbulence modelling and simulations. International Journal of Heat and Fluid Flow 21, pp. 252-263, 2000.

One equation model – The Spalart-Allmaras model

- Before introducing the SA model, it is worth mentioning that this model does not actually solves a transport equation for the turbulent eddy viscosity ν_T.
- It solves a modified version of the turbulent eddy viscosity.
- Namely, modified eddy viscosity $\tilde{\nu}$.
- Therefore, we deal with the following general transport equation,

$$\underbrace{\nabla_t \tilde{\nu}}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \tilde{\nu}}_{\text{Convection}} = \underbrace{P^{\tilde{\nu}}}_{\text{Production}} + \underbrace{\epsilon^{\tilde{\nu}}}_{\text{Dissipation}} + \underbrace{D^{\tilde{\nu}}}_{\text{Diffusion}} + \underbrace{S^{\tilde{\nu}}}_{\text{Source terms}}$$

- Also, as it is an artificial method, the structure of the production and dissipation terms is slightly different from that of two and more equations models.

One equation model – The Spalart-Allmaras model

- The closure equation of the standard SA model is given as follows,

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = c_{b1}\left(1 - f_{t2}\right)\tilde{S}\tilde{\nu} - \left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + \frac{1}{\sigma}\frac{\partial}{\partial x_j}\left[\left(\nu + \tilde{\nu}\right)\frac{\partial \tilde{\nu}}{\partial x_j}\right] + \frac{c_{b2}}{\sigma}\frac{\partial \tilde{\nu}}{\partial x_i}\frac{\partial \tilde{\nu}}{\partial x_i}$$

- Where $\tilde{\nu}$ is the modified eddy viscosity.
- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_T = \tilde{\nu} f_{v1}$$

- Where f_{v1} can be interpreted as a wall damping function [1].

[1] G. Mellor, H. Herring. Two methods of calculating turbulent boundary layer behavior based on numerical solutions of the equations of motion. 1968.

One equation model – The Spalart-Allmaras model

- With the following closure relationships,

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$\chi = \frac{\tilde{\nu}}{\nu}$$

$$f_{t2} = c_{t3} e^{-c_{t4}\chi^2}$$

$$\tilde{S} = \Omega + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$

d is the minimum distance to the nearest wall

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

$$\Omega = \sqrt{2 W_{ij} W_{ij}}$$

Magnitude of the vorticity tensor

$$W_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

Anti-symmetric part of the velocity gradient (vorticity tensor)

$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{1/6}$$

$$g = r + c_{w2}\left(r^6 - r\right)$$

$$r = \min\left[\frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}, 10\right]$$

One equation model – The Spalart-Allmaras model

- And with the following closure coefficients,

$$c_{b1} = 0.1355 \qquad c_{b2} = 0.622$$

$$c_{v1} = 7.1 \qquad \sigma = 2/3$$

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{(1 + c_{b2})}{\sigma} \qquad c_{w2} = 0.3$$

$$c_{w3} = 2 \qquad \kappa = 0.41$$

$$c_{t3} = 1.2 \qquad c_{t4} = 0.5$$

One equation model – The Spalart-Allmaras model

- In the previous relationships, W_{ij} is the rotation tensor (anti-symmetric part of the velocity gradient) and d is the distance from the closest wall.
- Notice that the modified eddy viscosity equation depends on the distance from the closest wall, as well as on the gradient of the modified eddy viscosity gradient.
- Since $d \rightarrow \infty$ far from the walls, this model also predicts no decay of the eddy viscosity in a uniform stream.
- Inspection of the transport equation reveals that κd has been used as length scale.
- The length scale κd is also used in the term $\tilde{S}$, which is related to the vorticity.
- To avoid possible numerical problems, the vorticity parameter $\tilde{S}$ must never be allowed to reach zero or go negative. In references [1], a limiting method is reported.
- Many implementations of the SA model ignore the term f_{t2}, which was added to provide more stability when the trip term is used.
- Based on studies described in reference [2], the use of this form as opposed to the SA version with the trip term probably makes very little difference.
- The form of the Spalart-Allmaras model with the trip term included is given in reference [3].

[1] S. Allmaras, F. Johnson, P. Spalart. Modifications and Clarifications for the Implementation of the Spalart-Allmaras Turbulence Model. 2012.
[2] C. Rumsey. Apparent Transition Behavior of Widely-Used Turbulence Models. 2007.
[3] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1994.

One equation model – The Spalart-Allmaras model

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = \underbrace{c_{b1} \tilde{S} \tilde{\nu}}_{\text{Production}} - \underbrace{c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2}_{\text{Dissipation}} + \underbrace{\frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \overbrace{\frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}}^{\text{Extra diffusion source term - Wake profile spreading}}}_{\text{Diffusion}}$$

- The closure equations of the SA turbulence model have been derived using empirical relationships, dimensional analysis, and experimental and numerical data.
- It is an artificial (or synthetic) model with a theoretical background less rigorous than that of any of the turbulence models that we previously studied.
- Despite being a one equation model, the SA model performs very well for a specific group of application, namely, compressible high-speed external aerodynamics in aerospace applications.
- Note that this model has source terms (production and destruction) that are non-zero in the freestream, even when vorticity is zero.
- The source terms are, however, very small, proportional to $1/d^2$.

One equation model – The Spalart-Allmaras model

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = \underbrace{c_{b1} \tilde{S} \tilde{\nu}}_{\text{Production}} - \underbrace{c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2}_{\text{Dissipation}} + \underbrace{\frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \overbrace{\frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}}^{\text{Extra diffusion source term - Wake profile spreading}}}_{\text{Diffusion}}$$

- Notice that in this form of the **solvable** equations, we have dropped the term f_{t2}.
- Which implies that $c_{t3} = 0$.
- Many implementations of the SA model ignore the term f_{t2}, which was a numerical fix in the original model to slightly delay transition so that the trip term could be activated appropriately. So, it is argued that if the trip is not used, then f_{t2} is not necessary [1, 2].
- Based on previous studies [3], the use of this form as opposed to the version that retains the term f_{t2}, probably makes very little difference, at least at reasonably high Reynolds numbers.

[1] L. Eca, M. Hoekstra, A. Hay, D. Pelletier, A Manufactured Solution for a Two-Dimensional Steady Wall-Bounded Incompressible Turbulent Flow, International Journal of Computational Fluid Dynamics, Vol. 21, Nos. 3-4, pp. 175-188, 2007.
[2] B. Aupoix, P. Spalart, Extensions of the Spalart-Allmaras Turbulence Model to Account for Wall Roughness, International Journal of Heat and Fluid Flow, Vol. 24, pp. 454-462, 2003. [3] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1994.
[3] C. Rumsey, Apparent Transition Behavior of Widely-Used Turbulence Models, International Journal of Heat and Fluid Flow, Vol. 28, pp. 1460-1471, 2007.

One equation model – The Spalart-Allmaras model

- As for the previous turbulence models, we also need to define boundary conditions at the walls.
- The numerical values depends on the version of the turbulence model and the specific wall treatment method used.
- The standard SA model is wall resolving.
- However, over the years the model has been improved so it can deal with wall functions and y^+ insensitive treatments.
- When using commercial solvers (Ansys Fluent in our case) you do not need to be concerned about the boundary conditions at the walls because this is done automatically by the solver and the wall treatment implementation.
- Have in mind that there are many wall function treatment implementations, each one having different capabilities and limitations. We will talk more about this later.
- The wall boundary conditions for the turbulent variables can be estimated as follows [1,2],

$$\tilde{\nu} = 0 \qquad \nu_t = 0$$

[1] P. Spalart, S. Allmaras, A One-Equation Turbulence Model for Aerodynamic Flows, Recherche Aerospatiale, No. 1, pp. 5-21, 1994.
[2] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. AIAA Conference Paper AIAA-92-0439, 1992.

One equation model – The Spalart-Allmaras model

- The freestream conditions can be estimated using the following relationships [1,2],

$$\tilde{\nu}_{\text{farfield}} = 3\nu_{\infty} \quad to \quad 5\nu_{\infty} \qquad\qquad \nu_{t_{\text{farfield}}} = 0.210438\nu_{\infty} \quad to \quad 1.29423\nu_{\infty}$$

- If information related to the turbulent kinetic energy and turbulent dissipation is available, the freestream conditions can be estimated as follows,

$$\tilde{\nu} = \nu_t = \frac{C_{\mu} k^2}{\epsilon} \qquad\qquad \tilde{\nu} = \nu_t = \frac{k}{\omega}$$

- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
  - https://turbmodels.larc.nasa.gov/

[1] P. Spalart, S. Allmaras, A One-Equation Turbulence Model for Aerodynamic Flows, Recherche Aerospatiale, No. 1, pp. 5-21, 1994.
[2] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. AIAA Conference Paper AIAA-92-0439, 1992.

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
2. ~~Revisiting the closure problem~~
3. ~~Two equations models – The $k-\epsilon$ model~~
4. ~~Two equations models – The $k-\omega$ model~~
5. ~~One equation model – The Spalart-Allmaras model~~
6. **Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation**

Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

- The equations of the $k-\epsilon$ turbulence model can be generalized as follows,

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right] + L_k$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} f_1 \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} f_2 \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right] + L_\epsilon$$

$$\nu_t = f_\mu C_\mu \frac{k^2}{\epsilon}$$

- The circled terms have been added to generalized the equations.
- By changing the values of the coefficients f_1, f_2, f_μ, L_k, and L_ϵ we can obtain different formulations of the $k-\epsilon$ turbulence model.
- For example, by setting the coefficients f_1, f_2, f_μ, to one and the coefficients L_k and L_ϵ to zero, we recast the standard $k-\epsilon$ turbulence model.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- One of the drawbacks of the standard $k - \epsilon$ model is that it can only be used with wall functions.
- Actually, and without any modification to the standard $k - \epsilon$ turbulence model, the equations can be integrated in the viscous sublayer all the way down to wall, but the results will deteriorate.
- The standard $k - \epsilon$ turbulence model is a wall modeling model.
- This also applies to the RNG $k - \epsilon$ [1, 2] and realizable $k - \epsilon$ [3] turbulence models, as both models are variants of the standard $k - \epsilon$ turbulence model.
- In order to integrate the governing equations all the way down to the wall, we need to add a few modifications to the original formulation.
- The resulting formulations are known as low-Reynolds number $k - \epsilon$ turbulence models or lowRE or LRN.
- The terminology low-Reynolds number refers to the Reynolds number measured normal to the wall (something similar to the y^+ normal to the wall) and not the system Reynolds number.

References:

[1] V. Yakhot, S. Orszag. Renormalization group analysis of turbulence: 1. Basic theory. Journal of Scientific Computing. Vol. 1, pp. 3-51. 1986.

[2] V. Yakhot, S.A. Orszag, S. Thangam, and C.G. Speziale. Development of Turbulence Models for Shear Flows by a Double Expansion technique. Physics of Fluids A Fluid Dynamics, 4(7), 1992.

[3] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New k-epsilon Eddy-Viscosity Model for High Reynolds Number Turbulent Flows - Model Development and Validation. Computers Fluids, 24(3):227-238, 1995.

Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

- The first low-Reynolds number $k-\epsilon$ turbulence model was developed by Jones and Launder [1,2], and subsequently it has been modified by several authors [3,4].
- The primary modifications introduced by Jones and Launder [1,2] were to include turbulence Reynolds number dependency functions f_1, f_2, and f_μ.
- The purpose of these functions is to correct or damp the behavior of the turbulent viscosity as we approach to the walls.
- The main idea is getting asymptotically consistent near wall behavior.
- Furthermore, additional terms L_k and L_ϵ were added to the equations to account for the dissipation processes which may not be isotropic.
- Recall that the turbulence Reynolds number is related to the Reynolds number of the integral scales and can be computed as follows,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu} \approx \frac{\nu_t}{\nu}$$

References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

- Closure damping functions, coefficients, and extra source terms for the lowRE $k-\epsilon$ turbulence models.

Model	f_1	f_2	f_μ	L_k	L_ϵ
Standard	1.0	1.0	1.0	0	0
Jones-Launder [1,2]	1.0	$1-0.3\exp(-Re_T^2)$	$\exp\left[\frac{-2.5}{1+0.02Re_T}\right]$	$-2\nu\left(\frac{\partial k^{1/2}}{\partial x_j}\right)^2$	$2\nu\nu_t\left(\frac{\partial^2 u_i}{\partial x_j x_k}\right)^2$
Launder-Sharma [3]	1.0	$1-0.3\exp(-Re_T^2)$	$\exp\left[\frac{-3.4}{(1+0.02Re_T)^2}\right]$	$-2\nu\left(\frac{\partial k^{1/2}}{\partial x_j}\right)^2$	$2\nu\nu_t\left(\frac{\partial^2 u_i}{\partial x_j x_k}\right)^2$
Hoffman [4]	1.0	$1-0.3\exp(-Re_T^2)$	$\exp\left[\frac{-1.75}{1+0.02Re_T}\right]$	$-\frac{\nu}{y}\frac{\partial k}{\partial y}$	0
Nagano-Hishida [5]	1.0	$1-0.3\exp(-Re_T^2)$	$[1-\exp(-Re_T/26.5)]^2$	$-2\nu\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$\nu\nu_t(1-f_\mu)\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
Chien [6]	1.0	$1-\frac{0.4}{1.8}\exp(-Re_T^2/36)$	$1-\exp(-0.0115d^+)$ $d^+=d\rho u_\tau/\mu$	$-2\frac{\mu k}{\rho d^2}$	$-2\frac{\mu\epsilon}{\rho d^2}\exp(-d^+/2)$

References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] G. Hoffman. Improved form of the low Reynolds number k-epsilon turbulence model. Physics of Fluids, vol. 18(3), pp. 309-312,1975.
[5] Y. Nagado, M. Hishida. Improved form of the k-epsilon model for wall turbulent shear flows. Journal of Fluids Engineering, vol. 109, pp. 156-160, 1987.
[6] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

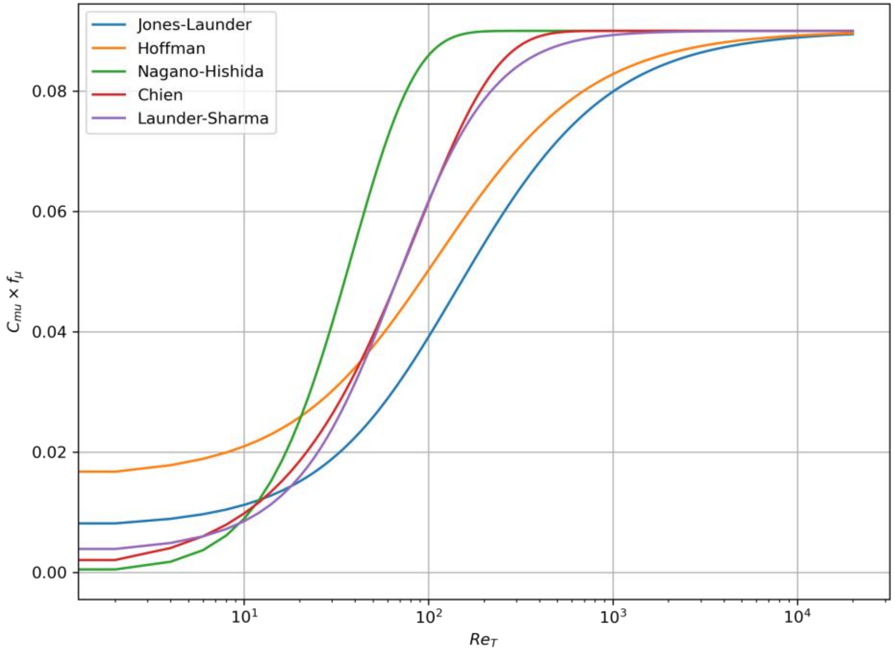
- Closure damping functions, coefficients, and extra source terms for the lowRE $k-\epsilon$ turbulence models.

Model	C_μ	$C_{\epsilon 1}$	$C_{\epsilon 2}$	σ_k	σ_ϵ
Standard	0.09	1.44	1.92	1.0	1.3
Jones-Launder [1,2]	0.09	1.44	1.92	1.0	1.3
Launder-Sharma [3]	0.09	1.44	1.92	1.0	1.3
Hoffman [4]	0.09	1.81	2.0	2.0	3.0
Nagano-Hishida [5]	0.09	1.45	1.9	1.0	1.3
Chien [6]	0.09	1.35	1.8	1.0	1.3

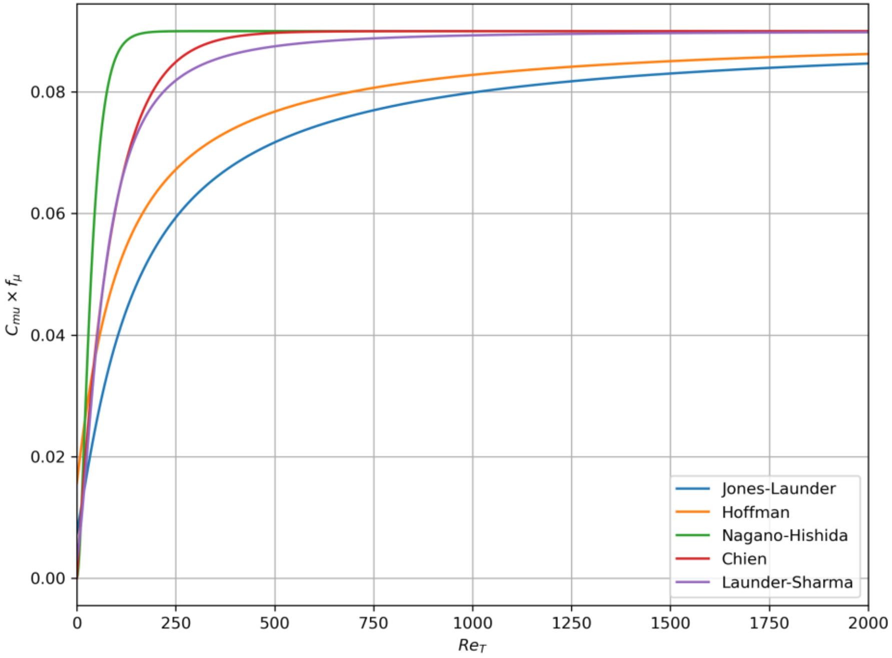
References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] G. Hoffman. Improved form of the low Reynolds number k-epsilon turbulence model. Physics of Fluids, vol. 18(3), pp. 309-312,1975.
[5] Y. Nagado, M. Hishida. Improved form of the k-epsilon model for wall turbulent shear flows. Journal of Fluids Engineering, vol. 109, pp. 156-160, 1987.
[6] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- Closure coefficient C_μ as a function of Re_T.
- These plots illustrate the damping effect towards the walls of the function f_μ of different lowRE turbulence models implementations.



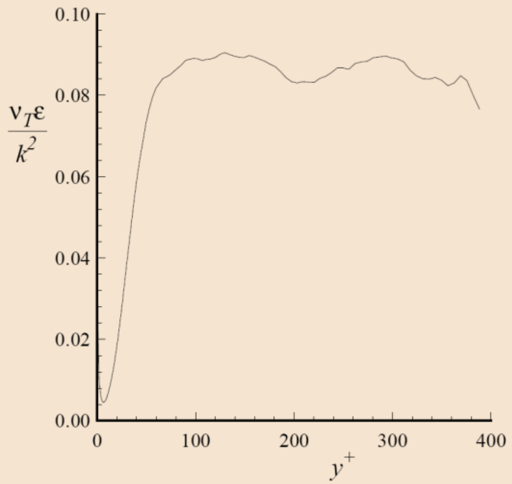
Semi-logarithmic scale



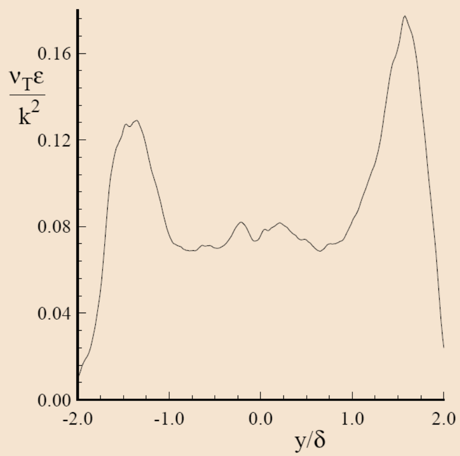
Linear scale

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- Experimental observations and numerical simulations show the value of C_μ decreases as y^+ decreases below 50, as illustrated in the figures below [1].
- The damping functions are specifically designed to have the proper behavior of C_μ as we approach to the walls.
- Depending on the formulation, it might be necessary to add extra source terms to balance or correct the equations.
- Also, we need to use the proper numerical values for the wall boundary conditions.



J. Kim, P. Moin, R. Moser. Turbulence statistics in fully developed channel flow at low Reynolds number. 1987.



M. Rogers, R. Moser. Direct simulation of a self-similar turbulent mixing layer. 1994.

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

- Let us address the wall boundary conditions of LRN models.
- Finding the right wall boundary conditions is not straightforward in the LRN formulation.
- Different LRN models will have different wall boundary conditions.
- The value of TKE at the wall for all the LRN models is zero. After all, there is no turbulence very close to the walls in the viscous sublayer,

$$k_{wall} = 0$$

- However, finding the value of the turbulent dissipation rate ϵ at the walls is rather complicated, not much is known about its value at the walls.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- We might argue that the value of ϵ is zero at the walls, because there is no turbulence in the viscous sublayer.
- But from the budgets of TKE and dissipation rate, it has been observed that the value of the dissipation at the wall is non-zero, and even sometimes it peaks at the walls.
- Basically, TKE is transported from the buffer layer towards the viscous sublayer where dissipation happens, and all this dissipation is due to molecular viscosity.
- We might also say, based on asymptotic analysis, that the dissipation rate value at the wall is equivalent to,

$$\epsilon/k \to 2\nu/y^2$$

- Any of the previous scenarios might be valid but they will also raise numerical instabilities when setting the wall boundary conditions.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- Besides any of the previous scenarios of possible boundary conditions for ϵ.
- If TKE is zero and the dissipation non-zero at the walls, the second term in the RHS of the transport equation of ϵ will become infinite (or very large).

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right] + L_k$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} f_1 \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} f_2 \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right] + L_\epsilon$$

$$\nu_t = f_\mu C_\mu \frac{k^2}{\epsilon}$$

Generalization of the $k-\epsilon$ turbulence model – The low-Reynolds formulation

- In addition, we must take into account the following considerations,
  - TKE and turbulent dissipation rate can not be prescribed arbitrarily because their development is governed by the turbulence transport equations in boundary layer.
  - In the $k-\epsilon$ family of turbulence models, the production of TKE and the dissipation rate, they are about the same order of magnitude in the log-layer.
  - And, as we approach to the walls, in the buffer layer, production and dissipation exhibit different behaviors, as illustrated in the figure below [1].
  - Therefore, TKE and ϵ must go to zero at the correct rate in order to balance the equations and to avoid production of turbulent viscosity production close to walls.
- We must somehow correct these issues (and other shortcomings) near the walls.

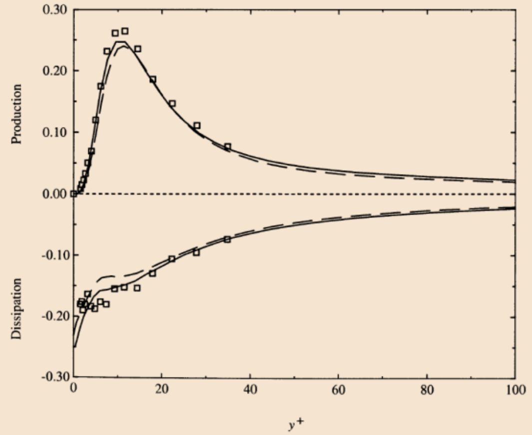


Fig. 4.11 Comparison of turbulent boundary layer and channel flow production and dissipation rates scaled with ν and u_τ. □, boundary layer measurements at $R_\tau = 1050$ [33]; —, DNS boundary layer at $R_\tau = 650$ [60]; – –, DNS channel flow $R_\tau = 590$ [49].

$$\nu_t = f_\mu C_\mu \frac{k^2}{\epsilon}$$

[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- In the Jones and Launder formulation [1,2], a dissipation variable named wall dissipation or modified dissipation $\tilde{\epsilon}$ is solved instead of turbulent dissipation rate ϵ.

$$\tilde{\epsilon} = \epsilon - \epsilon_0 = \epsilon - 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2$$

- This term is also known as dissipation value at the wall or,

$$\epsilon_0$$

- Each model will compute it in a different way.
- Its value tends to zero far from the walls, therefore,

$$\tilde{\epsilon} \approx \epsilon$$

- Some other authors call the term $\tilde{\epsilon}$ isotropic dissipation.
- It is important to stress that many LRN models use this variable instead of ϵ in the transport equation of the turbulent dissipation rate.
- Note that the circled term in the wall dissipation equation becomes very small away from the wall; therefore, this expression only modifies ϵ close to the walls.
- The following term in the transport equation of the dissipation rate can be updated as follows,

$$\frac{\epsilon^2}{k} \rightarrow \frac{\tilde{\epsilon}\epsilon}{k}$$

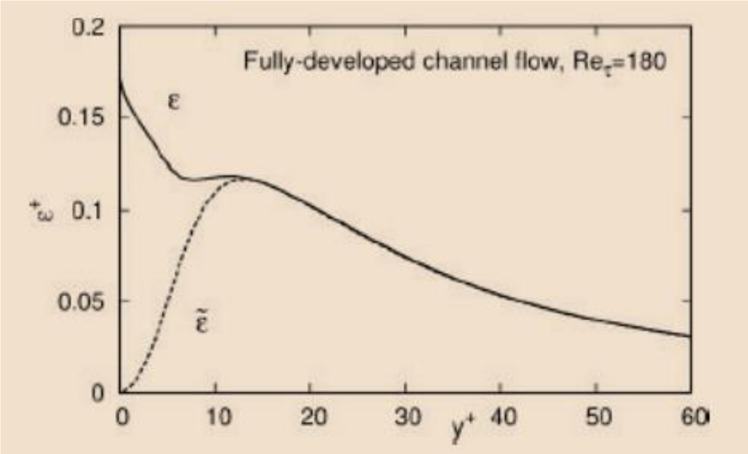
References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- The motivation of using the wall dissipation $\tilde{\epsilon}$ lies in its asymptotic behavior near the walls, as illustrated in the figure below.
- For the interested reader, in references [1,2] a detailed discussion of the asymptotic behavior is presented.
- Because of the quadratic near-wall variation of TKE at the wall, its wall limiting value is [1,2],

$$\tilde{\epsilon}\Big|_{y \to 0} = \epsilon - 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2 = \epsilon - 2\nu \frac{k}{y^2}$$

- Therefore, the wall value $\tilde{\epsilon}\Big|_{y \to 0}$ vanishes, as shown in the figure below.



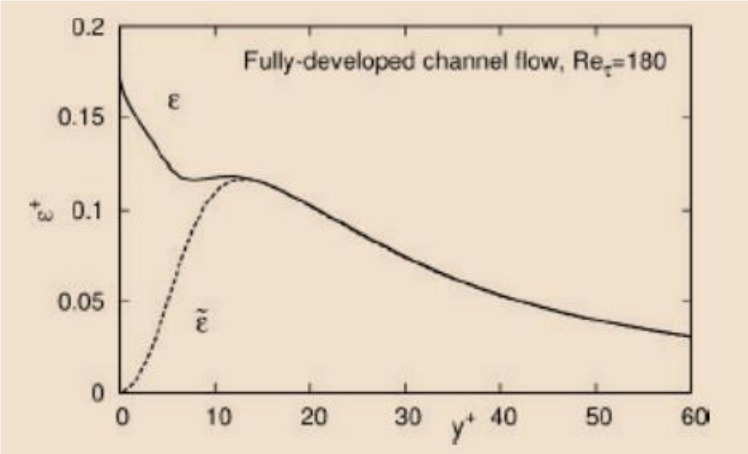
Comparison of dissipation and its “isotropic” form [2.]

[1] D. C. Wilcox. Turbulence Modeling for CFD. Third edition, DCW Industries, 2010.
[2] M. Leschziner. Statistical Turbulent Modeling for Fluid Dynamics. Imperial College Press, 2016.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- As we approach the wall, the value of $\tilde{\epsilon}$ vanishes, that is, $\tilde{\epsilon}_{wall} = 0$.
- This behavior decouples the wall boundary condition for $\tilde{\epsilon}$ from the solution of TKE.
- This coupling is a potential source of numerical instabilities if ϵ is the term to be solved.
- As it can be seen in the figure, $\tilde{\epsilon}$ is practically identical to ϵ outside of the viscous sub-layer.
- Now we can prescribe a consistent non-zero wall boundary condition for ϵ using $\tilde{\epsilon}$ [1,2].

$$\epsilon = \tilde{\epsilon} + \epsilon_0 = \tilde{\epsilon} + 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2$$



Comparison of dissipation and its "isotropic" form [2.]

[1] C. Speziale, R. Abid, E. Anderson. A Critical Evaluation of Two-Equation Models for Near Wall Turbulence. ICASE Report 90-46, 1990.
[2] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- The Jones-Launder [1,2], Launder-Sharma [3], and Chien [4] models build in the proper asymptotic behavior through introduction of the function ϵ_0 (dissipation at the wall).

$$\epsilon_0 = 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2$$

- Consequently, the boundary conditions appropriate at the surface are,

$$k = \tilde{\epsilon} = 0$$

- When this approach is taken, and extra source term needs to be added to the dissipation rate transport equation.
- This source term needs to be added in order to make sure that the correct value of the dissipation rate is recovered at the boundary. For example, in the Jones-Launder model [1], the source term is equal to,

$$-2\nu \left(\frac{\partial k^{1/2}}{\partial x_j} \right)^2$$

References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- By contrast, Lam and Bremhorst [1] deal directly with ϵ and specify the surface boundary condition on ϵ by requiring,

$$\epsilon = \nu \frac{\partial^2 k}{\partial y^2}$$

- Implementing this boundary condition is not so straightforward.
- And alternative to the previous boundary condition is the following one,

$$\frac{\partial \epsilon}{\partial y} = 0$$

- One advantage of the Lam and Bremhorst formulation is that it does not require extra source terms.
- This formulation is less robust than the previous ones.
- In reference [2], a review of several lowRE turbulence models is presented.

References:
[1] C. Lam, K. Bremhorst. Modified Form of k-epsilon Model for Predicting Wall Turbulence. ASME, Journal of Fluids Engineering, Vol. 103, pp. 456-460, 1981.
[2] V. Patel, W. Rodi, G. Scheuerer. Turbulence models for near-wall and low Reynolds number flows - A review. AIAA Journal, Volume 23, Number 9, September 1985.

Generalization of the $k - \epsilon$ turbulence model – The low-Reynolds formulation

- When using commercial solvers (Ansys Fluent in our case) you do not need to be concerned about the boundary conditions at the walls because this is done automatically by the solver and the wall treatment implementation.
- The wall boundary conditions of LRN models, are a source of confusion. We strongly recommended the interested reader to access the documentation of the model being used.
- The **NASA Turbulence Modeling Resource** is an excellent source of information related to turbulence models and validation cases,
    - https://turbmodels.larc.nasa.gov/