Roadmap to Lecture 5

1. ~~Governing equations of fluid dynamics~~
2. ~~RANS equations – Reynolds averaging~~
3. The Boussinesq hypothesis
4. The gradient diffusion hypothesis
5. Sample turbulence models

The Boussinesq hypothesis

- The RANS/URANS approach to turbulence modeling requires the Reynolds stress tensor τ^R to be appropriately modeled.

$$\tau^R = \tau_{ij}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

- We do not want to resolve the instantaneous fluctuations.
- Even if it is possible to derive governing equations for the Reynolds stress tensor τ^R (six new equations as the tensor is symmetric), it is much simpler to model this term.
- The approach of deriving the governing equations for the Reynolds stress tensor τ^R is known as Reynolds stress model (RSM).
- Probably, RSM is the most physically sound RANS model as it avoids the use of hypothesis/assumptions to model this term.

The Boussinesq hypothesis

- We will address RSM models in Lecture 6.
- If you are curious, this is how the **exact** Reynolds stress equations look like,

$$\underbrace{\frac{\partial \tau_{ij}^R}{\partial t}}_{1} + \underbrace{\bar{u}_k \frac{\partial \tau_{ij}^R}{x_k}}_{2} = \underbrace{-\left(\tau_{ik}^R \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}^R \frac{\partial \bar{u}_i}{\partial x_k}\right)}_{3} + \underbrace{2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{4} + \ldots$$

$$\ldots + \underbrace{\frac{1}{\rho}\left(\overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}}\right)}_{5} + \underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ij}^R}{\partial x_k}\right)}_{6} + \underbrace{\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right)}_{7}$$

1. Transient stress rate of change term.
2. Convective term.
3. Production term.
4. Dissipation rate (tensor dissipation).
5. Turbulent stress transport related to the velocity and pressure fluctuations.
6. Rate of viscous stress diffusion (molecular).
7. Diffusive stress transport resulting from the triple correlation of velocity fluctuations.

We get 6 new equations, but we also generate 22 new unknowns.

$$\overline{u_i' u_j' u_k'} \quad \rightarrow \quad 10 \text{ unknowns}$$

$$2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \quad \rightarrow \quad 6 \text{ unknowns}$$

$$\frac{1}{\rho}\left(\overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}}\right) \quad \rightarrow \quad 6 \text{ unknowns}$$

The Boussinesq hypothesis

- Modeling the Reynolds stress tensor is a much easier approach.
- A common approach used to model the Reynolds stress tensor τ^R, is to use the Boussinesq hypothesis.
- The Boussinesq hypothesis (or approximation, or assumption), was proposed by Boussinesq in 1877 [1, 2, 3, 4].
- This hypothesis (or assumption) simply states that, similar to fluid viscosity in laminar flows, a flow dependent turbulent viscosity may be added to the molecular agitation to represent turbulent mixing or diffusion, such as,

$$\tau_{Total} = (\mu + \mu_t) \nabla \bar{\mathbf{u}}$$

- Boussinesq simply stated that the turbulent stress tensor is proportional to the mean strain rate tensor, multiplied by a constant (or coefficient), which we will call turbulent eddy viscosity.
- It is important to highlight that Boussinesq did not propose a method to model the turbulent stress tensor (or the Reynolds stress tensor in modern turbulence models).
- In fact, this hypothesis was proposed before the seminal work of Reynolds [5].

[1] J. Boussinesq. Essai sur la théorie des eaux courantes, Mémoires présentés par divers savants à l'Académie des Sciences 23 (1): 1-680, 1877.
[2] F. Schmitt. About Boussinesq's turbulent viscosity hypothesis: historical remarks and a direct evaluation of its validity, Comptes Rendus Mécanique 335 (9-10): 617-627, 2007.
[3] D. C. Wilcox. Turbulence Modeling for CFD. DCW Industries, 2010.
[4] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[5] Osborne Reynolds. On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion. Philosophical Transactions of the Royal Society of London. A, Vol. 186,1895.

The Boussinesq hypothesis

- Reynolds [5] proposed to take the Navier-Stokes equations as a starting point to derive equations for the evolution of the mean (or average) velocity field.
- Furthermore, he obtained equations for the evolution of the kinetic energy contained in the mean velocity field, and for the energy contained in the small fluctuation motions.
- Without a doubt, the most important result of Reynolds work [5], is the procedure in which the velocity field is decomposed into a mean and a fluctuating part. This procedure is now referred to as Reynolds averaging.
- As Boussinesq, Reynolds did not propose a method to model the turbulent stress tensor or to solve the Navier-Stokes equations.
- **The Boussinesq hypothesis [1,2,3,4] and the Reynolds averaging [5] constitute the basis for modern simulation techniques like RANS and LES models.**

[1] J. Boussinesq. Essai sur la théorie des eaux courantes, Mémoires présentés par divers savants à l'Académie des Sciences 23 (1): 1-680, 1877.
[2] F. Schmitt. About Boussinesq's turbulent viscosity hypothesis: historical remarks and a direct evaluation of its validity, Comptes Rendus Mécanique 335 (9-10): 617-627, 2007.
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The Boussinesq hypothesis

- The Boussinesq hypothesis [1,2,3,4] and the Reynolds averaging [5] constitute the landmark contributions to the field of turbulence modeling.
- The Boussinesq hypothesis reduces the turbulence modeling process from finding the six turbulent stresses in the RSM equations to determining an appropriate value for the turbulent eddy viscosity μ_T or ν_T.
- This hypothesis is a brutal and flawed approximation to the actual physics, but it has been demonstrated that it is accurate if good standard practices are followed.
- It is considered to be flawed because it assumes that the eddy viscosity is the same for all the Reynolds stress components. That is, the eddy viscosity is considered isotropic.
- We should be aware limitations and deficiencies of the Boussinesq hypothesis. Over the years, many improvements and corrections have been formulated.

[1] J. Boussinesq. Essai sur la théorie des eaux courantes, Mémoires présentés par divers savants à l'Académie des Sciences 23 (1): 1-680, 1877.
[2] F. Schmitt. About Boussinesq's turbulent viscosity hypothesis: historical remarks and a direct evaluation of its validity, Comptes Rendus Mécanique 335 (9-10): 617-627, 2007.
[3] D. C. Wilcox. Turbulence Modeling for CFD. DCW Industries, 2010.
[4] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[5] Osborne Reynolds. On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion. Philosophical Transactions of the Royal Society of London. A, Vol. 186,1895.

The Boussinesq hypothesis

- The Boussinesq hypothesis is somehow similar to the hypothesis taken when dealing with Newtonian flows, where the viscous stresses are assumed to be proportional to the shear stresses, therefore, to the velocity gradient.
- Recall that the stress tensor of Newtonian flows can be written as follows,

$$-\left(\tau_{ij} + P\delta_{ij}\right) = -2\mu S_{ij}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

- Using index notation, the **Boussinesq hypothesis for incompressible flows** is written as follows,

$$\tau_{ij}^R = -\rho\overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

- This term is not intended in the original assumption.
- We will address the motive of this extra term later.

The Boussinesq hypothesis

- The Boussinesq hypothesis is somehow similar to the hypothesis taken when dealing with Newtonian flows, where the viscous stresses are assumed to be proportional to the shear stresses, therefore, to the velocity gradient.
- If we compare both tensors, they look very similar.

$$-\left(\tau_{ij} + P\delta_{ij}\right) = -2\mu S_{ij}$$

Viscous stress tensor

$$-\left(\tau_{ij}^{R} + \frac{2}{3}\rho k\delta_{ij}\right) = -2\mu_t S_{ij}$$

Reynolds stress tensor

Note:

$$\tau^{R} = -\rho\overline{\mathbf{u}'\mathbf{u}'}$$

- And where the strain rate tensor S_{ij} is equal to,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

- The strain rate tensor describes the rate of stretching and shearing.

The Boussinesq hypothesis

- **Do not confuse the Boussinesq hypothesis used in turbulence modeling with the completely different concept found in natural convection and buoyancy-driven flows, that is, the Boussinesq approximation.**
- In turbulence, maybe is better to talk about Boussinesq assumption instead of hypothesis or approximation.
- But have in mind that in the context of turbulence modeling, Boussinesq assumption, Boussinesq hypothesis, Boussinesq eddy-viscosity assumption, and Boussinesq approximation they all convey the same concept.
- From now on, we will consistently use the terminology Boussinesq hypothesis.

The Boussinesq hypothesis

- The Boussinesq hypothesis lies in the belief that the Reynolds stresses behave in a similar fashion as the Newtonian stress tensor.
- This constitutive equation is a linear stress–strain relation.
- And, as for a non-Newtonian flows, nonlinear models have been proposed (which we will study later).
- The Boussinesq hypothesis inherently assumes an equilibrium between Reynolds stress and mean rate of strain.
- This may be violated in some flows, where the Reynolds stress is not proportional to the mean rate of strain.
- Surprisingly, the Boussinesq hypothesis works remarkably well for a wide variety of flows.

The Boussinesq hypothesis

- Using index notation, the Boussinesq hypothesis for incompressible flows is written as follow,

$$\tau_{ij}^R = -\rho \overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

- Where δ_{ij} is the Kronecker delta and is define as follows,

$$\delta_{ij} \begin{cases} = 1 & \text{if} \quad i = j \\ = 0 & \text{otherwise} \end{cases}$$

The Boussinesq hypothesis

- In expanded form, the Boussinesq hypothesis is written as follows,

$$\tau_{ij}^{R} = -\rho\overline{u_i' u_j'} = \begin{bmatrix} 2\mu S_{11} - \frac{2}{3}\rho k & 2\mu S_{12} & 2\mu S_{13} \\ 2\mu S_{21} & 2\mu S_{22} - \frac{2}{3}\rho k & 2\mu S_{23} \\ 2\mu S_{31} & 2\mu S_{32} & 2\mu S_{33} - \frac{2}{3}\rho k \end{bmatrix}$$

The Boussinesq hypothesis

- Using vector notation, the Boussinesq hypothesis for incompressible flows is written as follow,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_t\left[\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right] - \frac{2}{3}\rho k \mathbf{I}$$

$$\bar{\mathbf{S}}^R = \frac{1}{2}\left[\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right]$$

$$k = \frac{1}{2}\overline{\mathbf{u}' \cdot \mathbf{u}'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is equivalent to the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The Boussinesq hypothesis

- The Boussinesq hypothesis is a common approach used to model the Reynolds stress tensor.
- This approach is widely used and accurate (to some extension) but is not the only one.
- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = 2\mu_t \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_t \left[\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right] - \frac{2}{3}\rho k \mathbf{I}$$

$\bar{\mathbf{S}}^R$ → Reynolds averaged strain-rate tensor.

$\mathbf{I}$ → identity matrix (or Kronecker delta).

k → turbulent kinetic energy.

μ_T → turbulent eddy viscosity.

- At the end of the day, we want to determine the turbulent eddy viscosity.
- Each turbulence model will compute this quantity in a different way.
- Remember, the turbulent eddy viscosity μ_T is not a fluid property, it is a property needed by the turbulence model.

The Boussinesq hypothesis

- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_t\left[\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right] - \frac{2}{3}\rho k \mathbf{I}$$

This term represent normal stresses, therefore, is analogous to the pressure term that arises in the viscous stress tensor

- The term circled in the Boussinesq hypothesis, is added in order for the hypothesis to be valid when traced.
- That is, the trace of the right-hand side must be equal to the trace of the left-hand side,

$$-\rho(\overline{\mathbf{u}'\mathbf{u}'})^{\mathrm{tr}} = \tau_{ii} = -2\rho k$$

- Hence, it is consistent with the definition of turbulent kinetic energy

$$k = \frac{1}{2}\overline{\mathbf{u}' \cdot \mathbf{u}'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

The Boussinesq hypothesis

- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_t \left[\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T\right] - \frac{2}{3}\rho k \mathbf{I}$$

This term represent normal stresses, therefore, is analogous to the pressure term that arises in the viscous stress tensor

- In order to evaluate the turbulent kinetic energy, usually a governing equation for k is derived and solved.
- Typically, two-equations models include such an option, as we will see in Lecture 6.
- The term circled in the Boussinesq hypothesis can be ignored if there is no governing equation for k.

The Boussinesq hypothesis

- In expanded form, the Boussinesq hypothesis is written as follows,

$$\tau_{ij}^R = -\rho\overline{u_i'u_j'} = \begin{bmatrix} 2\mu S_{11} - \frac{2}{3}\rho k & 2\mu S_{12} & 2\mu S_{13} \\ 2\mu S_{21} & 2\mu S_{22} - \frac{2}{3}\rho k & 2\mu S_{23} \\ 2\mu S_{31} & 2\mu S_{32} & 2\mu S_{33} - \frac{2}{3}\rho k \end{bmatrix}$$

- The contracted strain rate tensor (by setting $i = j$) or trace of the strain rate tensor without adding the term $2/3\,k$ is equal to,

$$S_{ii} = \frac{\partial \bar{u}_i}{\partial x_i} = 0$$

From the divergence-free constraint

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

- After taking the contraction or the trace of the Boussinesq hypothesis without the term $2/3\,k$, **we obtain the following identity that is false**,

$$-\rho\left(\overline{u_i'u_i'}\right)^{\mathrm{tr}} = \tau_{ii} = -2\rho k = \underbrace{0}_{RHS}$$

where

$$k = \frac{1}{2}\overline{u_i'u_i'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

The Boussinesq hypothesis

- In expanded form, the Boussinesq hypothesis is written as follows,

$$\tau_{ij}^R = -\rho\overline{u_i' u_j'} = \begin{bmatrix} 2\mu S_{11} - \frac{2}{3}\rho k & 2\mu S_{12} & 2\mu S_{13} \\ 2\mu S_{21} & 2\mu S_{22} - \frac{2}{3}\rho k & 2\mu S_{23} \\ 2\mu S_{31} & 2\mu S_{32} & 2\mu S_{33} - \frac{2}{3}\rho k \end{bmatrix}$$

- Instead, when taking the trace of the Boussinesq hypothesis and adding the term $2/3\,k$, we obtain the following identity that holds true,

$$-\rho\left(\overline{u_i' u_i'}\right)^{\mathrm{tr}} = \tau_{ii} = -2\rho k = \underbrace{-2\rho k}_{RHS} \qquad \text{where} \qquad k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

- The term $2/3\,k$ has a physical meaning, it represents normal stresses.
- Therefore, it is analogous to the pressure term that arises in the viscous stress tensor.

$$\tau_{ij}^R = -\rho\overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

← Normal stresses

The Boussinesq hypothesis

Final remarks

- In the previous discussion, we wrote down the incompressible Boussinesq hypothesis.
- The general Boussinesq hypothesis (valid for compressible and incompressible flows) can be written as follows,

$$\tau_{ij}^R = -\rho \overline{u_i' u_j'} = 2\mu_t \left(S_{ij} - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) - \frac{2}{3} \rho k \delta_{ij}$$

- For incompressible flows, the circled term becomes zero.
- If you are dealing with compressible flows, you also need to model the turbulent thermal flux.

$$q = -\rho c_p \overline{u_i' T'} = \Gamma_T \frac{\partial \overline{T}}{\partial x_i} = \frac{c_p \mu_t}{Pr_t} \frac{\partial \overline{T}}{\partial x_i}$$

Turbulent thermal diffusivity

The Boussinesq hypothesis

Final remarks

- Closure models based on the Boussinesq hypothesis are known as eddy viscosity models (**EVM**).
- As previously mentioned, the Boussinesq hypothesis lies in the belief that the Reynolds Stress tensor behaves in a similar fashion as the Newtonian viscous stress tensor.
- In spite of the theoretical weakness of the Boussinesq hypothesis, it does produce reasonable results for a large number of flows.
- The main disadvantage of the Boussinesq hypothesis as presented (linear model), is that it assumes that the turbulent eddy viscosity is an isotropic scalar quantity, which is not strictly true.
- There are more sophisticated methods where the eddy turbulent viscosity is treated as an anisotropic quantity or a tensor (non-linear models).

The Boussinesq hypothesis

Final remarks

- Another weakness of the EVM is that they do not have memory.
- That is, if we remove the mean rate strain tensor, the Boussinesq hypothesis predicts instantaneous zero turbulent shear stress.
- This does not correspond to experiments, where the rate of decay is an observable.
- There are more advanced models that to some extension account for this.
- Unlike linear EVM which use an isotropic eddy viscosity, RSM solves all components of the turbulent transport; therefore, RSM models are anisotropic.
- This is the main reason why the RSM models are more physically sound.

The Boussinesq hypothesis

Final remarks

- EVM models have significant shortcomings in complex, real-life turbulent flows.
- For example, EVM perform poorly in the following situations,
    - Flows with sudden changes in axial mean strain, *e.g.*, pipes with restrictions.
    - Flows with large extra strains, *e.g.*, curved surfaces, strong vorticity, swirling flows.
    - Rotating flows, *e.g.*, turbomachinery, wind turbines.
    - Impinging jets, *e.g.*, a jet hitting a wall.
    - Highly anisotropic flows and flows with secondary motions, *e.g.*, fully developed flows in non-circular ducts or square ducts.
    - Strongly three-dimensional boundary layers.
    - Non-local equilibrium and flow separation, *e.g.*, airfoil in stall, dynamic stall.

- Many EVM models has been developed, corrected, and improved over the years so they address the shortcomings of the Boussinesq hypothesis.
- Without no doubt, EVM models are the cornerstone of turbulence modeling.

Final remarks

- Gradient models, such as the Boussinesq hypothesis and the gradient diffusion hypothesis (that we will study next) play a central role in turbulence modeling.
- Many authors criticize a lot these hypotheses and question their validity.
    - And paradoxically, they still use these models.
    - These hypotheses are used widely and pervasively.
- But instead in focusing all efforts in questioning these hypotheses, it is better to understand why they produce reasonable results for a large number of flows, as stated by Saffman [1],

> *"The continual preaching against the eddy diffusivity hypothesis … has not served any useful purpose. The effort would have been better spent trying to understand the reasons for the apparent success and the circumstances in which it must (not ought to) fail."*

[1] P. Saffman. "Problems and progress in the theory of turbulence". Structure and mechanism of turbulence II, Lecture notes in physics, Vol. 7, 183, Springer-Verlag, Berlin, 1978.

Final remarks – Relationship for the turbulent eddy viscosity

- In most turbulence models, a relationship for the turbulent eddy viscosity is derived by using dimensional arguments.
    - This can be done by using any combination of dimensional groups.
    - Namely, velocity, length, time, etc.
    - In the end, we should have viscosity units.
- This relationship can be corrected later or validated based on empirical and physical arguments, *e.g.*,
    - asymptotic analysis,
    - canonical solutions,
    - analytical solutions,
    - consistency with experimental measurements,
    - and so on.

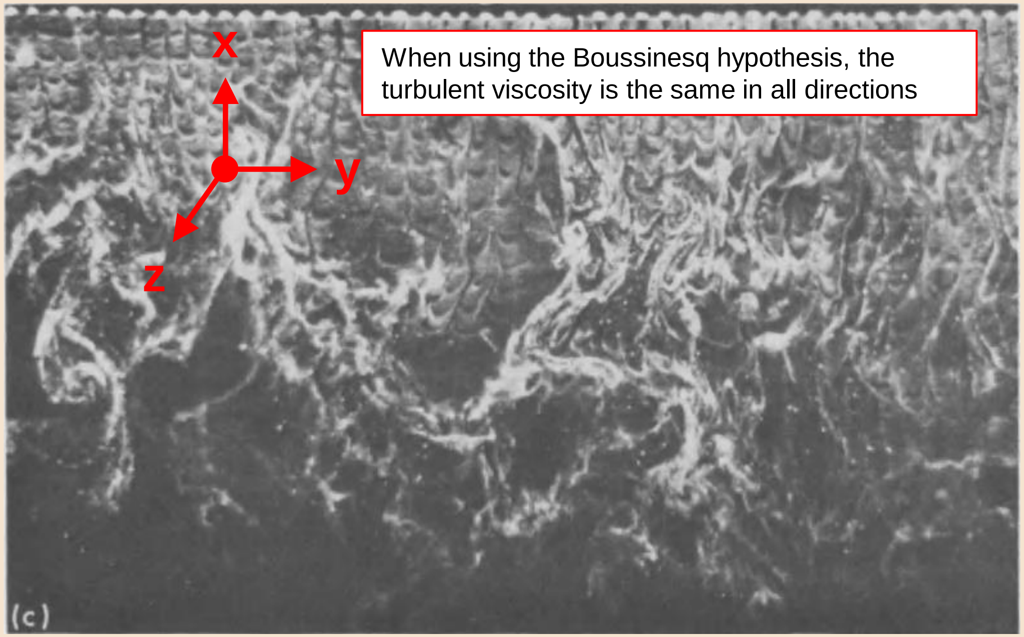
The Boussinesq hypothesis

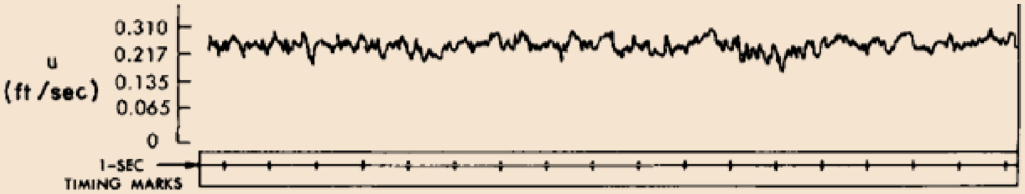
Final remarks – Relationship for the turbulent eddy viscosity

- It is also possible to use numerical arguments to correct, calibrate, and validate the relationship.
- To achieve this end, we rely on scale resolving simulations. Most of the time DNS simulations.
- Regardless of the approach used, we see a recurring behavior.
- Specifically, eddy viscosity and length scale are all related on the basis of dimensional arguments.
- Historically, dimensional analysis has been one of the most powerful tools available for deducing and correlating properties of turbulent flows.
- However, we should always be aware that while dimensional analysis is extremely useful, it unveils nothing about the physics underlying its implied relationships.

Final remarks – Turbulent boundary-layer flow structure

- Water flowing in a channel made visible by the pulsed hydrogen-bubble technique.
- In the bottom figure, the velocity fluctuations are illustrated.
- As it can be seen, the velocity fluctuations are large.
- In the top figure, we can clearly observe the strong three-dimensional characteristics of the flow. Note that the water is flowing from top to bottom.
- Resolving this kind of three-dimensional flows using EVM models is difficult.



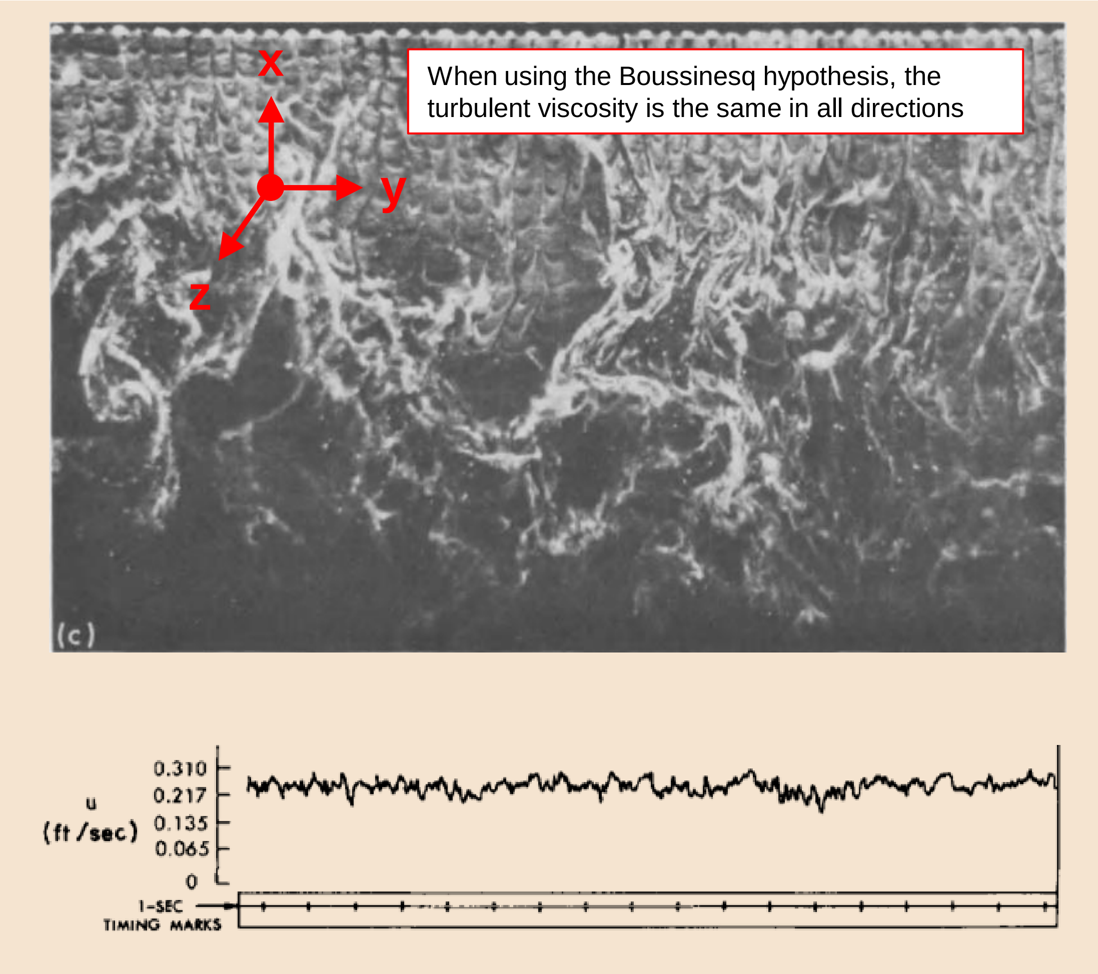


References:
P. Runstadler, S. Kline, W. Reynolds. An experimental investigation of the flow structure of the turbulent boundary layer. Rep. MD-8, Stanford Univ., Mech. Eng. Dept., Stanford, 1963.

The Boussinesq hypothesis

Final remarks – Turbulent boundary-layer flow structure

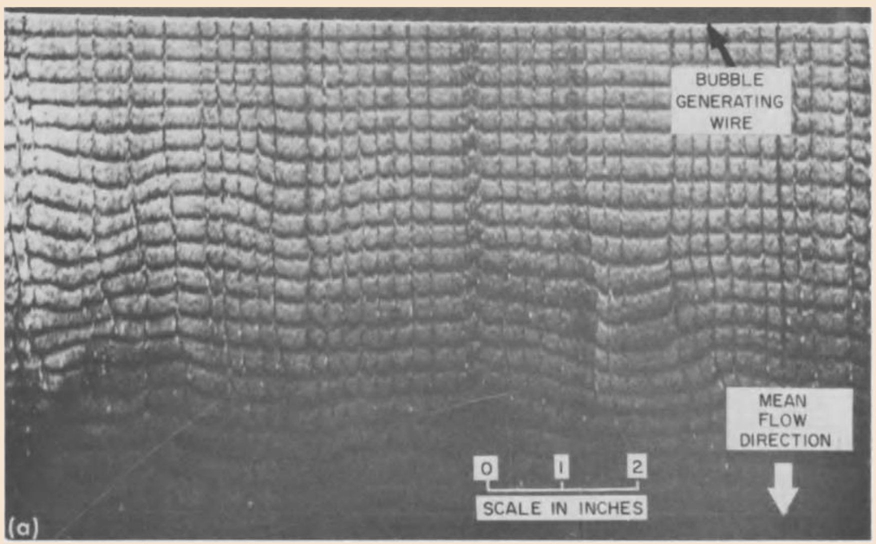
- The Boussinesq hypothesis will assign the same turbulent viscosity value in all directions (that is, to all Reynolds stress components), indifferently of the strong three-dimensional nature of the flow.
- Isotropy is the biggest weakness of the EVM models.
- However, many EVM models has been developed, corrected, and improved over the years so they address the shortcomings of the Boussinesq hypothesis.
- The complete sequence of images is shown in the next four slides.

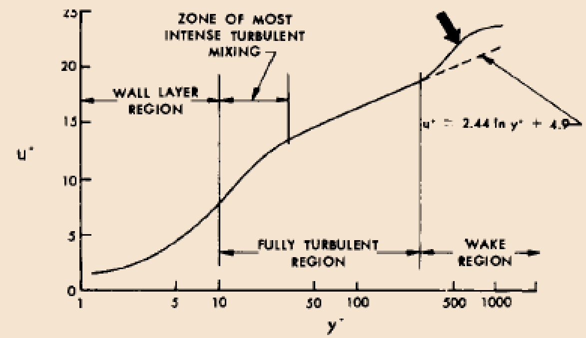


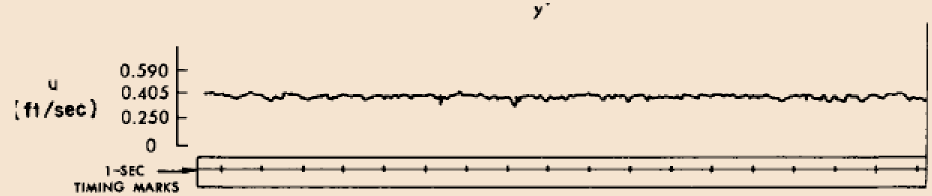
References:
P. Runstadler, S. Kline, W. Reynolds. An experimental investigation of the flow structure of the turbulent boundary layer. Rep. MD-8, Stanford Univ., Mech. Eng. Dept., Stanford, 1963.

Final remarks – Turbulent boundary-layer flow structure

- Water flowing in a channel made visible by the pulsed hydrogen-bubble technique.
- This figure shows the flow in the wake region.
- The velocity fluctuations are weak. There are no strong three-dimensional effects.







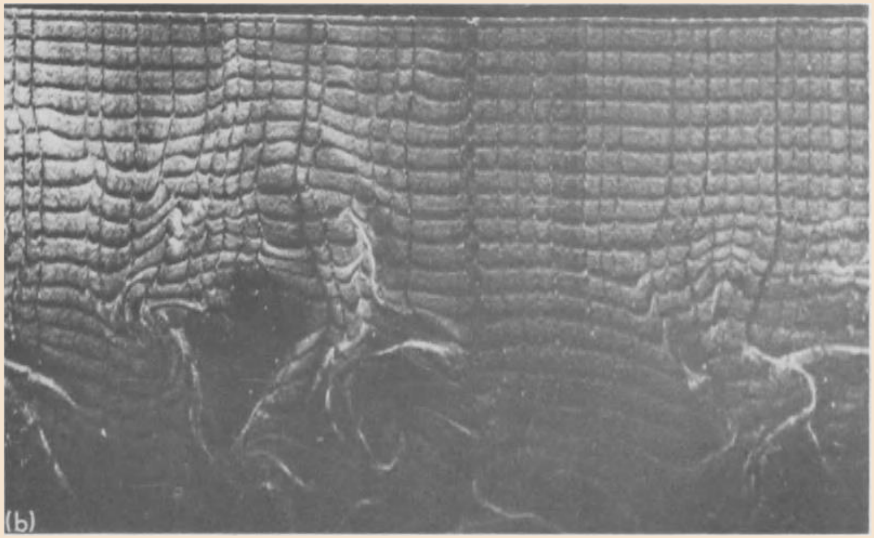
Flow conditions:
a. u = 0.430 ft/sec, y = 3.25 in., y+ = 531.

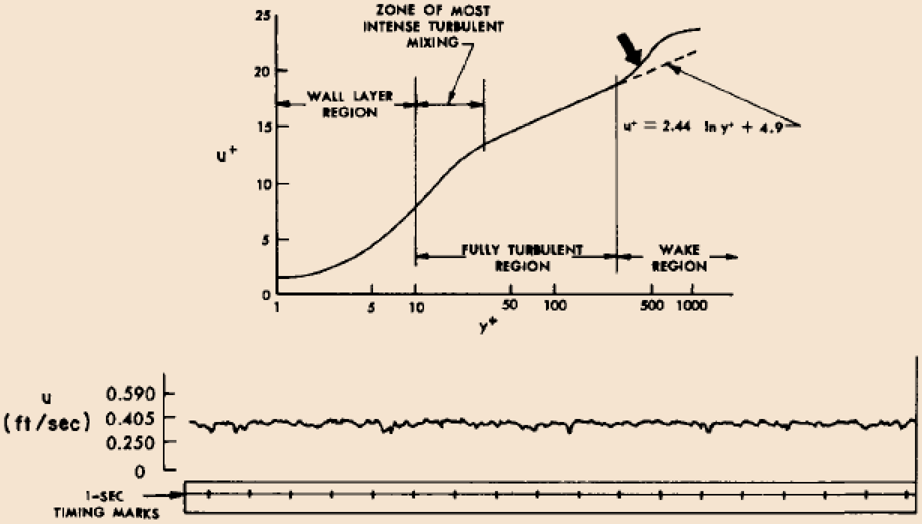
References:
P. Runstadler, S. Kline, W. Reynolds. An experimental investigation of the flow structure of the turbulent boundary layer. Rep. MD-8, Stanford Univ., Mech. Eng. Dept., Stanford, 1963.

The Boussinesq hypothesis

Final remarks – Turbulent boundary-layer flow structure

- Water flowing in a channel made visible by the pulsed hydrogen-bubble technique.
- This figure shows the flow in the wake region.
- The velocity fluctuations are larger than in the previous figure, and the three-dimensional effects are much stronger.





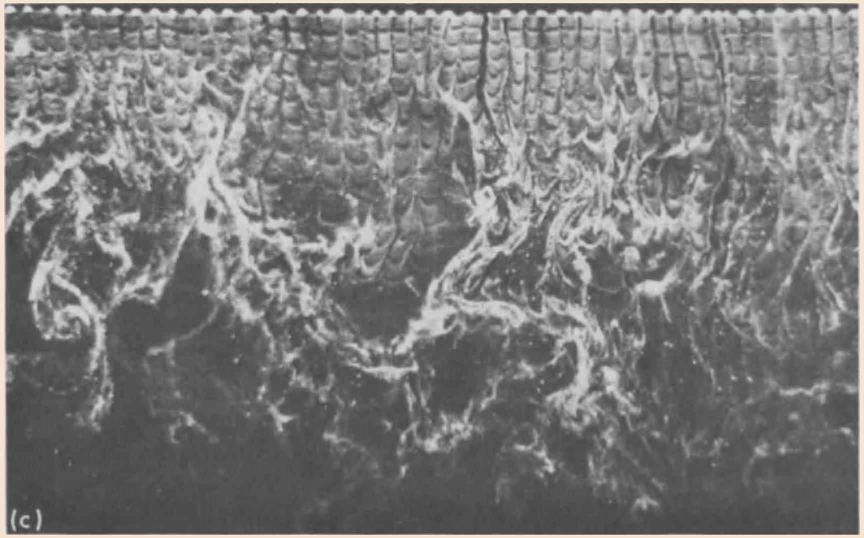
Flow conditions:
b. u = 0.430 ft/sec, y = 2.50 in., y+ = 407.

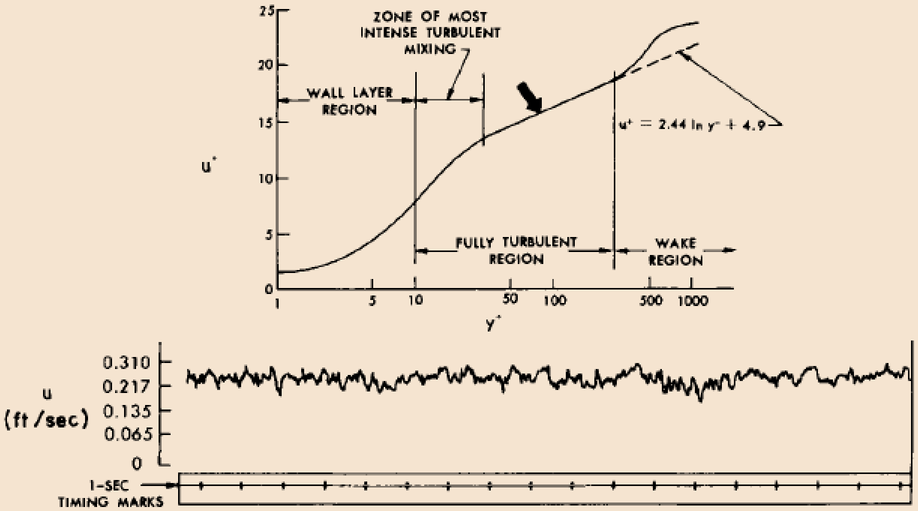
References:
P. Runstadler, S. Kline, W. Reynolds. An experimental investigation of the flow structure of the turbulent boundary layer. Rep. MD-8, Stanford Univ., Mech. Eng. Dept., Stanford, 1963.

The Boussinesq hypothesis

Final remarks – Turbulent boundary-layer flow structure

- Water flowing in a channel made visible by the pulsed hydrogen-bubble technique.
- This figure shows the flow in the logarithmic region.
- The velocity fluctuations are strong. This region of the boundary layer is very energetic.





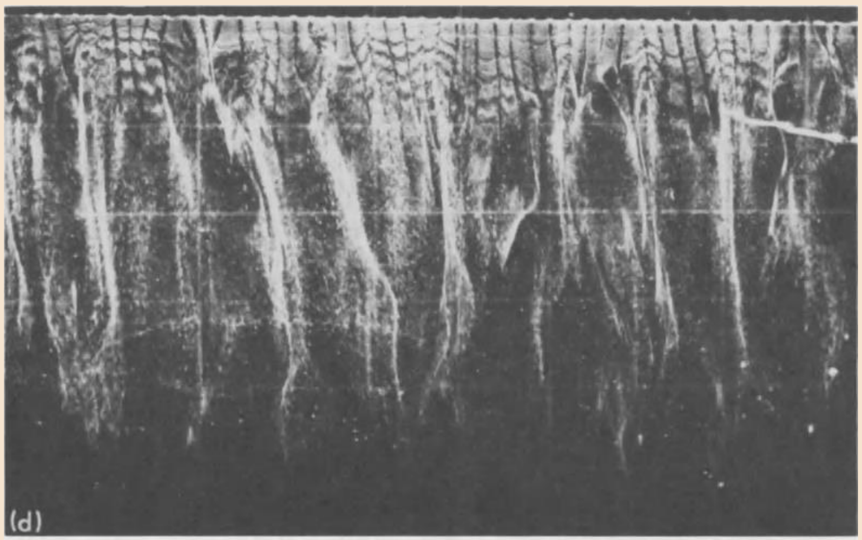
Flow conditions:
c. u = 0.430 ft/sec, y = 0.50 in., y+ = 82.

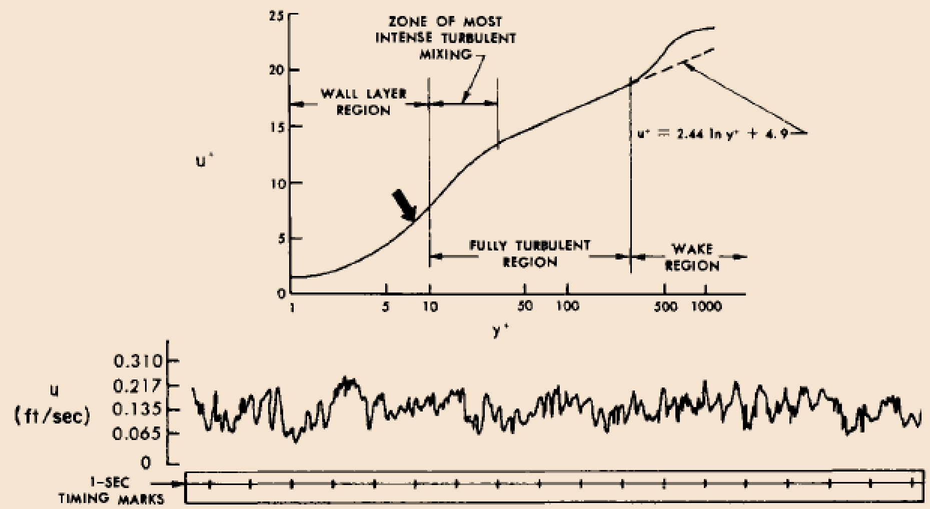
References:
P. Runstadler, S. Kline, W. Reynolds. An experimental investigation of the flow structure of the turbulent boundary layer. Rep. MD-8, Stanford Univ., Mech. Eng. Dept., Stanford, 1963.

The Boussinesq hypothesis

Final remarks – Turbulent boundary-layer flow structure

- Water flowing in a channel made visible by the pulsed hydrogen-bubble technique.
- This figure shows flow extremely close to the wall.
- The velocity fluctuations are still noticeable but small in comparison to the previous images. The strong three-dimensionality has disappeared (compare with figures b and c).





Flow conditions:
d. u = 0.430 ft/sec, y = 0.050 in., y+ = 8.

References:
P. Runstadler, S. Kline, W. Reynolds. An experimental investigation of the flow structure of the turbulent boundary layer. Rep. MD-8, Stanford Univ., Mech. Eng. Dept., Stanford, 1963.

Final touches to the incompressible RANS equations

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- Using vector notation, the **exact Navier-Stokes RANS/URANS** equations can be written as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \overline{p}) + \nu \nabla^2 \overline{\mathbf{u}} + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau}^R$$

- By using the Boussinesq hypothesis,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I}$$

where

$$\bar{\mathbf{S}}^R = \frac{1}{2}\left[\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T\right]$$

- And after doing some algebra, we can now write down the **exact NS RANS** equations in the form of **solvable equations**, as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}(\mu + \mu_t)\nabla \bar{\mathbf{u}}\right]$$

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- The **solvable RANS/URANS** equations, can be written as follows using vector notation,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}\overbrace{(\mu + \mu_t)}\nabla \bar{\mathbf{u}}\right]$$

Effective viscosity

Turbulent viscosity

Normal stresses arising from the Boussinesq hypothesis

- In the **solvable equations** we introduce approximations.
    - All terms are now expressed in function of mean quantities.
    - These are the equations that are actually solved by the solver.
- Instead, in the **exact equations**, we do not use approximations.
    - Fluctuating terms appear in the equations.

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- Or using index notation, the **exact Navier-Stokes RANS/URANS** equations can be written as follows,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j x_j} + \frac{1}{\rho}\frac{\partial \tau_{ij}^R}{\partial x_j}$$

- By using the Boussinesq hypothesis,

$$\tau_{ij}^R = -\rho \overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

- And after doing some algebra, we can now write down the **exact NS RANS** equations in the form of **solvable equations**, as follows,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = -\frac{1}{\rho}\left[\frac{\left(\partial \bar{p} + \partial \frac{2}{3}\rho k\right)}{\partial x_i}\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \mu_t\right)\frac{\partial \bar{u}_i}{\partial x_j}\right]$$

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- The problem now reduces to computing the turbulent eddy viscosity μ_T in the momentum equation.

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}(\mu + \mu_t)\nabla \bar{\mathbf{u}}\right]$$

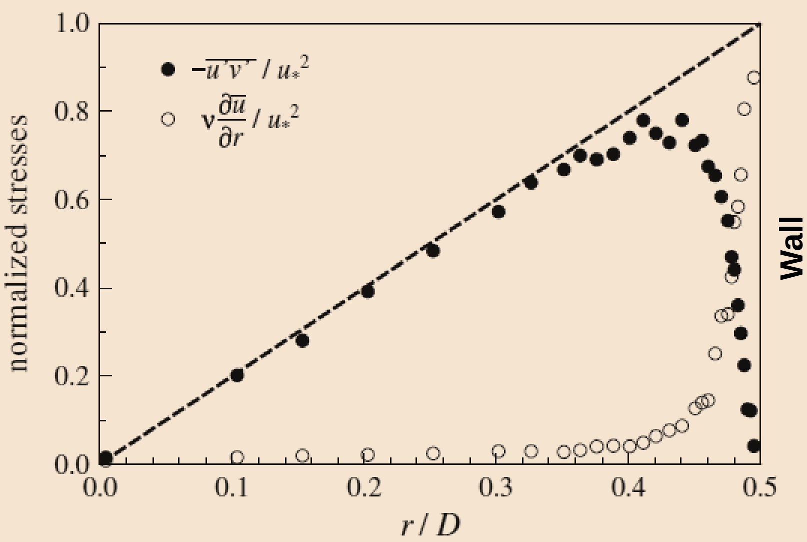
- This can be done by using any of the models that we will study in Lecture 6.
    - Zero equation models (algebraic models).
    - One equation models.
    - Two equation models.
    - Three, four, five, … , equation models.
    - Reynolds stress models.
    - And so on.

The Boussinesq hypothesis

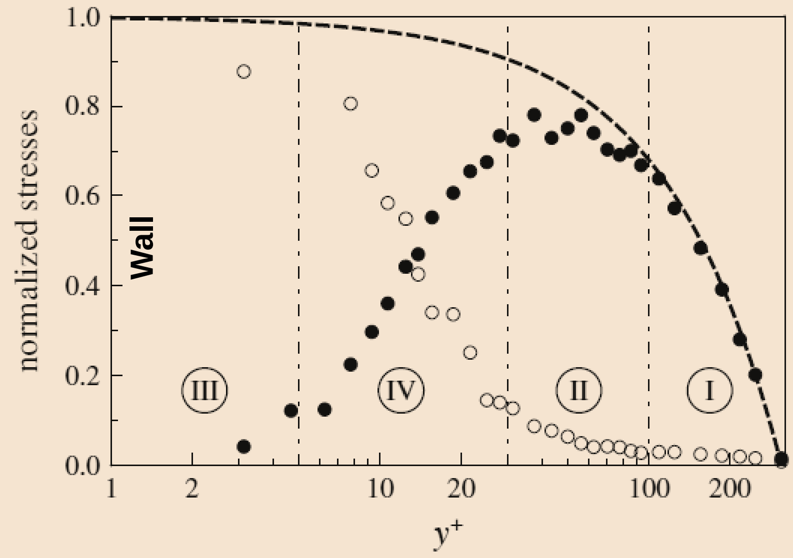
Turbulent flow in a pipe
Reynolds and viscous shear stresses distribution
Comparison of experimental and numerical results

The Boussinesq hypothesis

Reynolds and viscous shear stresses distribution – Turbulent flow in a pipe (experimental data)



The total normalized stress (using wall shear stress), as a function of the distance r from the centerline of a pipe with diameter D. The total stress consists of a contribution from the Reynolds stress (black circles) and the viscous stress (empty circles). Experimental data for a turbulent pipe flow at Re = 10000 [1].



The same data as in the left figure, but now as a function of the dimensionless distance y^+ from the pipe wall in a semi-log plot. In the figure, I = core region; II = logarithmic wall region; III = viscous sublayer; IV = buffer layer. Note that r = 0 corresponds to y^+ = 312 and r = 0.5D to y+ = 0 [1].

Exact RANS equation

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \bar{p}) + \underbrace{\overset{*}{\nu \nabla^2 \bar{\mathbf{u}}} - \frac{1}{\rho}\nabla \cdot \overset{**}{(\rho \overline{\mathbf{u}'\mathbf{u}'})}}_{\tau_{\text{Total}}}$$

Solvable RANS equation

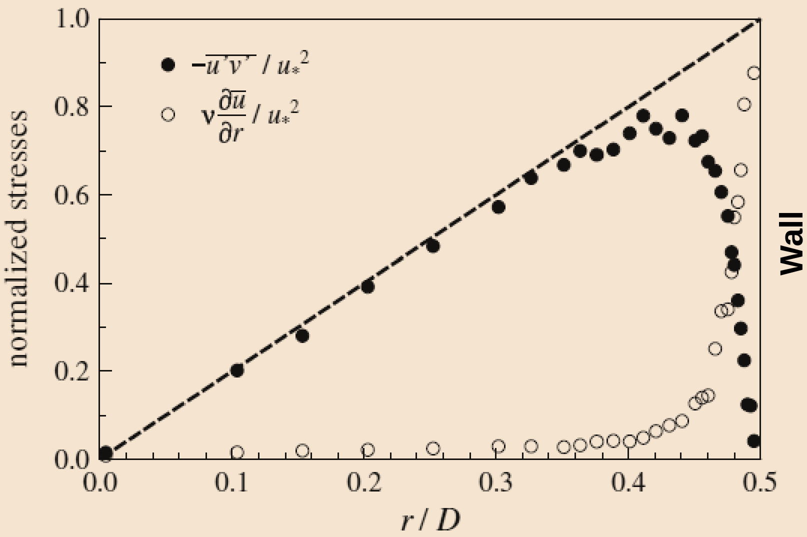
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \underbrace{\nabla \cdot \left[\frac{1}{\rho}(\overset{*}{\mu} + \overset{**}{\mu_t}) \nabla \bar{\mathbf{u}}\right]}_{\tau_{\text{Total}}}$$

* Viscous stress – Empty circles in the figure.
** Reynolds stress – Black circles in the figure.

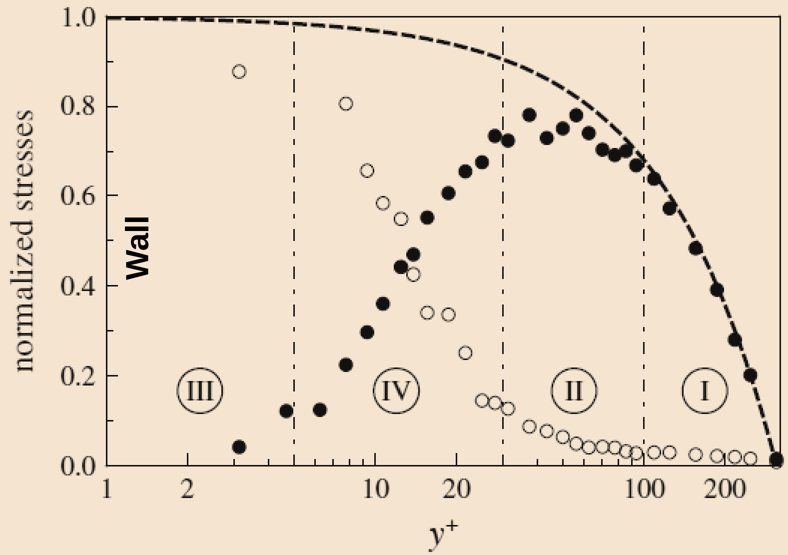
[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

The Boussinesq hypothesis

Reynolds and viscous shear stresses distribution – Turbulent flow in a pipe (experimental data)



The total normalized stress (using wall shear stress), as a function of the distance r from the centerline of a pipe with diameter D. The total stress consists of a contribution from the Reynolds stress (black circles) and the viscous stress (empty circles). Experimental data for a turbulent pipe flow at Re = 10000 [1].



The same data as in the left figure, but now as a function of the dimensionless distance y^+ from the pipe wall in a semi-log plot. In the figure, I = core region; II = logarithmic wall region; III = viscous sublayer; IV = buffer layer. Note that r = 0 corresponds to y^+ = 312 and r = 0.5D to y+ = 0 [1].

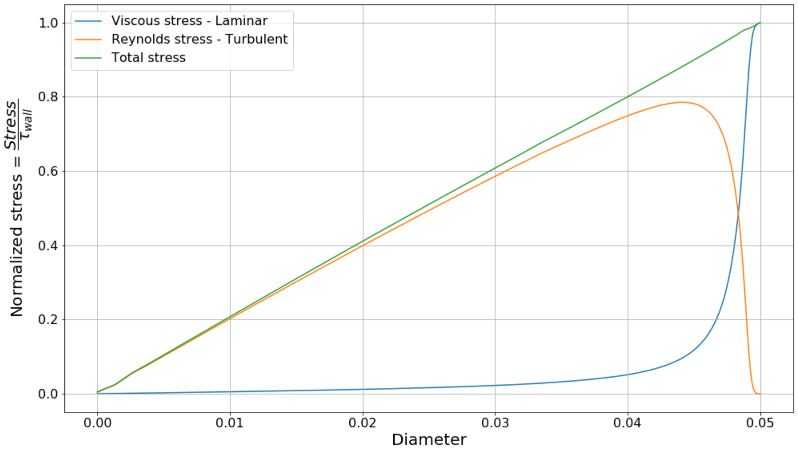
- Close to the walls, the viscous stress dominates, and as we get far from the wall, the Reynolds stress increases.
- In reference to the right figure. In region I and II the Reynolds stress dominates. In region III the viscous stress dominates. In region IV, both, the Reynolds stress and the viscous stress are important.
- The buffer layer is very energetic.

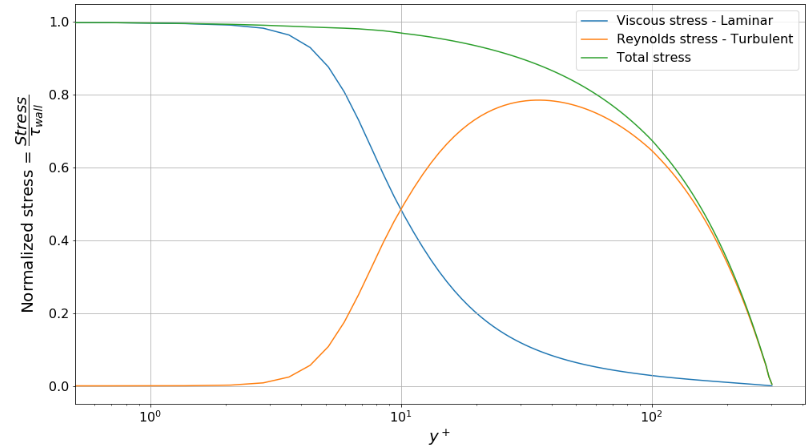
[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

The Boussinesq hypothesis

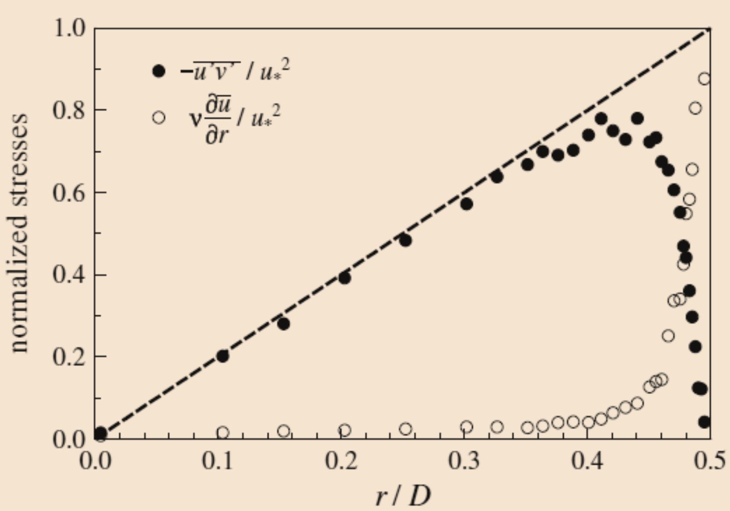
Reynolds and viscous shear stresses distribution
Turbulent flow in a pipe (experimental and numerical results)

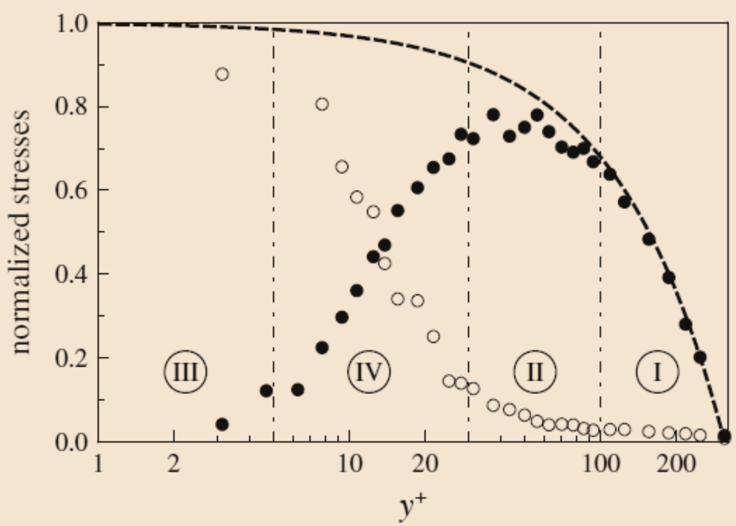
- Comparison of numerical results (top row) and experimental results (bottom row)





Results from tutorial 1



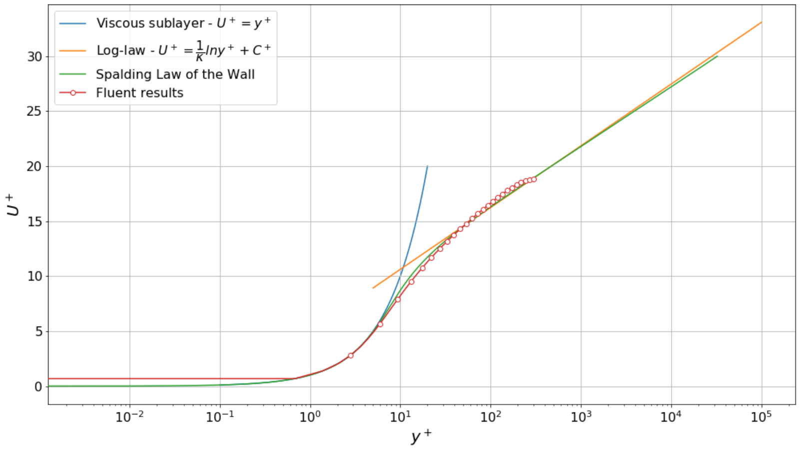


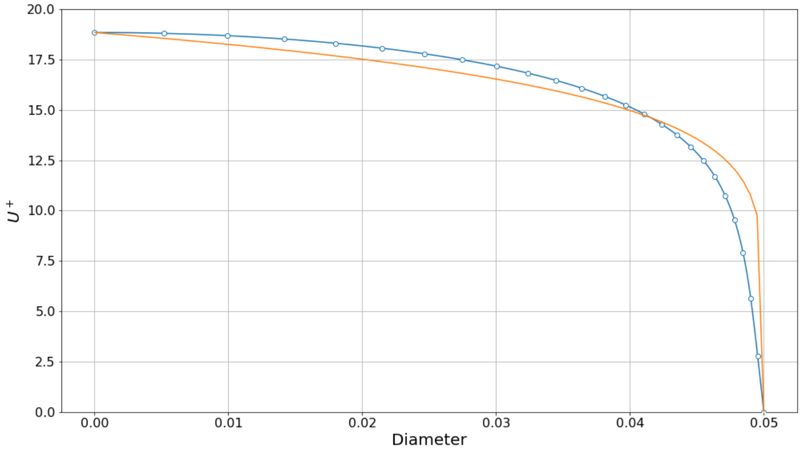
[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

The Boussinesq hypothesis

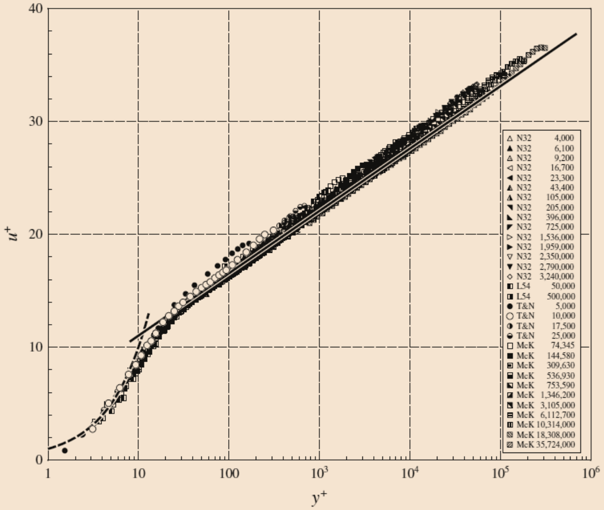
Reynolds and viscous shear stresses distribution
Turbulent flow in a pipe (experimental and numerical results)

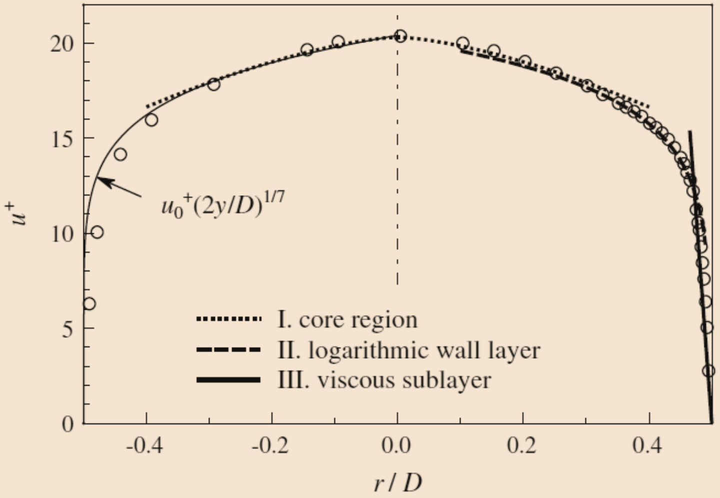
- Comparison of numerical results (top row) and experimental results (bottom row)





Results from tutorial 1





[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

Roadmap to Lecture 5

1. ~~Governing equations of fluid dynamics~~
2. ~~RANS equations – Reynolds averaging~~
3. ~~The Boussinesq hypothesis~~
4. **The gradient diffusion hypothesis**
5. Sample turbulence models

The gradient diffusion hypothesis

- When deriving the exact scalar transport RANS/URANS equation, a new term arose, the turbulent dispersion.

Scalar flux – Velocity-scalar covariance of turbulent diffusion

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \overline{\mathbf{u}} \overline{\phi} = \nabla \cdot \left(\Gamma_\phi \nabla \overline{\phi} - \rho \overline{\mathbf{u}' \phi'} \right) + S_\phi$$

- This extra term can be seen as the vector flux diffusing the transported quantity ϕ.
- This term has a similar meaning to the Reynolds stress tensor.
- And as for the Reynolds stress tensor, it requires modeling.
- The simplest model, and most widely used is the gradient diffusion hypothesis.
- Using this model, the scalar turbulent diffusion term $\rho \overline{\mathbf{u}' \phi'}$ is approximated as follows,

$$\rho \overline{\mathbf{u}' \phi'} = -\Gamma_T \nabla \overline{\phi}$$

- Mathematically speaking, the gradient diffusion hypothesis is analogous to Fourier’s law of heat conduction and Fick’s law of molecular diffusion.

The gradient diffusion hypothesis

- By using the gradient diffusion hypothesis, we can obtain the following equation,

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \overline{\mathbf{u}} \overline{\phi} = \nabla \cdot \left(\Gamma_L \nabla \overline{\phi} + \Gamma_T \nabla \overline{\phi} \right) + S_\phi$$

- After some minor algebra, we can write down the **exact scalar transport RANS/URANS** equations in the form of a **solvable equation**, as follows,

Effective diffusivity

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \overline{\mathbf{u}} \overline{\phi} = \nabla \cdot \left[\overbrace{\left(\Gamma_L + \Gamma_T \right)} \nabla \overline{\phi} \right] + S_\phi$$

Molecular diffusivity

Turbulent diffusivity

- At this point, specification of the turbulent eddy viscosity μ_T and the turbulent eddy diffusivity Γ_T solves the closure problem.

The gradient diffusion hypothesis

- The scalar transport RANS/URANS **solvable equations** can be written as follows,

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \overline{\mathbf{u}} \overline{\phi} = \nabla \cdot \left[\left(\Gamma_L + \Gamma_T \right) \nabla \overline{\phi} \right] + S_{\phi}$$

- Since turbulent transport of momentum and the scalar (heat, mass, concentration, and so on) is due to the same mechanism, *i.e.*, due to eddy mixing, it is often assumed that the eddy turbulent diffusivity Γ_T is proportional to the turbulent eddy viscosity μ_T.
- Recall that the molecular Prandtl number is defined as follows,

$$Pr = \frac{\nu}{\alpha}$$

- Molecular kinematic viscosity or eddy turbulent diffusivity (in turbulent flows).

- Thermal diffusivity or eddy turbulent diffusivity (in turbulent flows).
- This is equivalent to Γ_L or Γ_T in the scalar transport RANS/URANS solvable equations.
- In the literature, sometimes is defined as α, and sometimes as *k (d*o not confuse with TKE).
- Also, do not confuse the thermal conductivity *k* in the Pr definition or in the Fourier law of heat conduction.
- In summary, be careful with the notation.

- In turbulent flow simulations the turbulent Prandtl number must provided by the user
- The turbulent Prandtl number has a strong influence on the turbulent mixing and is problem dependent

The gradient diffusion hypothesis

- At this point, we can write the molecular (or laminar) and turbulent eddy diffusivities as follows,

$$\Gamma_L = \frac{\mu_L}{Pr_L} \qquad\qquad \Gamma_T = \frac{\mu_T}{Pr_T}$$

- Where Pr_L is the molecular Prandtl number (a property of the fluid), and Pr_T is the turbulent Prandtl number (a property of the flow).
- The Prandtl number is used when dealing with heat transfer.
- Instead, when dealing with species concentration or mass transfer, we use the Schmidt number Sc_T.
- So, if we know the eddy turbulent viscosity, we can prescribe the turbulent eddy diffusivity.

The gradient diffusion hypothesis

- Values of the turbulent Prandtl number Pr_T and of the turbulent Schmidt Sc_T are commonly found between,
    - $0.6 \leq Pr_T \leq 1$ – Prandtl number in heat transfer.
    - $0.6 \leq Sc_T \leq 1$ – Schmidt number in mass or species transport.
- Experimental measurements suggest that a value of $Pr_T \approx 0.9$ can be used in turbulent boundary layers, while $Pr_T \approx 0.7$ is often more suitable in free-shear flows.
- The recommended value often found in the literature is $Pr_T \approx 0.85$.
- However, have in mind that the values of the turbulent Prandtl number Pr_T and that of the turbulent Schmidt Sc_T greatly depends on the physics involved. No single value is valid for all flow conditions.
- The particular case of $Pr_T = 1$ or $Sc_T = 1$ corresponds to the Reynolds analogy, for which turbulent momentum and thermal transfers lead to similar turbulent boundary layer profiles for the mean velocity, temperature, and mass transfer.
- The same Reynolds analogy suggests that,

$$\Gamma_T = \frac{\mu_T}{Pr_T}$$

The gradient diffusion hypothesis

- It is worth mentioning that the gradient diffusion hypothesis and the Boussinesq hypothesis are both gradient based hypothesis and therefore very similar,

$$-\rho\overline{\mathbf{u}'\phi'} = \Gamma_T \nabla\overline{\phi} \qquad\qquad -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t\bar{\mathbf{S}}^R - \frac{2}{3}\rho k\mathbf{I}$$

- In a direct analogy to the Boussinesq hypothesis, in the gradient diffusion hypothesis the turbulent transport of the scalar is assumed to be proportional to the gradient of the transported quantity times a proportionality constant.
- While the gradient diffusion hypothesis seems to be a little bit simplistic, it does produce reasonable results for a large number of flows.
- The main deficiency of this hypothesis is the same as for the Boussinesq hypothesis, the model is isotropic.
- Despite the deficiencies of this hypothesis, it is used in more advanced turbulence models, such as the Reynolds stress models, to eliminate triple correlations and other terms, and thereby achieve closure.
- It is worth mentioning that more advanced scalar transport closures exists.

The gradient diffusion hypothesis

From the scalar transport RANS equation to the incompressible energy RANS equation

- Recall that the **exact** scalar transport RANS/URANS equation can be written as follows,

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \overline{\mathbf{u}} \overline{\phi} = \nabla \cdot \left(\Gamma_\phi \nabla \overline{\phi} - \rho \overline{\mathbf{u}' \phi'} \right) + S_\phi$$

- Let us use the enthalpy definition in the **exact** scalar transport RANS/URANS equation,

$$\overline{h} = c_p \overline{T}$$

- After substitution, we get the following equation,

$$\nabla_t \rho c_p \overline{T} + \nabla \cdot \rho c_p \overline{\mathbf{u}} \overline{T} = \nabla \cdot \left(k_L \nabla \overline{T} - \rho c_p \overline{\mathbf{u}' T'} \right) + S_T$$

Turbulent thermal heat flux

The gradient diffusion hypothesis

From the scalar transport RANS equation to the incompressible energy RANS equation

- In the following equation,

$$\nabla_t \rho c_p \overline{T} + \nabla \cdot \rho c_p \overline{\mathbf{u}}\overline{T} = \nabla \cdot \left(k_L \nabla \overline{T} - \rho c_p \overline{\mathbf{u}'T'} \right) + S_T$$

Turbulent thermal heat flux

- Then, the turbulent thermal flux q can be modeled as follows (gradient diffusion hypothesis),

$$q = -\rho c_p \overline{\mathbf{u}'T'} = k_T \nabla \overline{T}$$

- After substitution and regrouping, we get the incompressible **solvable** energy RANS equation,

$$\nabla_t \rho c_p \overline{T} + \nabla \cdot \rho c_p \overline{\mathbf{u}}\overline{T} = \nabla \cdot \left[\left(k_L + k_T \right) \nabla \overline{T} \right] + S_T$$

$$k_L = \frac{c_p \mu_L}{Pr_L}$$

Laminar thermal diffusivity

$$k_T = \frac{c_p \mu_T}{Pr_T}$$

Turbulent thermal diffusivity

The gradient diffusion hypothesis

From the scalar transport RANS equation to the incompressible energy RANS equation

- Recall that the incompressible **solvable** energy RANS equation can be written as follows,

$$\nabla_t \rho c_p \overline{T} + \nabla \cdot \rho c_p \overline{\mathbf{u}} \overline{T} = \nabla \cdot \left[(k_L + k_T) \nabla \overline{T} \right] + S_T$$

Laminar thermal diffusivity

Turbulent thermal diffusivity

- Note that the turbulent thermal diffusivity depends on the turbulent eddy viscosity and the turbulent Prandtl number.
- Evaluating k_T is not easy as it greatly depends on the turbulence model.
- For example, when using the $k - \epsilon$ turbulence model, the turbulent thermal diffusivity k_T is computed as follows,

$$k_T = \frac{c_p}{Pr_T} \frac{\rho C_\mu k^2}{\epsilon}$$

The gradient diffusion hypothesis

From the scalar transport RANS equation to the incompressible energy RANS equation

- At this point, the specification of k_T, together with μ_T, solve the closure problem.
- That is, if μ_T and Γ_T can somehow be specified, then the mean flow quantities in the solvable Navier-Stokes RANS equations and in the scalar transport RANS equation can be solved.
- At high Reynolds number, and far from walls, the turbulent eddy viscosity and the turbulent eddy diffusivity are found to scale with the integral velocity scale and integral length scale of the flow, independent from the molecular properties of the fluid. Consequently, the following ratios,

$$\frac{\nu_t}{\nu} \qquad \frac{\Gamma_t}{\Gamma}$$

- Both increase linearly with the Reynolds number. And in some circumstance, the molecular transport is negligible [1].

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.