• What is the circled term?

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho} (\nabla p) + \nabla \cdot \left(\nu \nabla \bar{\mathbf{u}} + \frac{1}{\rho} \boldsymbol{\tau}^R\right)$$

• The circled term represents the total stresses,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho} (\nabla p) + \nabla \cdot \left(\nu \nabla \bar{\mathbf{u}} + \frac{1}{\rho} \boldsymbol{\tau}^R\right)$$

• The total stresses can be decomposed in a laminar contribution and a turbulent contribution.

$$\boldsymbol{ au}_{Total} = \boldsymbol{ au}_{Laminar} + \boldsymbol{ au}_{Turbulent}$$

• Where each contribution is given by,

$$\boldsymbol{\tau}_{Laminar} = \nu \nabla \bar{\mathbf{u}}$$

$$oldsymbol{ au}_{Turbulent} = rac{1}{
ho} oldsymbol{ au}^R$$

 The laminar (or viscous) stresses can be computed from the molecular viscosity and the mean velocity gradient,

$$\boldsymbol{\tau}_{Laminar} = \nu \nabla \bar{\mathbf{u}}$$

• The turbulent stresses (Reynolds stresses), can be computed as follows,

$$oldsymbol{ au}_{Turbulent} = rac{1}{
ho} oldsymbol{ au}^R$$

- As we have seen, this term is a little bit trickier to compute.
- This term can be approximated or modeled (*e.g.*, by using the Boussinesq approximation).
- Or it can be resolved (DNS, LES, DES, experiments).

• We can approximate the Reynolds stresses by using the Boussinesq approximation,

$$\tau^{R} = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = 2\mu_{T}\overline{\mathbf{D}}^{R} - \frac{2}{3}\rho k\mathbf{I} = \mu_{T} \left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}} \right)^{\mathrm{T}} \right] - \frac{2}{3}\rho k\mathbf{I}$$

• Or we can directly resolve it,

$$\tau^{R} = \tau_{ij}^{R} = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = -\begin{pmatrix} \rho \overline{u'u'} & \rho \overline{u'v'} & \rho \overline{u'w'} \\ \rho \overline{v'u'} & \rho \overline{v'v'} & \rho \overline{v'w'} \\ \rho \overline{w'u'} & \rho \overline{w'v'} & \rho \overline{w'w'} \end{pmatrix}$$

• We can derive a transport equation for each term of the Reynolds stress tensor au^R , or we can resolve the velocity fluctuations using SRS simulations (DES, LES, DNS).

• If we make the assumptions that we are working with a 2D boundary layer, the total stress can be approximated as follows,

$$\boldsymbol{\tau}_{Total} = \nu \frac{\partial \bar{u}}{\partial y} + \overline{u'v'}$$

• With no simplifications (3D case), the total shear stress is computed as follows,

$$oldsymbol{ au}_{Total} =
u
abla ar{\mathbf{u}} + rac{1}{
ho} oldsymbol{ au}^R$$

• Statistical moments.

$$\mu = E(X) \qquad \qquad \text{Mean}$$

$$\sigma^2 = E[(X - \mu)^2] \qquad \qquad \text{Variance}$$

$$S = \frac{\mu_3}{\sigma^3} = \frac{E[(X - \mu)^3]}{\sigma^3} \qquad \qquad \text{Skewness}$$

$$T = \frac{\mu_4}{\sigma^4} = \frac{E[(X - \mu)^4]}{\sigma^4} \qquad \qquad \qquad \text{Kurtosis}$$

$$\frac{\mu_n}{\sigma^n} = \frac{E[(X - \mu)^n]}{\sigma^n}$$

- In lecture 5, we talked about first order and second order closure methods.
- Let us recall these methods.
- In analogy to the statistical moments,
 - First order closure methods (or first moment closure), approximate the solution using the mean velocity field.
 - They are based on the Boussinesq approximation,

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho} \left(\nabla p \right) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R$$

• Second order closure methods (or second moment closure), compute the solution using the mean turbulent field,

$$\overline{u_i'\mathcal{N}(u_j) + u_j'\mathcal{N}(u_i)} = 0$$

• Where

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k x_k} = 0$$

• Second order closure methods are based on the Reynolds stress equations,

$$\frac{\partial \tau_{ij}^R}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}^R}{x_k} = -\left(\tau_{ik}^R \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}^R \frac{\partial \bar{u}_i}{\partial x_k}\right) 2\nu \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} + \dots$$
$$\dots + \frac{1}{\rho} \left(\overline{u'_i \frac{\partial p'}{\partial x_j}} + \overline{u'_j \frac{\partial p'}{\partial x_i}}\right) + \frac{\partial}{\partial x_k} \left(\nu \frac{\partial \tau_{ij}^R}{\partial x_k}\right) + \frac{\partial}{\partial x_k} \left(\overline{u'_i u'_j u'_k}\right)$$

• The previous equations can be rewritten as,

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$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + \epsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k} \left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk} \right]$$
$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} \qquad \rho C_{ijk} = \rho \overline{u'_i u'_j u'_k} + \overline{p' u'_i} \delta_{jk} + \overline{p' u'_j} \delta_{ik} \qquad \Pi_{ij} = \frac{\overline{p'}}{\rho} \left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} \right)$$

- There are higher order closure methods.
- For example, the equations for the third order moments, read as,



• Notice that we keep multiplying by the mean fluctuating velocity (in analogy to the statistical moments).

• Recall that the turbulent kinetic energy can be computed as follows,

$$k = \frac{1}{2}\overline{u'_i u'_i} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right)$$

- If you are running SRS simulations, you need to compute the average of the product of the fluctuations.
- The product of the fluctuations are derived from the primitive variables.
- Therefore, before running, remember to set the unsteady statistics and define the products of the fluctuating quantities.



Velocity field 0.0e+00 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0e+00

Instantaneous field



Velocity field 0.0e+00 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0e+00

Mean field

- To stress this point.
 - All the correlations (or products) appearing in the Reynolds stress tensor, are computed from the instantaneous field.
 - Therefore, you need to compute the unsteady statistics and define these variables, that is, the products of the fluctuations.
 - Recall that the Reynolds stress tensor read as,

$$\tau^{R} = \tau^{R}_{ij} = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = - \begin{pmatrix} \rho \overline{u'u'} & \rho \overline{u'v'} & \rho \overline{u'w'} \\ \rho \overline{v'u'} & \rho \overline{v'v'} & \rho \overline{v'w'} \\ \rho \overline{w'u'} & \rho \overline{w'v'} & \rho \overline{w'w'} \end{pmatrix}$$

• And recall that any instantaneous quantity can be decomposed as follows,

$$\phi = \overline{\phi} + \phi'$$

• Therefore, the fluctuating part can be computed as follows,

$$\phi' = \phi - \overline{\phi}$$

• Two-point correlation.



$$R_{ij}(\mathbf{x}, t; \mathbf{r}) = \overline{u'_i(\mathbf{x}, t) u'_j(\mathbf{x} + \mathbf{r}, t)} \quad \longleftarrow \quad \text{Two-point velocity correlation tensor}$$

$$R_{ij}(\mathbf{x}, t; t') = \overline{u'_i(\mathbf{x}, t) u'_j(\mathbf{x}, t + t')} \quad \longleftarrow \quad \text{Autocorrelation tensor}$$

• Two-point correlation.



$$f(x;r) = \frac{R_{11}(x;r)}{\overline{u_{1}^{\prime 2}}}$$

Longitudinal velocity correlation coefficient

$$g(x;r) = \frac{R_{22}(x;r)}{\overline{u_2'^2}}$$

Lateral velocity correlation coefficient

$$L_{11} = \int_0^\infty f\left(x;r\right) dr_1$$

Longitudinal length scale

$$L_{22} = \int_0^\infty g\left(x;r\right) dr_2$$

Lateral length scale

• Time series of the measured stream wise and wall-normal velocity components measured by means of laser-Doppler velocimetry.



- Left image. Time series of the measured stream wise (u component) and wall-normal (v component) velocity measured by means of laser-Doppler velocimetry in a turbulent boundary layer over a flat plate
- Right image. The correlation of the fluctuations u' = u u and v' = v v. The ellipse represents the correlation between the two velocity components, i.e. u'v', or Reynolds stress. The main axes of the ellipse are proportional to σ_u and σ_v, respectively. The measurement was taken at a distance of y⁺ = 20 viscous wall-units from the plate.
- Images adapted from reference [1].

• The turbulence intensity is defined as follows,

$$I = \frac{\mathbf{u}_{RMS}}{U_{\infty}}$$

- Where RMS stands for root mean square.
- The RMS of the velocity is equivalent to the velocity fluctuations,

$$\mathbf{u}' = \mathbf{u}_{RMS} = \sqrt{\frac{1}{3} \left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right)}$$

- The standard deviation is the same as the root mean squared, if it is defined about zero.
- We use the RMS to compute the turbulent intensity because otherwise you will get turbulent intensity levels that are too high.
- When you see the quantity RMS in solvers, it means that the fluctuations are computed about zero.

 The normal Reynolds stresses can be normalized relative to the mean flow velocity, as follows,

$$\widehat{u} \equiv \frac{\sqrt{\overline{u'^2}}}{\overline{u}} \quad \widehat{v} \equiv \frac{\sqrt{\overline{v'^2}}}{\overline{u}} \quad \widehat{w} \equiv \frac{\sqrt{\overline{w'^2}}}{\overline{u}}$$

• These three quantities are known as relative intensities.



Turbulence intensities for a flat-plate boundary layer of thickness [1].

 $\overline{u'^2}: \overline{v'^2}: \overline{w'^2} \approx 4:2:3$

[1] P. Klebanoff. "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient". NACA TN 1247, 1955.