## References

1 K.-T. Yang, "Possible Similarity Solutions for Laminar Free Convection on Vertical Plates and Cylinders," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, Series E, 1960, pp. 230–236. See also discussion of this paper in March, 1961, issue of the Journal of Applied Mechanics, pp. 153–154.

2 R. Eichhorn, "The Effect of Mass Transfer on Free Convection," *Journal of Heat Transfer*—Trans. ASME, vol. 82, Series C, 1960, pp. 260–263.

3 A. S. Gupta, "Steady and Transient Free Convection of an Electrically Conducting Fluid From a Vertical Plate in the Presence of a Magnetic Field," *Applied Science Research*, Section A, vol. 9, 1960, pp. 319–333.


# A Single Formula for the "Law of the Wall"


## D. B. SPALDING[1]



### Summary

It is shown that experimental velocity distributions may be well fitted, in the laminar sublayer, the transition region, and the turbulent core, by the formula:

$$y^+ = u^+ + 0.1108\{e^{0.4u^+} - 1 - 0.4u^+ - (0.4u^+)^2/2! - (0.4u^+)^3/3! - (0.4u^+)^4/4!\}$$

Omission of the $(0.4u^+)^4$ term gives an equally good fit. The corresponding expressions for the ratio of turbulent shear stress to total shear stress agree with the measurements of Laufer [8][2] quite closely.


### Nomenclature

$u$ = time-mean velocity of fluid in $x$-direction

$u^+$ = $u\sqrt{\rho/\tau}$

$x$ = distance along the wall in the direction of flow

$y$ = distance from the wall

$y^+$ = $y\sqrt{\tau\rho}/\mu_{\text{molecular}}$

$\epsilon^+$ = $\mu_{\text{total}}/\mu_{\text{molecular}}$

$\mu_{\text{molecular}}$ = absolute viscosity of fluid in laminar motion

$\mu_{\text{total}}$ = ratio of shear stress to gradient of time-mean velocity

$\mu_{\text{turb}}$ = $\mu_{\text{total}} - \mu_{\text{molecular}}$

$\rho$ = density of fluid

$\phi$ = density of fluid divided by density of fluid adjacent to wall

$\tau$ = shear stress in fluid, assumed independent of $y$

### Introduction

**Purpose of note.** Numerous formulas have been proposed to describe the universal turbulent velocity profile, called by Coles [1] the "law of the wall." The present note discloses a new formula which is valid over the whole range of dimensionless distance $y^+$.[3] The new formula has a form which, on the one hand, permits analytical determination of several important boundary-layer parameters, and, on the other, may provide the vantage point for a new look at the theory of the turbulent boundary layer. These matters are only touched on briefly in the following.

**The universal turbulent velocity profile.** Prandtl's [12] postulate, that the velocity in the neighborhood of a wall should obey the relation:

$$u^+ = u^+(y^+) \tag{1}$$

has been confirmed experimentally by Nikuradse [10], and subsequently by many other authors.

The experimental relation has been described analytically in various ways, some of which are listed in Table 1. It will be noted that all the authors mentioned, except Reichardt [15] and van Driest [19], have found it necessary to use at least two expressions, valid for different ranges of $y^+$, in order to describe the profile adequately.

**Table 1 Formulas for the "law of the wall"[a]**

| Author | Range of validity | Formulas |
|---|---|---|
| Prandtl [11][b] | $0 \leqslant y^+ < 11.5$ | $u^+ = y^+$ |
| Taylor [18][b] | $11.5 \leqslant y^+$ | $u^+ = 2.5 \ln y^+ + 5.5$ |
| von Karman [7] | $0 \leqslant y^+ < 5$ | $u^+ = y^+$ |
| | $5 \leqslant y^+ < 30$ | $u^+ = 5 \ln y^+ - 3.05$ |
| | $30 \leqslant y^+$ | $u^+ = 2.5 \ln y^+ + 5.5$ |
| Reichardt [15] | $0 \leqslant y^+$ | $u^+ = 2.5 \ln(1 + 0.4y^+) + 7.8\{1 - e^{-y^+/11} - (y^+/11)e^{-0.33y^+}\}$ |
| Deissler [2] | $0 \leqslant y^+ < 26$ | $u^+ = \int_0^{y^+} \frac{dy^+}{1 + n^2u^+y^+(1 - e^{-n^2u^+y^+})}$, $n = 0.124$ |
| | $26 \leqslant y^+$ | $u^+ = 2.78 \ln y^+ + 3.8$ |
| van Driest [19] | $0 \leqslant y^+$ | $u^+ = \int_0^{y^+} \frac{2dy^+}{1 + \{1 + 0.64y^{+2}[1 - \exp(-y^+/26)]^2\}^{1/2}}$ |
| Rannie [13] | $0 \leqslant y^+ < 27.5$ | $u^+ = 14.54 \tanh(0.0688y^+)$ |
| | $27.5 \leqslant y^+$ | $u^+ = 2.5 \ln y^+ + 5.5$ |

[a] See also Hofmann [5], Reichardt [14], Rotta [16], Miles [9], Elrod [3], and Frank-Kamenetsky [21].

[b] These authors did not, at the dates in question, state the formulas attributed to them in the table. However, they did introduce the idea of a sharp division between a laminar sublayer and a fully turbulent core; when compared with experimental data, this idea leads directly to the formulas given.

**The problem.** A single formula, expressing the $u^+(y^+)$ relation over the whole range of the variables, is both more satisfying aesthetically and more convenient practically than the two-point formulas of Table 1. However, Reichardt's formula is rather complex in form, whereas van Driest's involves a quadrature requiring numerical evaluation. There is need for a simpler, easily evaluated formula.

Such a formula would preferably fit the experimental data closely, contain sufficient adjustable constants to permit modification in the light of new experimental data, and have an analytical form permitting easy integration of the various functions of the velocity distribution which arise in, for example, the theory of heat transfer through a turbulent boundary layer.

Looked at mathematically, our problem is to establish a formula which:

(i) passes through the point: $y^+ = 0$, $u^+ = 0$;
(ii) is tangential at this point to: $u^+ = y^+$;
(iii) is asymptotic at large $y^+$ to:[4]

$$u^+ = 2.5 \ln y^+ + 5.5 \tag{2}$$

(iv) fits the experimental points at intermediate $y^+$ values.

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Manuscript received by ASME Applied Mechanics Division, March 8, 1961.

[2] Numbers in brackets indicate References at end of Note.

[3] See Nomenclature at beginning of Note.

[4] Here the most popular constants for the logarithmic velocity profile have been accepted.

## The New "Law of the Wall"

**The simplest $y^+ (u^+)$ relation.** The previous efforts to find a single formula fitting the foregoing specification $u^+$ has been sought explicitly in terms of $y^+$. There is, however, no need to demand this; a relation giving $y^+$ explicitly in terms of $u^+$ is just as good, and indeed may even be better for some purposes.

Once this possibility is recognized, progress can be made swiftly. We now seek a $y^+(u^+)$ relation such that

$$\text{near } u^+ = 0: y^+ = u^+ \tag{3}$$

$$\text{and at large } u^+: y^+ = 0.1108e^{0.4u^+} \tag{4}$$

the latter equation being derived directly from equation (2).

The equation which immediately suggests itself is:

$$y^+ = u^+ + 0.1108(e^{0.4u^+} - 1 - 0.4u^+) \tag{5}$$

This satisfies requirements (3) and (4). Does it also fit the experimental data? This can be judged by reference to Fig. 1, which contains the experimental data of Laufer [8]. Evidently, equation (5) fits the data fairly well, but gives values of $u^+$ which are approximately 10 per cent low when $y^+$ lies between 10 and 50. Fig. 1 also contains, as broken curves, the asymptotic expressions (2) and (3).

**Improved $y^+(u^+)$ relations.** If we define a dimensionless "total" (i.e., "molecular plus turbulent" viscosity) $\epsilon^+$ by

$$\epsilon^+ = \mu_{\text{total}}/\mu_{\text{molecular}} \tag{6}$$

then the assumption that the shear stress is independent of distance from the wall, when combined with the definitions of $u^+$ and $y^+$, leads to the relation:

$$\epsilon^+ = \frac{dy^+}{du^+} \tag{7}$$

Equation (5) therefore implies the $\epsilon^+(u^+)$ relation:

$$\begin{aligned}\epsilon^+ &= 1 + 0.4 \times 0.1108(e^{0.4u^+} - 1)\\ &= 1 + 0.04432\left\{0.4u^+ + \frac{(0.4u^+)^2}{2!} + \ldots\right\}\end{aligned} \tag{8}$$

Now there are theoretical reasons (Reichardt, [15]; Hinze, [4]) against a growth of $\epsilon^+$ in the wall region with a power of $y^+$ which is less than 3, if the shear stress varies along the wall, and less than 4 if there is no such variation. Equation (8) satisfies neither requirement.[5] However, it is easy to see what must be done to the velocity distribution if either of these requirements is to be satisfied: the distribution formula becomes, respectively:

$$y^+ = u^+ + 0.1108\left\{e^{0.4u^+} - 1 - 0.4u^+ - \frac{(0.4u^+)^2}{2!} - \frac{(0.4u^+)^3}{3!}\right\} \tag{9}$$

or

$$y^+ = u^+ + 0.1108\left\{e^{0.4u^+} - 1 - 0.4u^+ - \frac{(0.4u^+)^2}{2!} - \frac{(0.4u^+)^3}{3!} - \frac{(0.4u^+)^4}{4!}\right\} \tag{10}$$

Curves corresponding to equations (9) and (10) are plotted in Fig. 1. They fit the experimental data rather better than does equation (5), but it is not possible to say which of the two gives the more precise fit. Whether the $(0.4u^+)^4$ term should be included or not will therefore probably have to be decided on other grounds.

[5] Nor, incidentally, do the expressions of Reichardt and van Driest which appear in Table 1.

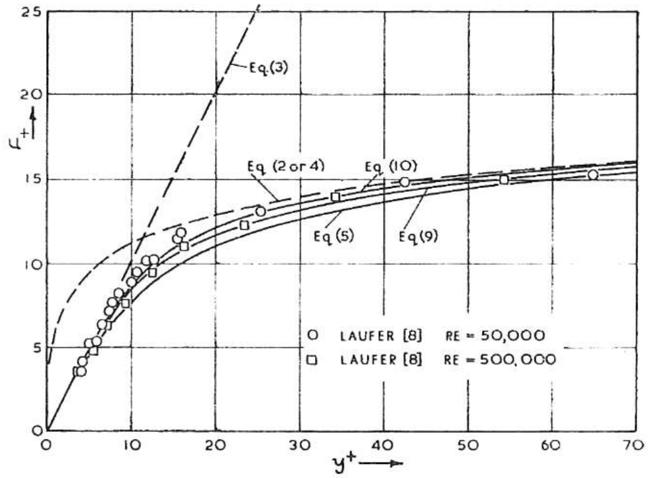


**Fig. 1 Experimental data of Laufer [8] for velocity distribution near the wall in turbulent pipe flow, compared with various analytical expressions**

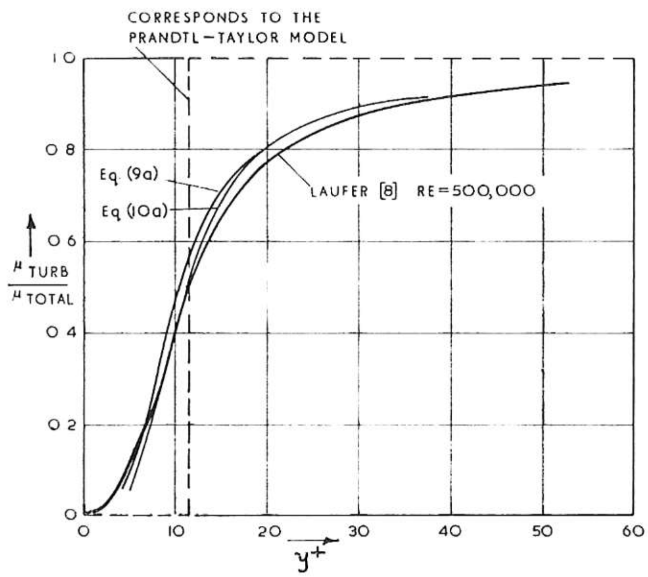


**Fig. 2 Experimental data of Laufer on turbulent-stress distribution near the wall in turbulent pipe flow, compared with various analytical expressions**

Laufer [8] has also made measurements of the ratio of the turbulent shear stress divided by the total shear stress near the wall. His measurements in a pipe flow, of Reynolds number 500,000, are shown in Fig. 2 as a bold line; $y^+$ is the abscissa and the viscosity ratio $\mu_{\text{turb}}/\mu_{\text{total}}$ is the ordinate. Also drawn in Fig. 2 are the corresponding relations deduced from equations (9) and (10). These are, respectively:

$$\frac{\mu_{\text{turb}}}{\mu_{\text{total}}} = 1\Bigg/\left[1 + 1/0.04432\left\{e^{0.4u^+} - 1 - 0.4u^+ - \frac{(0.4u^+)^2}{2!}\right\}\right] \tag{9a}$$

and

$$\frac{\mu_{\text{turb}}}{\mu_{\text{total}}} = 1\Bigg/\left[1 + 1/0.04432\left\{e^{0.4u^+} - 1 - 0.4u^+ - \frac{(0.4u^+)^2}{2!} - \frac{(0.4u^+)^3}{3!}\right\}\right] \tag{10a}$$

Comparison of these relations with the experimental curves shows that the former equation gives the better fit at low $y^+$, while the latter gives the better fit at high $y^+$. However, it is probable that both curves can be regarded as equally satisfactory when experimental scatter is taken into account.

Also plotted in Fig. 2, as a broken steplike curve, is the $\mu_{turb}/\mu_{total}$ distribution which corresponds to the assumption of a sharp boundary between a laminar sublayer and a fully turbulent outer region. Clearly this gives a very poor representation of the data.

**Further possible improvements.** Equation (10) fits the requirement that $\epsilon^+$ increases with the fourth power of $u^+$, and so of $y^+$, close to the wall. However, even if this is correct, there is no reason why the first nonzero term of the expansion should happen to be that which appears in the expansion of $0.1108e^{0.4u^+}$. In other words, it may be that further terms should appear inside the braces of equations (9) and (10) which have the effect of only partially canceling the corresponding terms in the exponential expansion. Discussion of such further developments will be deferred to a later publication.

## Discussion

**Practical use of the new formula.** Fig. 1 shows that equations (9) or (10) can be used to represent the "law of the wall" within the accuracy of the experimental data. Moreover, as just noted, the general form of these equations is sufficiently flexible to accommodate any further modifications of constants which experiment shows to be necessary. Of course, the constants 0.4 and 0.1108 must not be regarded as sacrosanct.

It should also be noted that the form of the equations is very suitable for analytical work involving such expressions as $\int u^+ dy^+$; for this integral can be written as $\int u^+(dy^+/du^+)du^+$, which can be evaluated in closed form, since $dy^+/du^+$ is easily obtained by differentiating the $y^+(u^+)$ relation. The way is therefore open to the analytical derivation of drag laws, for example, without the approximations which are usually introduced (e.g., "seventh-power" profiles). These possibilities will be elaborated elsewhere. (See, for example, Spalding [17].)

**Theoretical implications.** Equations (9) and (10) are presented solely as useful interpolation formulas; they are not based on any postulated mechanism of turbulent transport. Nevertheless, they provoke certain questions which it may be profitable to investigate further. Some of these will now be listed.

(i) Does (10), for example, satisfy a differential equation in which $u^+$ and $y^+$ appear *only* as differentials?

The answer is readily seen; it is:

$$\frac{d^6y^+}{du^{+6}} = 0.4\,\frac{d^5y^+}{du^{+5}} \tag{11}$$

Similarly, equation (5) satisfies the differential equation:

$$\frac{d^3y^+}{du^{+3}} = 0.4\,\frac{d^2y^+}{du^{+2}} \tag{12}$$

(ii) Such differential equations are reminiscent of those derived by Prandtl [12] and von Karman [7] as starting points for the logarithmic velocity profile. Can a physical significance be attached to these equations? Could they have been derived by postulation of a physical model followed by dimensional analysis?

(iii) The von Karman differential equation is derived from the consideration that the local "mixing length" must be related to local values of $(\partial u/\partial y)$, $(\partial^2 u/\partial y^2)$, and so forth. Is there any reason why $u$ should have been chosen as dependent and $y$ as independent variable in this analysis, other than the irrelevant one that we happen to perform experiments by fixing the position of the Pitot tube first and then taking the reading? If not, a relation of the mixing length to $(\partial y/\partial u)$, $(\partial^2 y/\partial u^2)$, and so on, is equally valid.

(iv) When the density varies such that density ratio $\phi$ is a known function of $u^+$, is it reasonable to calculate the velocity profile from a suitably modified version of (11)? This would run:

$$\frac{d^6y^+}{du^{+6}} = 0.4\phi(u^+)\cdot\frac{d^5u^+}{du^{+5}} \tag{13}$$

which can be evaluated by numerical quadrature without difficulty. This thought might lead to more[6] satisfactory theories of friction and heat transfer in compressible boundary layers. If equation (13) is not as suitable a starting point for analysis as that, for example, of van Driest [20], what is the physical reason for this?

It is not intended to suggest answers to these questions here. They are put forward solely to provoke thought and criticism.

## Conclusions

(*a*) Formulas have been presented [equations (9) and (10)] which represent adequately the experimental data for the universal turbulent velocity profile when the viscosity and density of the fluid are uniform.

(*b*) The formulas are flexible enough to permit further adjustment of constants in the light of new experimental data, and simple enough in form to permit analytical integration in important cases of interest.

(*c*) The formulas represent $y^+$ explicitly in terms of $u^+$ instead of vice versa. It appears possible that other aspects of turbulent boundary-layer analysis may be profitably re-examined with velocity as the independent variable.

## Acknowledgment

The author expresses his gratitude to Prof. J. O. Hinze of Delft, Holland, for his helpful comments on an earlier draft of this Note.

## References

1 D. Coles, "The Law of the Wake in the Turbulent Boundary Layer," *Journal of Fluid Mechanics*, vol. 1, 1956, pp. 191–226.

2 R. G. Deissler, "Analysis of Turbulent Heat Transfer, Mass Transfer and Friction in Smooth Tubes at High Prandtl and Schmidt Numbers," NACA Tech. Rep. 1210, 1955 (supersedes NACA Tech. Note 3145, 1954).

3 H. G. Elrod, "Note on the Turbulent Shear Stress Near a Wall," *Journal of the Aeronautical Sciences*, vol. 24, 1957, p. 468.

4 J. O. Hinze, "Turbulence," McGraw-Hill Book Company, New York, N. Y., 1959, p. 472.

5 E. Hofmann, *Forschung a.d. Geb. Ing.*, vol. 11A, 1940, p. 159.

6 T. von Karman, "Mechanische Ähnlichkeit u. Turbulenz," *Nachr. Ges. der Wiss. Göttingen, Math. Phys. Klasse*, vol. 58, 1930.

7 T. von Karman, "The Analogy Between Fluid Friction and Heat Transfer," Trans. ASME, vol. 61, 1939, pp. 705–710.

8 J. Laufer, "The Structure of Turbulence in Fully Developed Pipe Flow," NACA Tech. Rep. 1174, 1954.

9 J. W. Miles, *Journal of the Aeronautical Sciences*, vol. 24, 1957, p. 704.

10 J. Nikuradse, "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren," *Forschung a.d. Geb. Ing.*, No. 356, 1932.

11 L. Prandtl, "Eine Beziehung zwischen Wärmeaustausch und Strömungswiderstand der Flüssigkeit," *Z. Physik*, vol. 11, 1910, pp. 1072–1078.

12 L. Prandtl, "Über die ausgebildete Turbulenz," *Z. für angew. Math. Mech.*, vol. 5, 1925, p. 136.

13 W. D. Rannie, "Heat Transfer in Turbulent Shear Flow," *Journal of the Aeronautical Sciences*, vol. 23, 1956, p. 485.

14 H. Reichardt, "Die Wärmeübertragung in turbulenten Reibungschichten," *Z. angew. Math. Mech.*, vol. 20, 1940, p. 297.

15 H. Reichardt, "Vollständige Darstellung der turbulenten Geschwindigkeitsverteilung in glatten Leitungen," *Z. angew. Math. Mech.*, vol. 31, 1951, pp. 208–219.

16 J. Rotta, *Ing.-Archiv.*, vol. 18, 1950, p. 277.

17 D. B. Spalding, "Heat Transfer to a Turbulent Stream From a Surface With a Step-Wise Discontinuity in Wall Temperatures," paper to be presented at the joint ASME-I. Mech. E. Heat Transfer Conference, Boulder, Colo., August 28–September 1, 1961.

[6] Or, of course, less.

18 G. I. Taylor, "Conditions at the Surface of a Hot Body Exposed to the Wind," Brit. Aero. Res. Comm. R&M No. 272, 1916, p. 423.

19 E. R. van Driest, "On Turbulent Flow Near a Wall," *Journal of the Aeronautical Sciences*, vol. 23, 1956, p. 1007.

20 E. R. van Driest, "Turbulent Boundary Layer in Compressible Fluids," *Journal of the Aeronautical Sciences*, vol. 18, 1951, p. 145.

21 D. A. Frank-Kamenetsky, "Diffusion and Heat Exchange in Chemical Kinetics," Princeton University Press, 1955, p. 184. First published in the USSR, 1947.

# On Classical Normal Modes of a Damped Linear System

**MORRIS MORDUCHOW**[1]

It has been essentially shown by Rayleigh [1][2] that if the damping matrix of a linear vibrating system is a linear combination of the stiffness and inertia matrices, then the damped system will have principal modes which are exactly the same as those of the undamped system. Caughey [2] has recently developed more general conditions for the existence of classical normal modes with damping, including the above condition as a special case. In both [1] and [2], the analysis is based on the use of normal coordinates. The purpose of this Note is to demonstrate Rayleigh's condition (equation (2) below) in a straightforward manner without the use of normal co-ordinates and hence without asssuming a knowledge of the theory associated with transformations to such co-ordinates. This procedure, in addition to being instructive, will also lead to explicit results for the damping factor and natural frequency in any principal mode, and will be seen to yield some interesting implications. Finally, the method of analysis given here will be applied to a vibrating beam with simultaneous internal and external damping.

Let a dynamical system be governed by the equations

$$[m]\{\ddot{h}\} + [c]\{\dot{h}\} + [k]\{h\} = 0 \tag{1}$$

where $[m]$, $[c]$, and $[k]$ are square (inertia, damping, and stiffness, respectively) matrices of order $n$. Moreover, suppose

$$[c] = a[m] + b[k] \tag{2}$$

where $a$ and $b$ are any constants. To solve equations (1), let

$$\{h\} = \{H\}e^{pt} \tag{3}$$

where $\{H\}$ is independent of the time $t$, and $p$ is a constant. Then, if equation (2) holds, equation (1) reduces to

$$\left([m] + \left(\frac{1 + bp}{p^2 + ap}\right)[k]\right)\{H\} = 0 \tag{4}$$

Equation (4) is seen to be the same as the equation for no damping ($a = b = 0$), but with $(1/p^2)$ replaced by $(1 + bp)/(p^2 + ap)$. Hence the characteristic normalized vectors $\{H\}$ with damping will be the same as those without the damping. Moreover, if in the $k$th mode without damping $p_k^2 = -\omega_{k_o}^2$ (where $\omega_{k_o}$ denotes the undamped natural frequency in the $k$th mode), then for the $k$th mode with damping

$$\frac{1 + bp_k}{p_k^2 + ap_k} = -\frac{1}{\omega_{k_o}^2} \tag{5}$$

Thus

$$p_k = -d_k \pm i\omega_k \tag{6a}$$

where

$$d_k = \frac{a + \omega_{k_o}^2 b}{2}, \quad \omega_k = \omega_{k_o}\left[1 - \left(\frac{d_k}{\omega_{k_o}}\right)^2\right]^{1/2} \tag{6b}$$

Equations (6*a*, *b*) give the damping factor $d_k$ and the natural frequency $\omega_k$ for any principal mode with damping when condition (2) holds. In the latter case, in fact, a necessary and sufficient condition for *dynamic stability* of the system is that

$$a + \omega_{k_o}^2 b \geqslant 0 \tag{7}$$

for *each* undamped natural frequency $\omega_{k_o}$. [It is interesting to note that (7) can be satisfied even in cases when either $a$ or $b$ (but not both) is negative. Equations (3) through (6*b*) are valid whether $[c]$ is positive definite or not.]

Consider, finally, a *beam* subjected to an external damping load $f(x)\partial Y/\partial t$ and an internal damping load $(g/\omega)\partial/\partial t(EIY'')''$ (cf., e.g., [3]) per unit length, where $' \equiv \partial/\partial x$. Moreover, suppose $f(x) = c\rho(x)$, where $c$ is a constant,[3] and $\rho(x)$ is the mass per unit length of the beam. Let $Y(x, t) = y(x)e^{pt}$. Then the equation for the free bending vibrations reduces to:

$$(EI(x)y'')'' + \rho(x)\left(\frac{cp + p^2}{1 + \dfrac{g}{\omega}p}\right)y = 0 \tag{8}$$

Hence the principal mode shapes $y(x)$ will be the same as without any damping, and the value of $p$ in any mode will be such that

$$\frac{cp + p^2}{1 + \dfrac{g}{\omega_k}p} = -\omega_{k_o}^2 \tag{9}$$

where $\omega_{k_o}$ is the undamped natural frequency in the $k$th mode. Setting $p = -d_k + i\omega_k$, equation (9) implies

$$d_k = \frac{c}{2} + \frac{g}{2}\frac{\omega_{k_o}^2}{\omega_k} \tag{10a}$$

where

$$4\omega_k^4 + (c^2 - 4\omega_{k_o}^2)\omega_k^2 + 2cg\omega_{k_o}^2\omega_k + g^2\omega_{k_o}^4 = 0 \tag{10b}$$

To first powers of $g$,

$$\omega_k = \omega_{kc} - \frac{cg}{4}\frac{\omega_{k_o}^2}{\omega_{kc}^2} \tag{11}$$

where $\omega_{kc} = [\omega_{k_o}^2 - (c/2)^2]^{1/2}$ = the natural frequency in the $k$th mode for $g = 0$. In the case of internal damping only ($c = 0$), equations (10*a*) and (10*b*) yield:

$$d_k = \frac{g\omega_{k_o}}{\sqrt{2}}\left[1 + (1 - g^2)^{1/2}\right]^{-1/2} = \frac{\omega_{k_o}}{\sqrt{2}}[1 - (1 - g^2)^{1/2}]^{1/2} \tag{12a}$$

$$\omega_k = \frac{\omega_{k_o}}{\sqrt{2}}[1 + (1 - g^2)^{1/2}]^{1/2} \tag{12b}$$

## References

1 Lord Rayleigh, "Theory of Sound," vol. 1, Dover Publications, New York, N. Y., pp. 130–131.

2 T. K. Caughey, "Classical Normal Modes in Damped Linear Dynamic Systems," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, Series E, June, 1960, pp. 269–271.

3 M. Morduchow, "On Internal Damping of Rotating Beams," NACA Technical Note 1996, December, 1949.

4 M. Morduchow, "On Application of a Quasi-Static Variational Principle to a System With Damping," Journal of Applied Mechanics, vol. 21, Trans. ASME, vol. 76, March, 1954, pp. 8–10.

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[2] Numbers in brackets indicate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, March 2, 1961.

[3] In [4] it has been shown that a necessary, as well as sufficient, condition that the mode shapes be entirely unaffected by a damping load of the form $f(x)\partial Y/\partial t$ is that $f(x)$ be proportional to $\rho(x)$.