1 Governing Equations of Fluid Dynamics

The starting point of any numerical simulation are the governing equations of the physics of the problem to be solved. Hereafter, we present the governing equations of fluid dynamics and their simplification for the case of an incompressible viscous flow.

The equations governing the motion of a fluid can be derived from the statements of the conservation of mass, momentum, and energy [1, 2, 3]. In the most general form, the fluid motion is governed by the time-dependent three-dimensional compressible Navier-Stokes system of equations. For a viscous Newtonian, isotropic fluid in the absence of external forces, mass diffusion, finite-rate chemical reactions, and external heat addition; the conservation form of the Navier-Stokes system of equations in compact differential form and in primitive variable formulation (ρ, u, v, w, e_t) can be written as,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) &= -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{S_u}, \\ \frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) &= -\nabla \cdot q - \nabla \cdot (p\mathbf{u}) + \boldsymbol{\tau} \mathbf{:} \nabla \mathbf{u} + \mathbf{S}_{e_t}, \end{aligned} \tag{1.1}$$

where $\boldsymbol{\tau}$ is the viscous stress tensor and is given by,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \tag{1.2}$$

The set of equations 1.1 can be rewritten in matrix-vector form as follows,

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{e}_i}{\partial x} + \frac{\partial \mathbf{f}_i}{\partial y} + \frac{\partial \mathbf{g}_i}{\partial z} = \frac{\partial \mathbf{e}_v}{\partial x} + \frac{\partial \mathbf{f}_v}{\partial y} + \frac{\partial \mathbf{g}_v}{\partial z}, \tag{1.3}$$

where $\mathbf{q}$ is the vector of the conserved flow variables given by,

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix}, \tag{1.4}$$

and $\mathbf{e}_i$, $\mathbf{f}_i$ and $\mathbf{g}_i$ are the vectors containing the inviscid fluxes (or convective fluxes) in the x, y and z directions and are given by,

$$\mathbf{e}_i = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (\rho e_t + p)\, u \end{bmatrix}, \quad \mathbf{f}_i = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vw \\ (\rho e_t + p)\, v \end{bmatrix}, \quad \mathbf{g}_i = \begin{bmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ (\rho e_t + p)\, w \end{bmatrix}, \tag{1.5}$$

where $\mathbf{u}$ is the velocity vector containing the u, v and w velocity components in the x, y and z directions and p, ρ and e_t are the pressure, density and total energy per unit mass respectively.

The vectors $\mathbf{e}_v$, $\mathbf{f}_v$ and $\mathbf{g}_v$ contain the viscous fluxes in the x, y and z directions and are defined as follows,

$$
\mathbf{e}_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x \end{bmatrix},
$$

$$
\mathbf{f}_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - q_y \end{bmatrix}, \tag{1.6}
$$

$$
\mathbf{g}_v = \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - q_z \end{bmatrix},
$$

where the heat fluxes q_x, q_y and q_z are given by the Fourier's law of heat conduction as follows,

$$
\begin{aligned}
q_x &= -k\,\frac{\partial T}{\partial x}, \\
q_y &= -k\,\frac{\partial T}{\partial y}, \\
q_z &= -k\,\frac{\partial T}{\partial z},
\end{aligned} \tag{1.7}
$$

and the viscous stresses τ_{xx}, τ_{yy}, τ_{zz}, τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz} and τ_{zy}, are given by the following relationships,

$$
\begin{aligned}
\tau_{xx} &= \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right), \\
\tau_{yy} &= \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right), \\
\tau_{zz} &= \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right), \\
\tau_{xy} = \tau_{yx} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\
\tau_{xz} = \tau_{zx} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\
\tau_{yz} = \tau_{zy} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).
\end{aligned} \tag{1.8}
$$

In equations 1.6-1.8, T is the temperature, k is the thermal conductivity and μ is the molecular viscosity. In order to derive the viscous stresses in eq. 1.8 the Stokes hypothesis was used [1, 4, 5, 6].

Examining closely equations 1.3-1.8 and counting the number of equations and unknowns, we clearly see that we have five equations in terms of seven unknown flow field variables u, v, w, ρ, p, T, and e_t. It is obvious that two additional equations are required to close the system. These two additional equations can be obtained by determining relationships that exist between the thermodynamic variables (p, ρ, T, e_i) through the assumption of thermodynamic equilibrium. Relations of this type are known as equations of state, and they provide a mathematical relationship between two or more state functions (thermodynamic variables). Choosing the specific internal energy e_i and the density ρ as the two independent thermodynamic variables, then equations of state of the following form are required,

$$p = p\left(e_i, \rho\right), \qquad T = T\left(e_i, \rho\right). \tag{1.9}$$

For most problems in aerodynamics and gasdynamics, it is generally reasonable to assume that the gas behaves as a perfect gas (a perfect gas is defined as a gas whose intermolecular forces are negligible), *i.e.*,

$$p = \rho R_g T, \tag{1.10}$$

where R_g is the specific gas constant and is equal to $287 \frac{m^2}{s^2 K}$ for air. Assuming also that the working gas behaves as a calorically perfect gas (a calorically perfect gas is defined as a perfect gas with constant specific heats), then the following relations hold,

$$e_i = c_v T, \qquad h = c_p T, \qquad \gamma = \frac{c_p}{c_v}, \qquad c_v = \frac{R_g}{\gamma - 1}, \qquad c_p = \frac{\gamma R_g}{\gamma - 1}, \tag{1.11}$$

where γ is the ratio of specific heats and is equal to 1.4 for air, c_v the specific heat at constant volume, c_p the specific heat at constant pressure and h is the enthalpy. By using eq. 1.10 and eq. 1.11, we obtain the following relations for pressure p and temperature T in the form of eq. 1.9,

$$p = \left(\gamma - 1\right) \rho e_i, \qquad T = \frac{p}{\rho R_g} = \frac{\left(\gamma - 1\right) e_i}{R_g}, \tag{1.12}$$

where the specific internal energy per unit mass $e_i = p/(\gamma - 1)\rho$ is related to the total energy per unit mass e_t by the following relationship,

$$e_t = e_i + \frac{1}{2}\left(u^2 + v^2 + w^2\right). \tag{1.13}$$

In our discussion, it is also necessary to relate the transport properties (μ, k) to the thermodynamic variables. Then, the molecular viscosity μ is computed using Sutherland's formula,

$$\mu = \frac{C_1 T^{\frac{3}{2}}}{\left(T + C_2\right)}, \tag{1.14}$$

where for the case of the air, the constants are $C_1 = 1.458 \times 10^{-6} \frac{kg}{ms\sqrt{K}}$ and $C_2 = 110.4K$.

The thermal conductivity of the fluid (k) is determined from the Prandtl number ($Pr = 0.72$ for air) which in general is assumed to be constant and is equal to,

$$k = \frac{c_p \mu}{Pr}, \tag{1.15}$$

where c_p and μ are given by equations eq. 1.11 and eq. 1.14 respectively.

2 Simplification of the Navier-Stokes System of Equations: Incompressible Viscous Flow Case

Equations 1.3-1.6, with an appropriate equation of state and boundary and initial conditions, governs the unsteady three-dimensional motion of a viscous Newtonian, compressible fluid. In many applications the fluid density may be assumed to be constant. This is true not only for liquids, whose compressibility may be neglected, but also for gases if the Mach number is below 0.3 [2, 7]; such flows are said to be incompressible. If the flow is also isothermal, the viscosity is also constant. In this case, the governing equations written in compact conservation differential form and in primitive variable formulation (u, v, w, p) reduce to the following set,

$$\begin{aligned} \nabla \cdot (\mathbf{u}) &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) &= \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}, \end{aligned} \tag{2.1}$$

where ν is the kinematic viscosity and is equal $\nu = \mu/\rho$. The previous set of equations in expanded three-dimensional Cartesian coordinates is written as follows,

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\ \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right). \end{aligned} \tag{2.2}$$

The set of equations 2.2 governs the unsteady three-dimensional motion of a viscous, incompressible and isothermal flow. This simplification is generally not of a great value, as the equations are hardly any simpler to solve. However, the computing effort may be much smaller than for the full equations (due to the reduction of the unknowns and the fact that the energy equation is decoupled from the system of equation), which is a justification for such a simplification. The set of equations 2.1 can be rewritten in matrix-vector form as follows,

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{e}_{\boldsymbol{i}}}{\partial x} + \frac{\partial \mathbf{f}_{\boldsymbol{i}}}{\partial y} + \frac{\partial \mathbf{g}_{\boldsymbol{i}}}{\partial z} = \frac{\partial \mathbf{e}_{\boldsymbol{v}}}{\partial x} + \frac{\partial \mathbf{f}_{\boldsymbol{v}}}{\partial y} + \frac{\partial \mathbf{g}_{\boldsymbol{v}}}{\partial z}, \tag{2.3}$$

where $\mathbf{q}$ is the vector containing the primitive variables and is given by,

$$\mathbf{q} = \begin{bmatrix} 0 \\ u \\ v \\ w \end{bmatrix}, \tag{2.4}$$

and $\mathbf{e}_{\boldsymbol{i}}$, $\mathbf{f}_{\boldsymbol{i}}$ and $\mathbf{g}_{\boldsymbol{i}}$ are the vectors containing the inviscid fluxes (or convective fluxes) in the x, y and z directions and are given by,

$$\mathbf{e}_{\boldsymbol{i}} = \begin{bmatrix} u \\ u^2 + p \\ uv \\ uw \end{bmatrix}, \qquad \mathbf{f}_{\boldsymbol{i}} = \begin{bmatrix} v \\ vu \\ v^2 + p \\ vw \end{bmatrix}, \qquad \mathbf{g}_{\boldsymbol{i}} = \begin{bmatrix} w \\ wu \\ wv \\ w^2 + p \end{bmatrix}. \tag{2.5}$$

The viscous fluxes (or diffusive fluxes) in the x, y and z directions, $\mathbf{e}_{\boldsymbol{v}}$, $\mathbf{f}_{\boldsymbol{v}}$ and $\mathbf{g}_{\boldsymbol{v}}$ respectively, are defined as follows,

$$\mathbf{e}_{\boldsymbol{v}} = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix}, \qquad \mathbf{f}_{\boldsymbol{v}} = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \end{bmatrix}, \qquad \mathbf{g}_{\boldsymbol{v}} = \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{bmatrix}. \tag{2.6}$$

Since we made the assumptions of an incompressible flow, appropriate expressions for shear stresses must be used, these expressions are given as follows,

$$\begin{aligned}
\tau_{xx} &= 2\mu\frac{\partial u}{\partial x}, \\
\tau_{yy} &= 2\mu\frac{\partial v}{\partial y}, \\
\tau_{zz} &= 2\mu\frac{\partial w}{\partial z}, \\
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \\
\tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right), \\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),
\end{aligned} \tag{2.7}$$

where we used Stokes hypothesis [4, 1, 5, 6] in order to derive the viscous stresses in eq. 2.7. Equation 2.7 can be written in compact vector form as $\boldsymbol{\tau} = 2\mu\mathbf{S}$, where,

$$\mathbf{S} = \frac{1}{2}\left[\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right], \tag{2.8}$$

represents the strain-rate tensor. We can further decompose the velocity gradient tensor as follows,

$$\nabla\mathbf{u} = [\mathbf{S} + \Omega], \tag{2.9}$$

where $\mathbf{S}$ represents the symmetric part of the velocity gradient tensor (or the strain-rate tensor), and Ω represents the anti-symmetric part of the velocity gradient tensor (or the spin tensor, also know as vorticity). In eq. 2.9, the skew or anti-symmetric part of the velocity gradient tensor is given by,

$$\Omega = \frac{1}{2}\left[\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}}\right]. \tag{2.10}$$

Equations 2.3-2.6, are the governing equations of an incompressible, isothermal, viscous flow written in conservation form.

3 Reynolds Averaging

The starting point for deriving the RANS equations is the Reynolds decomposition [3, 8, 9, 10, 11, 12] of the flow variables of the governing equations. This decomposition is accomplished by representing the instantaneous flow quantity ϕ by the sum of a mean value part (denoted by a bar over the variable, as in $\bar{\phi}$) and a time-dependent fluctuating part (denoted by a prime, as in ϕ'). This concept is illustrated in figure 1 and is mathematically expressed as follows,

$$\phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x})}_{mean\,value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\,part}. \tag{3.1}$$

Hereafter, $\mathbf{x}$ is the vector containing the Cartesian coordinates x, y, and z in $\mathbb{N}=3$ (where $\mathbb{N}$ is equal to the number of spatial dimensions). A key observation in eq. 3.1 is that $\bar{\phi}$ is independent of time, implying that any equation deriving for computing this quantity must be steady state.

In eq. 3.1, the mean value $\bar{\phi}$ is obtained by an averaging procedure. There are three different forms of the Reynolds averaging:

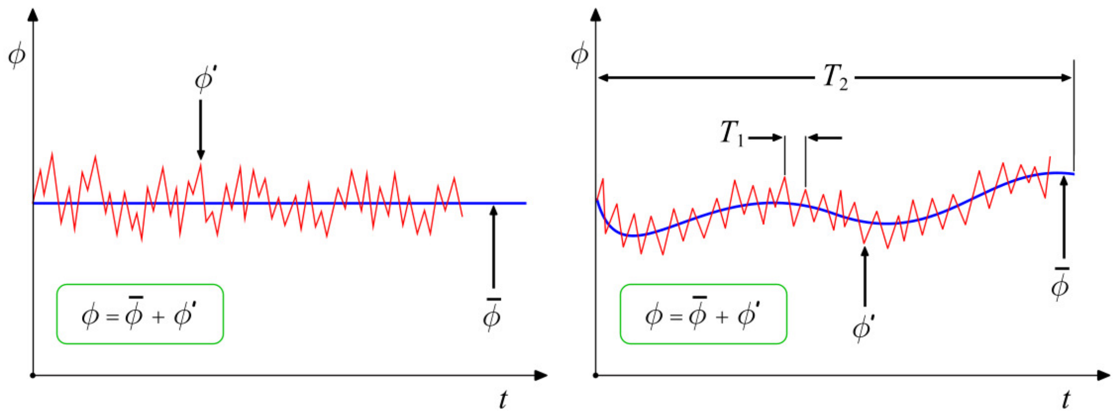

Figure 1: *Time averaging for a statistically steady turbulent flow (left) and time averaging for an unsteady turbulent flow (right).*

1. Time averaging: appropriate for stationary turbulence, *i.e.*, statically steady turbulence or a turbulent flow that, on average, does not vary with time.

$$\bar{\phi}(\mathbf{x}) = \lim_{T\to+\infty} \frac{1}{T} \int_{t}^{t+T} \phi(\mathbf{x}, t)\, \mathrm{d}t, \tag{3.2}$$

here t is the time and T is the averaging interval. This interval must be large compared to the typical time scales of the fluctuations; thus, we are interested in the limit $T \to \infty$. As a consequence, $\bar{\phi}$ does not vary in time, but only in space.

2. Spatial averaging: appropriate for homogeneous turbulence.

$$\bar{\phi}(t) = \lim_{\mathcal{CV}\to\infty} \frac{1}{\mathcal{CV}} \int_{\mathcal{CV}} \phi(\mathbf{x}, t)\, \mathrm{d}\mathcal{CV}, \tag{3.3}$$

with $\mathcal{CV}$ being a control volume. In this case, $\bar{\phi}$ is uniform in space, but it is allowed to vary in time.

3. Ensemble averaging: appropriate for unsteady turbulence.

$$\bar{\phi}(\mathbf{x}, t) = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \bar{\phi}(\mathbf{x}, t), \tag{3.4}$$

where $\mathcal{N}$, is the number of experiments of the ensemble and must be large enough to eliminate the effects of fluctuations. This type of averaging can be applied to any flow (steady or unsteady). Here, the mean value $\bar{\phi}$ is a function of both time and space (as illustrated in figure 1).

We use the term Reynolds averaging to refer to any of these averaging processes, applying any of them to the governing equations yields to the Reynolds-Averaged Navier-Stokes (RANS) equations. In cases where the turbulent flow is both stationary and homogeneous, all three averaging are equivalent. This is called the ergodic hypothesis.

If the mean flow $\bar{\phi}$ varies slowly in time, we should use an unsteady approach (URANS); then, equations eq. 3.1 and eq. 3.2 can be modified as

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}, t) + \phi'(\mathbf{x}, t), \tag{3.5}$$

and

$$\bar{\phi}(\mathbf{x},t) = \frac{1}{T}\int_{t}^{t+T} \phi(\mathbf{x},t)\mathrm{d}t, \quad T_1 << T << T_2, \tag{3.6}$$

where T_1 and T_2 are the characteristics time scales of the fluctuations and the slow variations in the flow, respectively (as illustrated in figure 1). In eq. 3.6 the time scales should differ by several order of magnitude, but in engineering applications very few unsteady flows satisfy this condition.

Before deriving the RANS equations, let us recall the following averaging rules,

$$\begin{aligned}
\bar{\phi'} &= 0,\\
\bar{\bar{\phi}} &= \bar{\phi},\\
\bar{\bar{\phi}} = \overline{\bar{\phi}+\phi'} &= \bar{\phi},\\
\overline{\phi+\varphi} &= \bar{\phi}+\bar{\varphi}\,,\\
\overline{\bar{\phi}\varphi} = \bar{\bar{\phi}}\bar{\varphi} &= \bar{\phi}\bar{\varphi},\\
\overline{\bar{\phi}\varphi'} = \bar{\phi}\bar{\varphi'} &= 0,\\
\overline{\phi\varphi} &= \overline{(\bar{\phi}+\phi')(\bar{\varphi}+\varphi')}\\
&= \overline{\bar{\phi}\bar{\varphi}+\bar{\phi}\varphi'+\bar{\varphi}\phi'+\phi'\varphi'}\\
&= \overline{\bar{\phi}\bar{\varphi}}+\overline{\bar{\phi}\varphi'}+\overline{\bar{\varphi}\phi'}+\overline{\phi'\varphi'}\\
&= \bar{\phi}\bar{\varphi}+\overline{\phi'\varphi'},\\
\overline{\phi'^2} &\neq 0,\\
\overline{\phi'\varphi'} &\neq 0,\\
\overline{\frac{\partial\phi}{\partial x}} &= \frac{\partial\bar{\phi}}{\partial x},\\
\overline{\frac{\partial\phi}{\partial t}} &= \frac{\partial\bar{\phi}}{\partial t},\\
\overline{\int\phi ds} &= \int\bar{\phi}ds
\end{aligned} \tag{3.7}$$

4 Incompressible Reynolds Averaged Navier-Stokes Equations

Let us recall the Reynolds decomposition for the flow variables of the incompressible Navier-Stokes equations eq. 2.1,

$$\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \bar{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t),\\
p(\mathbf{x},t) &= \bar{p}(\mathbf{x}) + p'(\mathbf{x},t),
\end{aligned} \tag{4.1}$$

we now substitute eq. 4.1 into the incompressible Navier-Stokes equations eq. 2.1 and we obtain for the continuity equation,

$$\nabla\cdot(\mathbf{u}) = \nabla\cdot\left(\bar{\mathbf{u}}+\mathbf{u}'\right) = \nabla\cdot(\bar{\mathbf{u}}) + \nabla\cdot\left(\mathbf{u}'\right) = 0. \tag{4.2}$$

Then, time averaging this equation results in,

$$\nabla\cdot(\bar{\bar{\mathbf{u}}}) + \nabla\cdot\left(\bar{\mathbf{u}'}\right) = 0, \tag{4.3}$$

and using the averaging rules stated in eq. 3.7, it follows that,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0. \tag{4.4}$$

We next consider the momentum equation of the incompressible Navier-Stokes equations eq. 2.1. We begin by substituting eq. 4.1 into eq. 2.1 in order to obtain,

$$\frac{\partial (\bar{\mathbf{u}} + \mathbf{u}')}{\partial t} + \nabla \cdot \left(\left(\bar{\mathbf{u}} + \mathbf{u}' \right) \left(\bar{\mathbf{u}} + \mathbf{u}' \right) \right) = \frac{-\nabla \left(\bar{p} + p' \right)}{\rho} + \nu \nabla^2 \left(\bar{\mathbf{u}} + \mathbf{u}' \right), \tag{4.5}$$

by time averaging eq. 4.5, expanding and applying the rules set in eq. 3.7, we obtain

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot \left(\bar{\mathbf{u}}\bar{\mathbf{u}} + \overline{\mathbf{u}'\mathbf{u}'} \right) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}}, \tag{4.6}$$

or after rearranging,

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} - \nabla \cdot \left(\overline{\mathbf{u}'\mathbf{u}'} \right). \tag{4.7}$$

By setting $\boldsymbol{\tau}^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right)$ in equation 4.7, and grouping with equation 4.4, we obtain the following set of equations,

$$\begin{aligned} &\nabla \cdot (\bar{\mathbf{u}}) = 0, \\ &\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R. \end{aligned} \tag{4.8}$$

The set of equations eq. 4.8 are the incompressible Reynolds-Averaged Navier-Stokes (RANS) equations. Notice that in eq. 4.8 we have retained the term $\partial \bar{\mathbf{u}} / \partial t$, despite the fact that $\bar{\mathbf{u}}$ is independent of time for statistically steady turbulence, hence this expression is equal to zero when time average. In practice, in all modern formulations of the RANS equations the time derivative term is included. In references [3, 8, 9, 13, 14], a few arguments justifying the retention of this term are discussed. For not statistically stationary turbulence or unsteady turbulence, a time-dependent RANS or unsteady RANS (URANS) approach is required, an URANS computation simply requires retaining the time derivative term $\partial \bar{\mathbf{u}} / \partial t$ in the computation.

The incompressible Reynolds-Averaged Navier-Stokes (RANS) equations eq. 4.8 are identical to the incompressible Navier-Stokes equations eq. 2.1 with the exception of the additional term $\boldsymbol{\tau}^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right)$, where $\boldsymbol{\tau}^R$ is the so-called Reynolds-stress tensor. Notice that by doing a check of dimensions, it will show that $\boldsymbol{\tau}^R$ it is not actually a stress; it must be multiplied by the density ρ, as it is done consistently in this manuscript, in order to have dimensions corresponding to the stresses. On the other hand, since we are assuming that the flow is incompressible, that is, ρ is constant, we might set the density equal to unity, thus obtaining implicit dimensional correctness. Moreover, because we typically use kinematic viscosity ν, there is an implied division by ρ. The Reynolds-stress tensor represents the transfer of momentum due to turbulent fluctuations. In $3\mathbb{D}$, the Reynolds-stress tensor $\boldsymbol{\tau}^R$ consists of nine components

$$\boldsymbol{\tau}^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = - \begin{pmatrix} \rho \overline{u'u'} & \rho \overline{u'v'} & \rho \overline{u'w'} \\ \rho \overline{v'u'} & \rho \overline{v'v'} & \rho \overline{v'w'} \\ \rho \overline{w'u'} & \rho \overline{w'v'} & \rho \overline{w'w'} \end{pmatrix}. \tag{4.9}$$

However, since u, v and w can be interchanged, the Reynolds-stress tensor forms a symmetrical second order tensor containing only six independent components. By inspecting the set of equations eq. 4.8 we can count ten unknowns, namely; three components of the velocity (u, v, w), the pressure (p), and six components of the Reynolds stress $\left(\boldsymbol{\tau}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right)\right)$, in terms of four equations, hence the system is not closed. The fundamental problem of turbulence modeling based on the Reynolds-averaged Navier-Stokes equations is to find six additional relations in order to close the system of equations eq. 4.8.

5 Boussinesq Approximation

The Reynolds averaged approach to turbulence modeling requires that the Reynolds stresses in eq. 4.8 to be appropriately modeled (however, it is possible to derive its own governing equations, but it is much simpler to model this term). A common approach uses the Boussinesq hypothesis to relate the Reynolds stresses $\boldsymbol{\tau}^R$ to the mean velocity gradients such that,

$$\boldsymbol{\tau}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T\bar{\mathbf{S}}^R - \frac{2}{3}\rho k\mathbf{I} = \mu_T\left[\nabla\bar{\mathbf{u}} + (\nabla\bar{\mathbf{u}})^{\mathrm{T}}\right] - \frac{2}{3}\rho k\mathbf{I}, \tag{5.1}$$

where $\bar{\mathbf{S}}^R$ denotes the Reynolds-averaged strain-rate tensor,

$$\bar{\mathbf{S}}^R = \frac{1}{2}(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^{\mathrm{T}}), \tag{5.2}$$

$\mathbf{I}$ is the identity matrix, μ_T is called the turbulent eddy viscosity, and,

$$k = \frac{1}{2}\overline{\mathbf{u}'\cdot\mathbf{u}'}, \tag{5.3}$$

is the turbulent kinetic energy.

Basically, we have assumed that the fluctuating Reynolds stresses are proportional to the gradient of the average quantities (similarly to Newtonian flows). The second term in eq. 5.1, namely,

$$\frac{2}{3}\rho k\mathbf{I}, \tag{5.4}$$

is added in order for the Boussinesq approximation to be valid when traced. That is, the trace of the right hand side in eq. 5.1 must be equal to that of the left hand side

$$-\rho(\overline{\mathbf{u}'\mathbf{u}'})^{\mathrm{tr}} = -2\rho k, \tag{5.5}$$

hence it is consistent with the definition of turbulent kinetic energy (eq. 5.3). In order to evaluate k, usually a governing equation for k is derived and solved, typically two-equations models include such an option.

The turbulent eddy viscosity μ_T (in contrast to the molecular viscosity μ), is a property of the flow field and not a physical property of the fluid. The eddy viscosity concept was developed assuming that a relationship or analogy exists between molecular and turbulent viscosities. In spite of the theoretical weakness of the turbulent eddy viscosity concept, it does produce reasonable results for a large number of flows.

The Boussinesq approximation reduces the turbulence modeling process from finding the six turbulent stress components $\boldsymbol{\tau}^R$ to determining an appropriate value for the turbulent eddy viscosity μ_T.

One final word of caution, the Boussinesq approximation discussed here, should not be associated with the completely different concept of natural convection.

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