Quick review of index and vector notation

- These two sets of equations are the same, we simply wrote them using different notations.

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

Vector notation

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j}$$

Index notation

- Let us illustrate index and vector notation using operators that commonly appears in the governing equations of fluid dynamics.

- In our notation, the indices $i,\ j,\ k$ can take the following values,

$$\left.\begin{matrix} i \\ j \\ k \end{matrix}\right\} 1, 2, 3$$

- For example, the vectors x_i and u_j, are defined as follows,

$$x_i \begin{cases} x_1 = x \\ x_2 = y \\ x_3 = z \end{cases} \qquad u_j \begin{cases} u_1 = u \\ u_2 = v \\ u_3 = w \end{cases}$$

- The second rank tensor $\boldsymbol{\tau}_{ij}$ is defined as follows,

$$\boldsymbol{\tau}_{ij} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

- One free index results in a vector.
- It represents a gradient.

$$\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_3}$$

- In vector notation, is equivalent to,

$$\frac{\partial \phi}{\partial x_i} = \mathrm{grad}\,\phi = \nabla \phi$$

- The gradient will increase the rank of a tensor.
- That is, a zero-rank tensor (scalar), will become a first-rank tensor (vector), and a first-rank tensor will become a second-rank tensor (tensor).

- One repeated index results in a scalar.
- It is the sum over the index.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

- In vector notation, is equivalent to,

$$\frac{\partial u_i}{\partial x_i} = \operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u}$$

- The divergence will decrease the rank of a tensor.
- That is, a second-rank tensor (tensor), will become a first-rank tensor (vector), and a first-rank tensor will become a zero-rank (tensor).

- Two free indices results in a second rank tensor.

$$u_i u_j = \begin{pmatrix} u_1 u_1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2 u_2 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3 u_3 \end{pmatrix} \rightarrow \mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \mathbf{a}\mathbf{b}$$

$$\frac{\partial \phi_i}{\partial x_j} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \\ \frac{\partial \phi_3}{\partial x_1} & \frac{\partial \phi_3}{\partial x_2} & \frac{\partial \phi_3}{\partial x_3} \end{pmatrix} \rightarrow \nabla \phi$$

Where ϕ is a vector

$$\frac{\partial \phi_j}{\partial x_i} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_3}{\partial x_1} \\ \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_3}{\partial x_2} \\ \frac{\partial \phi_1}{\partial x_3} & \frac{\partial \phi_2}{\partial x_3} & \frac{\partial \phi_3}{\partial x_3} \end{pmatrix} \rightarrow \nabla \phi^T$$

Where ϕ is a vector

- Two free indices results in a tensor.

$$\tau_{ij}^R = -\rho\overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

$$\delta_{ij} \begin{cases} = 1 & \text{if} \quad i = j \\ = 0 & \text{otherwise} \end{cases}$$

Kronecker delta

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Strain rate tensor

$$\tau_{ij}^R = -\rho\overline{u_i' u_j'} = \begin{bmatrix} 2\mu S_{11} - \frac{2}{3}\rho k & 2\mu S_{12} & 2\mu S_{13} \\ 2\mu S_{21} & 2\mu S_{22} - \frac{2}{3}\rho k & 2\mu S_{23} \\ 2\mu S_{31} & 2\mu S_{32} & 2\mu S_{33} - \frac{2}{3}\rho k \end{bmatrix}$$

- In vector notation, is equivalent to,

$$-\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I}$$

$$\mathbf{S} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$$

Strain rate tensor

- Two repeated indices (j) and one free indices (i) results in a tensor.
- Summation in j and it will form a vector in i.

$$\frac{\partial(\overline{u_i u_j})}{\partial x_j} = \frac{\partial(\overline{u_i u_1})}{\partial x_1} + \frac{\partial(\overline{u_i u_2})}{\partial x_2} + \frac{\partial(\overline{u_i u_3})}{\partial x_3}$$

→ Summation in j

$$\frac{\partial(\overline{u_i u_1})}{\partial x_1} + \frac{\partial(\overline{u_i u_2})}{\partial x_2} + \frac{\partial(\overline{u_i u_3})}{\partial x_3} =$$

$$\left(\frac{\partial(\overline{u_1 u_1})}{\partial x_1} + \frac{\partial(\overline{u_1 u_2})}{\partial x_2} + \frac{\partial(\overline{u_1 u_3})}{\partial x_3}\right)\vec{i} +$$

$$\left(\frac{\partial(\overline{u_2 u_1})}{\partial x_1} + \frac{\partial(\overline{u_2 u_2})}{\partial x_2} + \frac{\partial(\overline{u_2 u_3})}{\partial x_3}\right)\vec{j} +$$

$$\left(\frac{\partial(\overline{u_3 u_1})}{\partial x_1} + \frac{\partial(\overline{u_3 u_2})}{\partial x_2} + \frac{\partial(\overline{u_3 u_3})}{\partial x_3}\right)\vec{k}$$

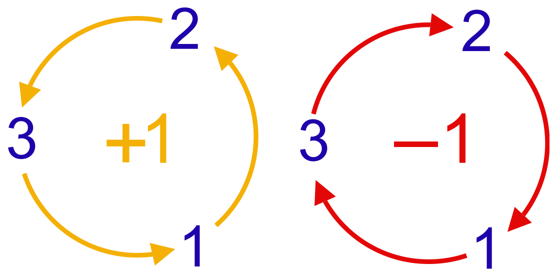
→ Vector in i

- Permutation or Levi-Civita operator.

$$\varepsilon_{ijk} \begin{cases} = 0 & \text{if any two of } i, j, k \text{ are the same} \\ = 1 & \text{for even permutation} \\ = -1 & \text{for odd permutation} \end{cases}$$

$$\text{for even permutation} = 123, 312, 231$$

$$\text{for odd permutation} = 321, 132, 213$$



- Using the Levi-Civita operator in the following way,

$$\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

- Results in the following vector,

$$\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \underbrace{\varepsilon_{i11}\frac{\partial u_1}{\partial x_1}}_{\substack{j=1\\k=1\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i12}\frac{\partial u_2}{\partial x_1}}_{\substack{j=1\\k=2\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=+1}} + \underbrace{\varepsilon_{i13}\frac{\partial u_3}{\partial x_1}}_{\substack{j=1\\k=3\\i=1\to\varepsilon=0\\i=2\to\varepsilon=-1\\i=3\to\varepsilon=0}} + \dots$$

$$\underbrace{\varepsilon_{i21}\frac{\partial u_1}{\partial x_2}}_{\substack{j=2\\k=1\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=-1}} + \underbrace{\varepsilon_{i22}\frac{\partial u_2}{\partial x_2}}_{\substack{j=2\\k=2\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i23}\frac{\partial u_3}{\partial x_2}}_{\substack{j=2\\k=3\\i=1\to\varepsilon=+1\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \dots$$

$$\underbrace{\varepsilon_{i31}\frac{\partial u_1}{\partial x_3}}_{\substack{j=3\\k=1\\i=1\to\varepsilon=0\\i=2\to\varepsilon=+1\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i32}\frac{\partial u_2}{\partial x_3}}_{\substack{j=3\\k=2\\i=1\to\varepsilon=-1\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i33}\frac{\partial u_3}{\partial x_3}}_{\substack{j=3\\k=3\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}}$$

- Using the Levi-Civita operator in the following way,

$$\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

- And after some algebra, we obtain the following vector,

$$\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\vec{i} + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\vec{j} + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\vec{k}$$

- In vector notation, is equivalent to,

$$\operatorname{curl}\mathbf{u} = \nabla \times \mathbf{u}$$

- A second rank tensor can also be decomposed into a deviatoric part (anisotropic part) and a spherical part (isotropic part).
- This procedure is known as additive decomposition of a second rank tensor.

Index notation	Vector notation
$\mathrm{dev}\left(\tau_{ij}\right) = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}$	$\mathrm{dev}\left(\tau\right) = \tau - \frac{1}{3}\mathrm{tr}\left(\tau\right)\mathbf{I}$
$\mathrm{sph}\left(\tau_{ij}\right) = \frac{1}{3}\tau_{kk}\delta_{ij}$	$\mathrm{sph}\left(\tau\right) = \frac{1}{3}\mathrm{tr}\left(\tau\right)\mathbf{I}$
$\tau_{ij} = \mathrm{dev}\left(\tau_{ij}\right) + \mathrm{sph}\left(\tau_{ij}\right)$	$\tau = \mathrm{dev}\left(\tau\right) + \mathrm{sph}\left(\tau\right)$

- Notice that the deviatoric part (dev) is traceless, and the spherical part (sph) is the trace of the second rank tensor
- This kind of decomposition is often used in continuous mechanics.

- A few additional operators in index and vector notation that you will find in the governing equations of fluid dynamics.
    - Strain rate tensor,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

-
    - In vector notation, is equivalent to,

$$\mathbf{S} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$$

- A few additional operators in index and vector notation that you will find in the governing equations of fluid dynamics.
    - Laplacian,

$$\frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial^2 u_1}{\partial x_1^2} & \frac{\partial^2 u_1}{\partial x_2^2} & \frac{\partial^2 u_1}{\partial x_3^2} \\ \frac{\partial^2 u_2}{\partial x_1^2} & \frac{\partial^2 u_2}{\partial x_2^2} & \frac{\partial^2 u_2}{\partial x_3^2} \\ \frac{\partial^2 u_3}{\partial x_1^2} & \frac{\partial^2 u_3}{\partial x_2^2} & \frac{\partial^2 u_3}{\partial x_3^2} \end{bmatrix}$$

    - In vector notation, is equivalent to,

$$\nabla \cdot \nabla \mathbf{u} = \nabla^2 \mathbf{u} = \Delta \mathbf{u}$$

- The Laplacian operator will not change the rank of a tensor.

- A few additional operators in index and vector notation that you will find in the governing equations of fluid dynamics.
    - Every second-rank tensor, *e.g.*, the gradient of a vector, can be decomposed into a symmetric part and an anti-symmetric part.

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{Anti-symmetric part}}$$

    - In vector notation, is equivalent to,

$$\nabla \mathbf{u} = \underbrace{\frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}\left(\nabla \mathbf{u} - \nabla \mathbf{u}^T\right)}_{\text{Anti-symmetric part}}$$

    - The **symmetric** part is equivalent to the **strain rate tensor** (stretching and shearing) and the **anti-symmetric** part is equivalent to the **spin tensor** (vorticity).

- A few additional operators in index and vector notation that you will find in the governing equations of fluid dynamics.
  - The dot product of two second-rank tensor (also known as single dot product or tensor product of two tensor), is written using index and vector notation as follows,

$$\mathbf{C} = C_{ij} = A_{ik} B_{kj} = \mathbf{A} \cdot \mathbf{B} = (\mathbf{A} \cdot \mathbf{B})_{ij}$$

  - This is a matrix multiplication, which in the case of second-rank tensors will results in another second rank tensor.
  - Very often, especially when working with turbulence models, you will find expressions as the following ones (in this case we are illustrating the product of three second rank tensor),

Matrix (or second rank tensor) multiplication

$$C_{ii} = \underbrace{\overbrace{\Omega_{ij} \Omega_{jk}} S_{ki}}$$

- This matrix multiplication results in a square matrix i by i (a second rank tensor)

$$c = \underbrace{|\Omega_{ij} \Omega_{jk} S_{ki}|}$$

- This is the magnitude of the second rank tensor resulting from this matrix multiplication.
- The result is a scalar.
- The magnitude of a second rank tensor is defined as

$$|\mathbf{S}| = (2\mathbf{S} : \mathbf{S})^{1/2} \qquad |S_{ij}| = (2 S_{ij} S_{ij})^{1/2}$$

- A few additional operators in index and vector notation that you will find in the governing equations of fluid dynamics.
  - Very often, especially when working with turbulence models, you will find the following vector notation,

$$c = \mathbf{A} : \mathbf{B}$$

  - This is the double dot product of two second rank tensors, and the output is a scalar.
  - It is evaluated as the sum of the 9 products of the tensor components.
  - Using index notation, it is written as follows,

$$\begin{aligned} \phi = A_{ij}B_{ij} = & A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} + \\ & A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} + \\ & A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33} \end{aligned}$$

  - This operator is used to reduce the product of two second rank tensors to a scalar value, for example, the production term in turbulence models can be written as follows,

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \tau : \nabla \mathbf{u}$$