Appendix 3

1. **Budgets discussion**

Budgets discussion

Turbulent kinetic energy budget

- The TKE budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- For $y^+ > 30$ it is seen that the TKE distribution is maintained almost exclusively as a balance between production and dissipation.
- The largest production rate occurs at $y^+ \approx 12$, which is close to the peak in TKE itself.
- The peak of dissipation is at the wall and has a local plateau of the wall.
- Turbulent transport of kinetic energy is important mainly near the wall. It is negative in the range $8 < y^+ < 30$ and positive for $y^+ < 8$, suggesting that much of the turbulent energy produced in the buffer layer is transferred towards the boundary.
- At the wall surface, the rate of viscous diffusion is balanced by its viscous dissipation. In other words, molecular diffusion brings energy towards the surface, where it is dissipated.
- Outside the viscous sublayer, in the buffer layer, the energy balance is more complex and involves transfer, production, dissipation, and pressure work.

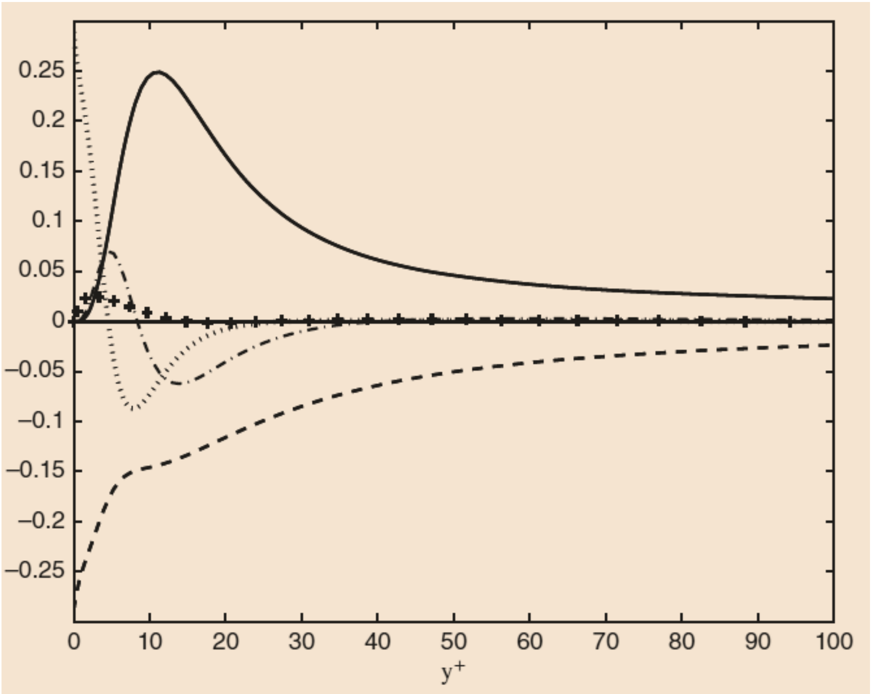

Figure 7.9 Turbulent kinetic energy budget in channel flow $R_\tau = 5186$ [10] scaled with ν and u_τ: —, production; – –, dissipation; +, pressure work; · · ·, viscous diffusion; – · –, turbulent transport.

Figure reproduced from reference [1,2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Turbulent dissipation rate budget

- The turbulent dissipation rate budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- Each term in the exact equation of ϵ owns a high degree of complexity; therefore, it is difficult to provide a clear physical interpretation of their trends.
- From the figure it is seen that the turbulent vortex stretching, P_ϵ^4 , balances the dissipation rate, Υ_ϵ , away from the walls.
- The production terms P_ϵ^1 and P_ϵ^2 are significant within approximately 25 wall units of the surface.
- Somewhat less important is P_ϵ^3 and the sum of the pressure and transport terms, $\Pi_\epsilon + T_\epsilon$.
- Near the wall, $-\Upsilon_\epsilon$ has a local minimum off the surface.
- At the boundary, $-\Upsilon_\epsilon$ and D_ϵ are in balance.
- The factors affecting ϵ near the boundary are complicated and a very great challenge to model.

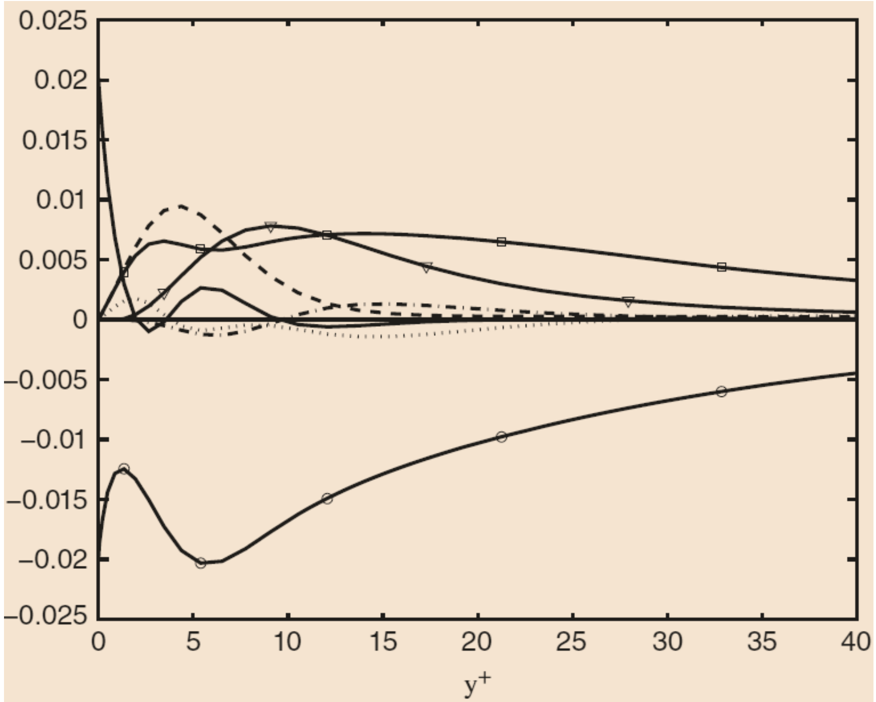

Figure 7.10 ϵ equation budget in channel flow at $R_\tau = 590$ [20] scaled with ν and u_τ. −−, P_ϵ^1; ▽, P_ϵ^2; − · −, P_ϵ^3; □, P_ϵ^4; ○, $-\Upsilon_\epsilon$; —, D_ϵ; · · ·, $\Pi_\epsilon + T_\epsilon$.

Figure reproduced from reference [1,2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Turbulent dissipation rate budget

- In the previous budget discussion, the terms of the turbulent dissipation rate equation are defined as follows [1,2],

$$\frac{D\epsilon}{Dt} = P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3 + P_\epsilon^4 + \Pi_\epsilon + T_\epsilon + D_\epsilon - \Upsilon_\epsilon$$

$$P_\epsilon^1 = -\epsilon_{ij}^c \frac{\partial \overline{U}_i}{\partial x_j}$$

$$P_\epsilon^2 = -\epsilon_{ij} \frac{\partial \overline{U}_i}{\partial x_j}$$

$$P_\epsilon^3 = -2\nu \overline{u_k \frac{\partial u_i}{\partial x_j}} \frac{\partial^2 \overline{U}_i}{\partial x_k \partial x_j}$$

$$P_\epsilon^4 = -2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_j}}$$

$$\Pi_\epsilon = -\frac{2\nu}{\rho} \frac{\partial}{\partial x_i} \left(\overline{\frac{\partial p}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \right)$$

$$T_\epsilon = -\nu \frac{\partial}{\partial x_k} \left(\overline{u_k \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \right)$$

$$D_\epsilon = \nu \nabla^2 \epsilon$$

$$\Upsilon_\epsilon = 2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} \right)^2}.$$

$$\epsilon_{ij}^c = 2\nu \overline{\frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}}$$

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$$

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Reynolds stresses and turbulent kinetic energy budget

- The Reynolds stress budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- The Individual normal stresses u^2, v^2, w^2 and TKE are shown in the figure.
- The sum of these budgets, divided by 2, gives the TKE budget.
- In the figure, TKE peaks relatively close to the wall around $y^+ = 15$, in the buffer layer. Clearly, this an indication that turbulent is at maximum in this region.
- The largest contribution to TKE comes primarily from u^2.
- The components v^2 and w^2 are relatively much smaller.
- The component v^2, which reflects the magnitude of the velocity fluctuations normal to the wall, is heavily damped by the presence of the walls.
- In the core region, far from the walls, there is a tendency to isotropy, which suggest that the influence of the wall has diminished.

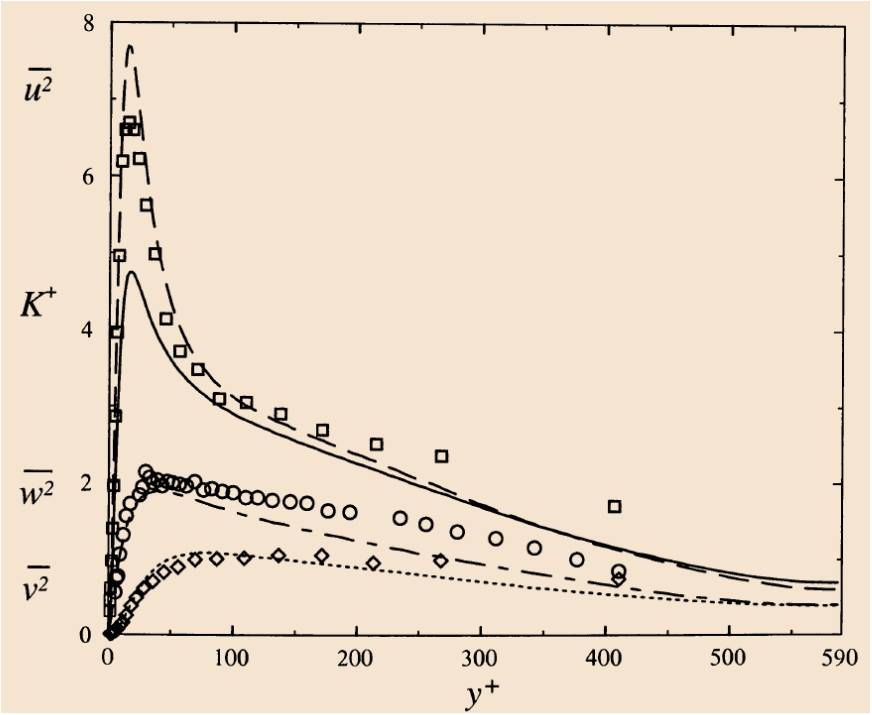

Fig. 4.7 Kinetic energy and normal stresses. Channel flow DNS [49], boundary layer measurements [33]. — —and □, $\overline{u^2}^+$; · · · and ◇, $\overline{v^2}^+$; — · —and ○, $\overline{w^2}^+$; —, K^+.

Figure reproduced from reference [1,2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Reynolds stress budget

- The particular anisotropic form taken by the normal stresses near the wall is a signature of an underlying dynamics of the turbulent motion.
- Experiments in channel and boundary layer flow [1] (refer to the figure) show that, with increasing Reynolds number, the peak in u^{+}_{rms} is less sharp and a plateau appears in the fully turbulent layer.
- However, the maximum value is reached in about the same y^+ value.
- This behavior is seen even at the very high Reynolds numbers in the atmospheric boundary layer, as determined in a field experiment in the Utah desert [1].

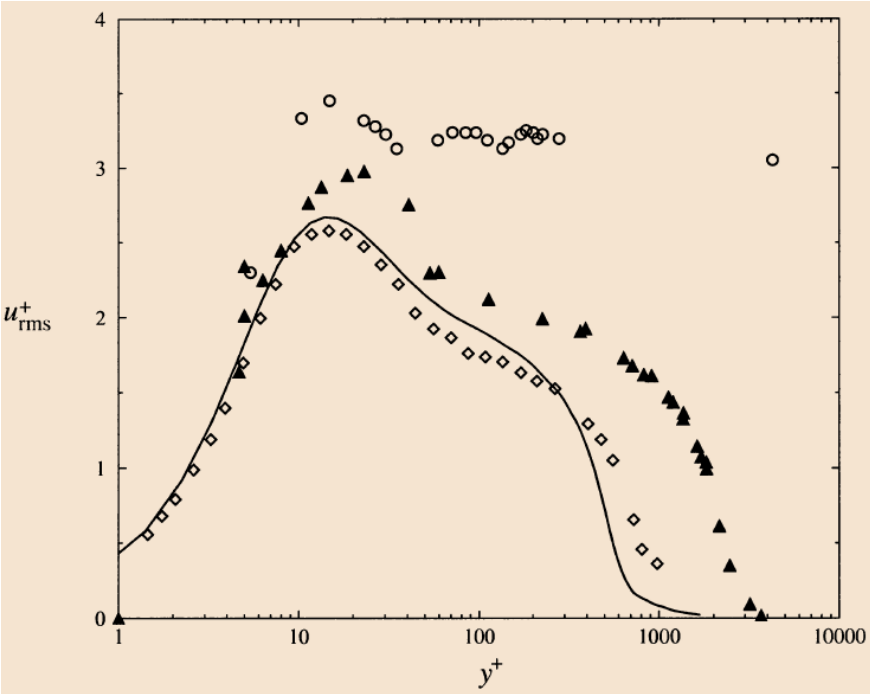

Fig. 4.8 Measurement and DNS of u^+_{rms} in channel and boundary layer flows at different Reynolds numbers. — [60], $R_\tau = 650$; ◇ [33], $R_\tau = 1050$; ▲ [38], $R_\tau = 2750$; ○, [21], $R_\tau = O(10^6)$.

Figure reproduced from reference [1,2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Reynolds stress u^2 budget

- The Reynolds stress budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- The individual normal stress u^2 is shown in the figure.
- The budget of u^2 has a lot in common with that of TKE.
- For example, viscous diffusion matches dissipation at the surface.
- The most noticeable difference is the extra loss deriving from the pressure-strain term.
- The v^2 and w^2 budgets, on the other hand, are noticeably different. As we will see next.

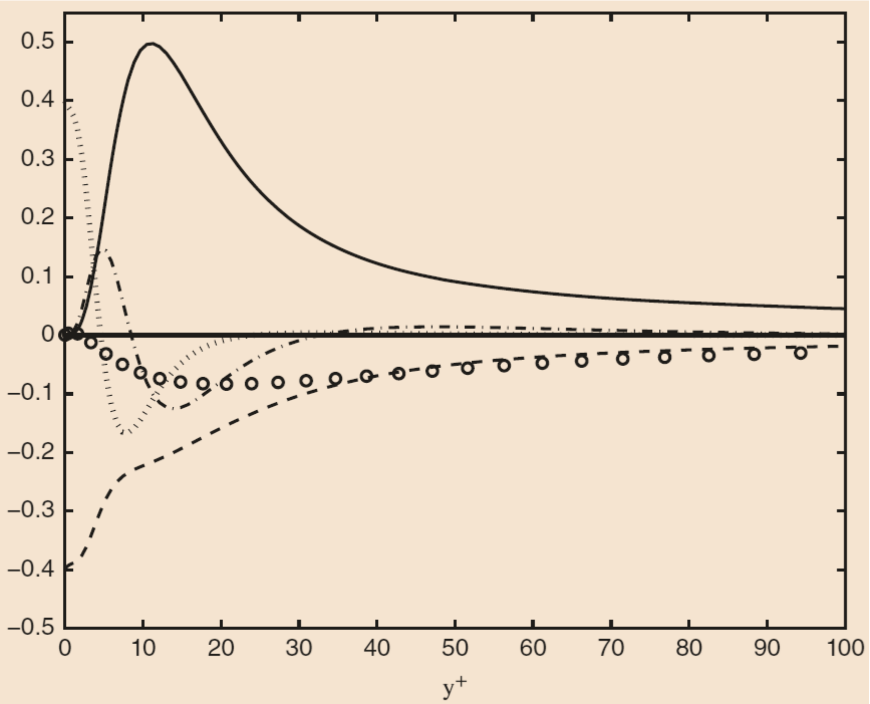

Figure 7.11 $\overline{u^2}$ budget in channel flow for $R_\tau = 5186$ [10]. —, production; – –, dissipation; o, pressure strain; · · ·, viscous diffusion; – · –, turbulent transport.

Figure reproduced from reference [1,2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Reynolds stress v^2 budget

- The Reynolds stress budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- The individual normal stress v^2 is shown in the figure.
- The v^2 and w^2 budgets are noticeably different from that of the normal stress u^2.
- In particular, losses due to dissipation are balanced mostly by the pressure-strain term, which thus takes on the role of primary production term.
- All terms, except from the pressure contributions, are zero at the walls.
- In fact, there is no direct production of v^2 or w^2 from the mean flow.
- Rather, they are produced as a by-product of the redistribution of energy from u^2 to the other normal Reynolds stress components.

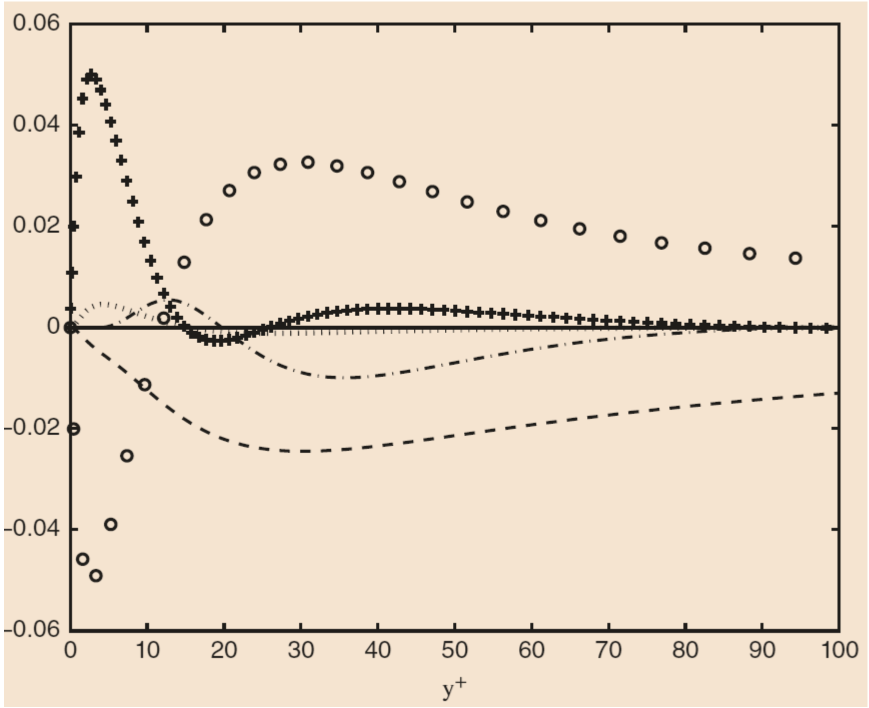

Figure 7.12 $\overline{v^2}$ budget in channel flow for $R_\tau = 5186$ [10]. ——, dissipation; o, pressure strain; +, pressure work; · · ·, viscous diffusion; — · —, turbulent transport.

Figure reproduced from reference [1.2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Reynolds stress w^2 budget

- The Reynolds stress budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- The individual normal stress w^2 is shown in the figure.
- The w^2 balance is similar to that of u^2 next to the wall in that dissipation is equal and opposite to viscous diffusion.
- And as for the balance of v^2, losses due to dissipation are balanced mostly by the pressure-strain term, which thus takes on the role of primary production term.

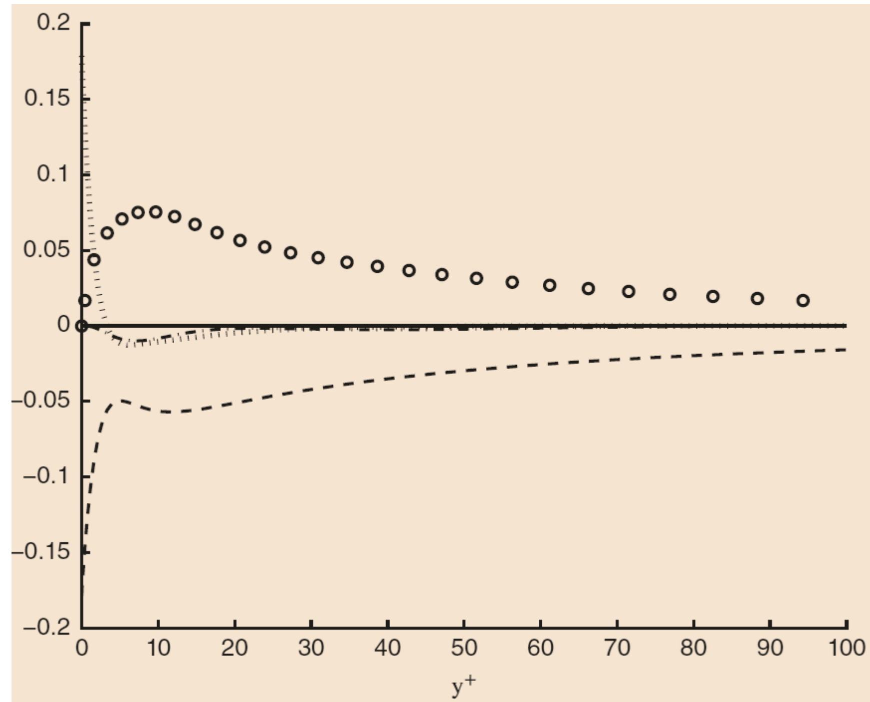

Figure 7.13 $\overline{w^2}$ budget in channel flow for $R_\tau = 5186$ [10]. ——, dissipation; ∘, pressure strain; · · ·, viscous diffusion; – · –, turbulent transport.

Figure reproduced from reference [1,2]

References:

[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.

[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Pressure strain in the normal Reynolds stresses budget

- The Reynolds stress budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- The redistribution role of pressure-strain in each of the normal Reynolds stresses is shown in the figure.
- The sum of these terms is identically zero.
- That is, the pressure-strain has no effect on the turbulent kinetic energy.
- It is interesting to note that moving towards the core of the flow, the lost energy in u^2 is redirected in equal amounts to v^2 and w^2.
- Near the surface, the energy redistribution is entirely into the spanwise direction (sub-index $_{33}$), with much of it coming from the wall normal direction (sub-index $_{22}$), and some minor contribution from the streamwise component (sub-index $_{11}$).
- For the plots of the budget v^2, it is evident that the main source of v^2 is the pressure work term.

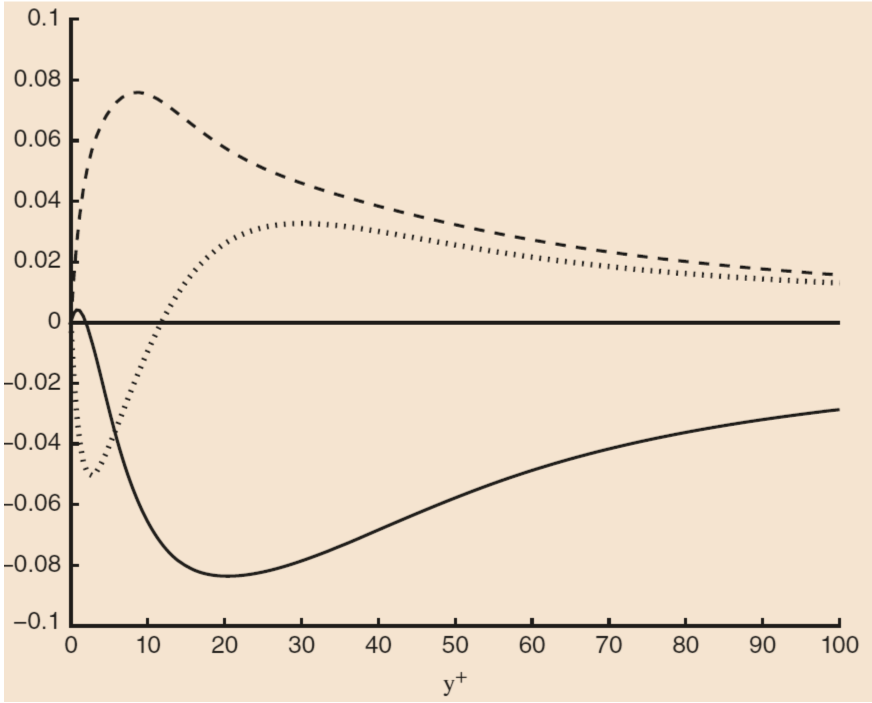

Figure 7.14 Pressure-strain term in normal Reynolds stress equations at $R_\tau = 5186$ [10]: —, Π_{11}; · · ·, Π_{22}; − −, Π_{33}.

Figure reproduced from reference [1,2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Reynolds stress uv budget

- The Reynolds stress budget plotted in the figure [1,2] represents the contributions of each term in the right-hand side of the exact equation.
- The shear stress uv is shown in the figure.
- Here, since uv < 0, the production term is negative.
- Interestingly enough, this is balanced by the pressure-strain term, since unlike the case of normal stresses, the dissipation is relatively insignificant.
- As in the case of the v^2 balance, the sum of the pressure terms nearly cancels in the vicinity of the wall.
- This is an argument for using the combined pressure terms, instead of considering them separately.
- It is important to note that in the Reynolds stresses with components normal to the walls, the pressure work and pressure strain plays an important role in generating and redistributing the turbulent energy

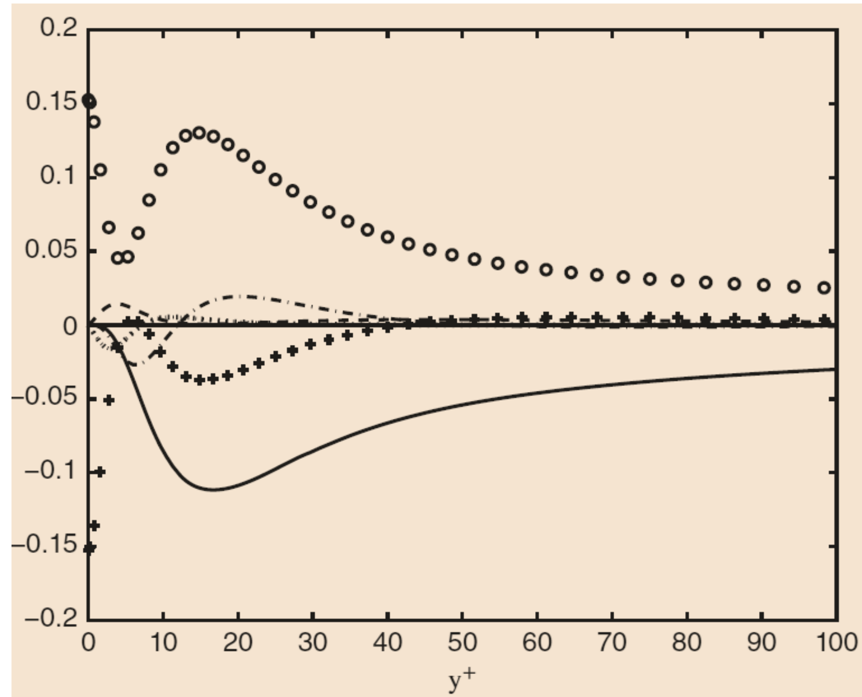

Figure 7.15 $\overline{uv}$ budget in channel flow for $R_\tau = 5186$ [10]. —, production; −−, dissipation; o, pressure strain; +, pressure work; · · ·, viscous diffusion; − · −, turbulent transport.

Figure reproduced from reference [2]

References:
[1] P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement and Prediction. 2002.
[2] P. Bernard. Turbulent Fluid Flow. Wiley, 2019.

Budgets discussion

Final comments

- The budgets discussed represent the contributions of each individual term of the exact transport equations to the physics of turbulent flows.
- The insight obtained form these budgets are used for model development, validation, and calibration.
- For $y^+ > 30 \sim 60$ it is seen that the budget balance is maintained almost exclusively as a balance between production and dissipation or a balance between pressure work and dissipation.
- The budget balance in the buffer layer is very complex and involves transfer, production, dissipation, and pressure work.
- TKE peaks in the buffer layer. The buffer layer is very energetic.
- With increasing Reynolds number, the peak in TKE and the normal Reynolds stress u^2 is less sharp and a plateau appears in the fully turbulent layer.
- Turbulent energy is produced directly in u^2 from the mean shearing and is then redistributed to the other normal Reynolds stress components through the action of the pressure force.
- The viscous and turbulent transport terms in every budget account for the spatial redistribution of energy without its production.
- The transport normal to the wall, as in uv and v^2, is know as wall blocking effect or viscous damping.
- Due to wall blocking, the transport normal to the wall is inhibit by the boundary presence. Therefore, the eddy viscosity must be decreased for consistent results.