Turbulence and CFD models: Theory and applications

Roadmap to Lecture 6

Part 2

1. Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation
2. Revisiting the closure problem
3. Two equations models – The $k-\epsilon$ model
4. Two equations models – The $k-\omega$ model
5. One equation model – The Spalart-Allmaras model

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Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- Let us recall the **exact** Reynolds stress transport equation,

$$\underbrace{\frac{\partial \tau_{ij}}{\partial t}}_{1} + \underbrace{\bar{u}_k \frac{\partial \tau_{ij}}{x_k}}_{2} = \underbrace{- \left(\tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} \right)}_{3} + \underbrace{2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}}_{4} + \dots$$

$$\dots + \underbrace{\frac{1}{\rho} \left(\overline{u'_i \frac{\partial p'}{\partial x_j}} + \overline{u'_j \frac{\partial p'}{\partial x_i}} \right)}_{5} + \underbrace{\frac{\partial}{\partial x_k} \left(\nu \frac{\partial \tau_{ij}}{\partial x_k} \right)}_{6} + \underbrace{\frac{\partial}{\partial x_k} \left(\overline{u'_i u'_j u'_k} \right)}_{7}$$

1. Transient term.
2. Convective term.
3. Production term.
4. Dissipation term (tensor dissipation).
5. Turbulent stress transport related to the velocity and pressure fluctuations.
6. Viscous stress diffusion (molecular)
7. Diffusive stress transport resulting from the triple correlation of velocity fluctuations.

- Nota tha in our notation $\tau_{ij} = \tau^R_{ij} = - \left(\overline{u'_i u'_j} \right)$.
- The terms 1, 2, 3 require no modeling when using the RSM approach.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- Let us recall the **exact** turbulent kinetic energy equation TKE, which is obtained by taking the trace (or contraction $i = j$ in index notation) of the Reynolds stress transport equation,

$$\underbrace{\frac{\partial \tau_{ii}}{\partial t}}_{1} + \underbrace{\bar{u}_k \frac{\partial \tau_{ii}}{\partial x_k}}_{2} = -\underbrace{2\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}}_{3} + \underbrace{\epsilon_{ii}}_{4} + \underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ii}}{\partial x_k}\right)}_{5} + \underbrace{\frac{2}{\rho}\left(\overline{u_i' \frac{\partial p'}{\partial x_i}}\right)}_{6} + \underbrace{\frac{\partial}{\partial x_k}(\overline{u_i' u_i' u_k'})}_{7}$$

1. Transient rate of change term.
2. Convective term.
3. Production term arising from the product of the Reynolds stress and the velocity gradient.
4. Dissipation rate (scalar dissipation).
5. Rate of viscous stress diffusion (molecular).
6. Turbulent transport associated with the eddy pressure and velocity fluctuations.
7. Diffusive turbulent transport resulting from the triple correlation of velocity fluctuations.

- And recall that,

$$k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$\tau_{ii} = \tau_{ii}^R = -\left(\overline{\mathbf{u}'\mathbf{u}'}\right)^{\mathrm{tr}} = -\left(\overline{u_i' u_i'}\right) = -2k$$

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- We can now substitute $\tau_{ii} = -\left(\overline{u_i' u_i'}\right) = -2k$ and simplify to obtain the following equation,

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Production}} - \underbrace{\epsilon}_{\text{Dissipation}} + \frac{\partial}{\partial x_j} \left[\underbrace{\nu \frac{\partial k}{\partial x_j}}_{\text{Molecular diffusion}} - \underbrace{\frac{1}{2} \overline{u_i' u_i' u_j'}}_{\text{Turbulent transport}} - \underbrace{\frac{1}{\rho} \overline{p' u_j'}}_{\text{Pressure diffusion}} \right]$$

- Where ϵ is the dissipation rate (per unit mass) and is given by the following relation,

$$\epsilon = \frac{\epsilon_{ii}}{2} = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}$$

Note:

$\tau_{ij} = \tau_{ij}^R = -\left(\overline{u_i' u_j'}\right)$

- This is the **exact** turbulent kinetic energy transport equation.
- To derive the **solvable** equation, we need to use approximations in place of the terms that contain fluctuating quantities.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- The **solvable** turbulent kinetic energy equation TKE can be written as follows,

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Production}} - \underset{\text{Dissipation}}{\epsilon} + \underbrace{\frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]}_{\text{Diffusion}}$$

- The Reynolds stresses can be modeled using the Boussinesq approximation.
- The term related to the turbulent transport and the pressure diffusion can be modeled as follows (gradient diffusion hypothesis),

$$\frac{1}{2}\overline{u_i' u_i' u_j'} + \frac{1}{\rho}\overline{p' u_j'} = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$$

- The term related to the dissipation rate can be modeled by adding an additional transport equation, which will be derived later.
- All the approximations added are based on DNS simulations, experimental data, analytical solutions, or engineering intuition.

Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation

- The **exact** form of the Reynolds stress transport equation, the turbulent kinetic energy transport equation, and the transport equations of the additional turbulent quantities (dissipation rate, specific rate of dissipation, and so on) share some similarities. Namely,
    - a production term (eddy factory),
    - a dissipation or destruction term (where eddies are destroyed – eddy graveyard –),
    - a turbulence diffusion term (transport, diffusion, and redistribution due to turbulence),
    - Plus, any additional source term, such as buoyancy or gravity forces.

- Therefore, the transport equations of the turbulent quantities can be expressed in the following way,

$$\underbrace{\nabla_t \phi}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \phi}_{\text{Convection}} = \underbrace{P^{\phi}}_{\text{Production}} + \underbrace{\epsilon^{\phi}}_{\text{Dissipation}} + \underbrace{D^{\phi}}_{\text{Diffusion}} + \underbrace{S^{\phi}}_{\text{Source terms}}$$

- Where ϕ represents the transported turbulent quantity.

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Revisiting the closure problem

- The **solvable** RANS/URANS equations can be written as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}\left(\mu + \mu_t\right)\nabla \bar{\mathbf{u}}\right]$$

Turbulent viscosity

- In this case, the **solvable** Navier-Stokes RANS/URANS equations were obtained after substituting the Boussinesq approximation into the **exact** RANS/URANS equations.
- The problem now reduces to computing the turbulent eddy viscosity in the momentum equation.
- Each turbulence model computes the turbulent eddy viscosity in a different way,

$$\mu_t = f(k, \epsilon, \omega, l, t, v, \ldots)$$

Revisiting the closure problem

- To compute the turbulent eddy viscosity, many approaches are available.
- To name a few,
    - Zero equation models.
    - One equation models.
    - Two equation models.
    - Three, four, five, … , equation models.
    - Reynolds stress models.
    - And so on.

- Hereafter, we will address the most widely used approaches.
- At this point, it only rest exploring some RANS/URANS turbulence models.
- The order in which we are going to present the turbulence models does not reflect the accuracy, importance, number of equations, release date, type of approximations used, or efficiency of the models.
- It is an order that we think follows the derivation of the exact equations.