Turbulence and CFD models: Theory and applications

Roadmap to Lecture 6

Part 6

1. Vorticity based models
2. Third-order and higher order moment closure methods
3. Non-linear eddy viscosity models

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Vorticity based models

- So far, we have used the Boussinesq hypothesis to model the Reynolds stress tensor.
- However, we must be aware that different approaches do exist.
- For example, an entirely different approach for dealing with RANS was originally considered by Taylor [1] and subsequent authors [2,3,4].
- To avoid the appearance of the Reynolds stress tensor, they proposed the use of the following identity,

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial k}{\partial x_i} - \epsilon_{ijk} \overline{u_j \omega_k}$$

- Using this identity, we can write the momentum equation of the RANS equations in the vorticity transport form,

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_i}{\partial x_j} = - \frac{\partial (\bar{p}/\rho + k)}{\partial x_i} + \nu \nabla^2 \bar{u}_i + \epsilon_{ijk} \overline{u_j \omega_k}$$

- In this approach, a model must be sought for the vorticity flux term $\overline{u_j \omega_k}$.
- Closures schemes based on this approach remain largely undeveloped.

References:

[1] G. Taylor. The transport of vorticity ad heat through fluids in turbulent motion. Proc. Roy. Soc., 135A, 1932.

[2] J. Hinze. Turbulence. McGraw-Hill. 1975.

[3] B. Perot, P. Moin. A new approach to turbulence modeling. Center for Turbulence Research. Proc. Summer Program. 1996.

[4] S. Goldstein. A Note on the Vorticity-Transport Theory of Turbulent Motion. Mathematical Proceedings of the Cambridge Philosophical Society, 31(3). 1935.

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Third-order and higher order moment closure methods.

- We have seen that in order to derive the Reynolds stress transport equations, we need to multiply the Navier-Stokes operator $\mathcal{N}(u_i)$, by the velocity fluctuations, as follows,

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k x_k} = 0$$

$$\overline{u_i' \mathcal{N}(u_j) + u_j' \mathcal{N}(u_i)} = 0$$

- Basically, we are multiplying the exact momentum equations by the velocity fluctuations in order to obtain governing equations for $\tau_{ij} = -\overline{u_i' u_j'}$.
- In doing so, we are increasing the order of closure of the equations, from first-order moment closure to second-order moment closure (in analogy to statistical moments).
- In theory, we can continue increasing the order of the moment closure up to infinite.
- So, we can derive third-order moment closure equations and so on.

Third-order and higher order moment closure methods.

- However, as we keep increasing the moment, higher order correlations will keep appearing in the equations.
- For example, in the **exact** Reynolds stress transport equations, which are second-order moment closure equations, a triple correlation appears, namely,

$$\overline{u_i' u_j' u_k'}$$

- We could derive a set of governing equations for this triple correlation, but the resulting equations will contain quadruple correlations.
- Therefore, it is easier to model this term.
- In the third-order moment closure equations, the quadruple correlation is expressed as follows,

$$\overline{u_i' u_j' u_k' u_l'}$$

- It is worth noting that third-order moment closure models do exist, but they are not widely diffused, and they do not guarantee better results.

Third-order and higher order moment closure methods.

- For example, the equations for the third order moments, read as,

$$\frac{\partial \overline{u_i u_j u_l}}{\partial t} + \overline{U}_k \frac{\partial \overline{u_i u_j u_l}}{\partial x_k} = \overline{u_i u_j} \frac{\partial \overline{u_l u_k}}{\partial x_k} + \overline{u_j u_l} \frac{\partial \overline{u_i u_k}}{\partial x_k} + \overline{u_l u_i} \frac{\partial \overline{u_j u_k}}{\partial x_k}$$
$$- \overline{u_j u_l u_k} \frac{\partial \overline{U}_i}{\partial x_k} - \overline{u_i u_j u_k} \frac{\partial \overline{U}_l}{\partial x_k} - \overline{u_l u_i u_k} \frac{\partial \overline{U}_j}{\partial x_k}$$
$$\underbrace{- \frac{\partial \overline{u_i u_j u_l u_k}}{\partial x_k}}_{T_{ijl}} \underbrace{- \frac{1}{\varrho} \left(\overline{u_l u_j \frac{\partial p}{\partial x_i}} + \overline{u_i u_j \frac{\partial p}{\partial x_l}} + \overline{u_l u_i \frac{\partial p}{\partial x_j}} \right)}_{\Pi_{ijl}}$$
$$\underbrace{-2\nu \left(\overline{u_i \frac{\partial u_j}{\partial x_k} \frac{\partial u_l}{\partial x_k}} + \overline{u_j \frac{\partial u_i}{\partial x_k} \frac{\partial u_l}{\partial x_k}} + \overline{u_l \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \right)}_{\epsilon_{ijl}} + \nu \frac{\partial^2 \overline{u_i u_j u_l}}{\partial x_k \partial x_k}.$$

- For the interested reader, works related to third-order moment closure turbulence models can be found in references [1,2,3,4].

References:
[1] R. Amano, J. Chai. Closure models of turbulent third-order momentum and temperature fluctuations. NASA-CR-180421. 1987
[2] R. Amano, P. Goel. A study of Reynolds-Stress closure model. NASA-CR-174342. 1985.
[3] R. Amano, P. Goel. Improvement of the second- and third-moment modeling of turbulence: A study of Reynolds-stress closure model. NASA-CR-176478. 1986.
[4] R. Amano, J. Chai, J. Chen. Higher order turbulence closure models. NASA-CR-183236. 1988.

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Non-linear eddy viscosity models

- One approach to achieving a more appropriate description of the Reynolds-stress tensor without introducing any additional transport equations (as in the RSM models) is to add extra high order terms to the Boussinesq approximation.

$$\rho\overline{u_i'u_j'} = \frac{2}{3}\rho k \delta_{ij} - 2\mu_t S_{ij} + f(S_{ij}, \Omega_{ij})$$

- Where $f(S_{ij}, \Omega_{ij})$ is a nonlinear function dependent on the mean strain rate S_{ij} tensor and spin tensor (rotation) Ω_{ij}.
- Recall that the mean strain rate tensor and spin tensor are defined as follows,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

- These models are known as nonlinear eddy viscosity models (NLEVM).
- This idea was originally proposed by Lumley [1,2], and many NLEVM has been proposed since then.

References:
[1] J. Lumley. Toward a turbulent constitutive equation. Journal of Fluid Mechanics. 1970.
[2] J. Lumley. Computational modeling of turbulent flows. Advances in Applied Mechanics. 1978.

Non-linear eddy viscosity models

- The NLEVM approach can be seen as a remedy to the deficiencies of the EVM.
- Where the main deficiencies of the EVM are:
  - Inability to proper describe the anisotropic behavior in shear layers. In the EVM the normal stresses are isotropic, $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = (2/3)k$.
  - Flow in ducts with secondary motions.
  - Overpredicting production of turbulent kinetic energy in stagnation points.
  - Failure to reproduce the asymmetric behavior of the velocity profiles in the presence of streamlined geometries (strong curvature).
  - Underpredicting turbulent viscosity in the presence of system rotation (strong vortices).
- In comparison to the EVM, the NLEVM models are more computational expensive (as they need to solve more terms and are wall resolving).
- They are also harder to convergence.
- However, they do offer improved prediction capabilities for certain complex turbulent flows.
- Despite the many apparent advantages of NLEVM, they are not widely used.
- EVM still are the workhorse of turbulence modeling but there is a trend to shift towards this models.

Non-linear eddy viscosity models

- The NLEVM are usually quadratic or cubic.
- Let us briefly discussed the NLEVM by Shih et al [1], which is cubic.
- In this model, the Reynolds stresses are computed as follows,

$$-\rho\overline{u_i'u_j'} = -\frac{2}{3}\rho k\delta_{ij} + \mu_t 2S_{ij}^* + A_3\frac{\rho k^3}{\epsilon^2}\left[\bar{S}_{ik}\bar{\Omega}_{kj} - \bar{\Omega}_{ik}\bar{S}_{kj}\right]$$

$$-2A_5\frac{\rho k^4}{\epsilon^3}\left[\bar{\Omega}_{ik}\bar{S}_{kj}^2 - \bar{S}_{ik}^2\bar{\Omega}_{kj} + \bar{\Omega}_{ik}\bar{S}_{km}\bar{\Omega}_{mj} - \frac{1}{3}\bar{\Omega}_{kl}\bar{S}_{lm}\bar{\Omega}_{mk}\delta_{ij} + I_s S_{ij}^*\right]$$

- Where I_s , S_{ij}^* , and S_{ij}^2 are given by the following relationships,

$$I_s = \frac{1}{2}\left[\bar{S}_{kk}\bar{S}_{mm} - \bar{S}_{kk}^2\right] \qquad S_{ij}^* = \bar{S}_{ij} - \frac{1}{3}\bar{S}_{kk}\delta_{ij} \qquad S_{ij}^2 = S_{ik}S_{kj}$$

References:

[1] TH. Shih, J. Zhu, WW. Liou, K-H. Chen, N-S. Liu, J. Lumley. Modeling of turbulent swirling flows. NASA-TM-113112. 1997.

Non-linear eddy viscosity models

- The NLEVM model by Shih et al [1], is particularly suited for swirling flows.
- It was developed to deal with aircraft engine combustors that generally involve turbulent swirling flows in order to enhance fuel-air mixing and flame stabilization.
- The model includes third order terms, so it offers extra accuracy.
- The method also satisfy the constraints of rapid distortion theory (RDT) and realizability.
- All the coefficients appearing in the nonlinear constitutive equation are calibrated using DNS and experimental data.
- The coefficient C_μ is not constant, it depends on the strain rate tensor.
- The value of the turbulent kinetic energy k and the dissipation rate ϵ are obtained from low-RE $k-\epsilon$ turbulence models (wall resolving).
- Also, the value of the turbulent eddy viscosity is computed using the relations from low-RE $k-\epsilon$ turbulence models.

References:
[1] TH. Shih, J. Zhu, WW. Liou, K-H. Chen, N-S. Liu, J. Lumley. Modeling of turbulent swirling flows. NASA-TM-113112. 1997.

Non-linear eddy viscosity models

- The mathematical framework to derive NLEVM is quite complex.
- Another way to derive non-linear models is by using algebraic stress models (ASM) or Explicit algebraic Reynolds stress model (EARSM).
- As for NLEVM, the mathematical formalism behind ASM and EARSM models is quite complex and will not address it here.
- In the cubic formulations of NLEVM, the quadratic terms allow for anisotropic effects to be modelled and the cubic terms allow modeling of the consequences of streamline curvature.
- These models also involve variable C_μ coefficient formulations based on **S** and Ω , which helps avoid excessive turbulence prediction at stagnation points.
- In these models, the realizability conditions are always enforced.
- These models are usually wall resolving.

Non-linear eddy viscosity models

- Pope [1,2], maybe was the first one to derive a general effective-viscosity hypothesis that serves as the base for NLEVM and ASM models.
- Pope proposed an explicit expression for the normalized anisotropy tensor b_{ij}.

$$b_{ij} = \mathcal{B}_{ij}\left(\mathbf{S}, \Omega\right) = \sum_{1}^{10} G^n \mathcal{T}_{ij}^n$$

- He assumed that the anisotropic part Reynolds stress tensor can be expressed in function of the strain-rate tensor, **S**, and the vorticity tensor, Ω.
- Furthermore, he showed that the coefficients, G^n, in the expression of the anisotropy tensor b_{ij} can be a function of not more than five invariants $\mathcal{T}_{ij}^n$ [1,2].

References:
[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.

Non-linear eddy viscosity models

- Pope's general effective-viscosity hypothesis is the starting point for developing ASM, EARSM, and NLEVM models.
- The mathematical formalism is quite complex and is outside the scope of this discussion.
- The derivation of the invariants is addressed in references [1,2,3].
- The three-dimensional effective-viscosity hypothesis constitutive expression for the ASM model can be written as follows,

$$
\begin{aligned}
\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij} = a_{ij} = C_\mu k \mathbf{S} + C_1 k \left(\mathbf{S} \cdot \Omega - \Omega \cdot \mathbf{S} \right) + \\
C_2 k \left[\Omega^2 - \frac{1}{3} tr \left(\Omega^2 \right) \mathbf{I} \right] + \\
C_3 k \left[\mathbf{S} \cdot \Omega^2 + \Omega^2 \cdot \mathbf{S} - \frac{2}{3} tr \left(\Omega^2 \cdot \mathbf{S} \right) \mathbf{I} \right] + \\
C_4 k \left(\Omega \cdot \mathbf{S} \cdot \Omega^2 - \Omega^2 \cdot \mathbf{S} \cdot \Omega \right)
\end{aligned}
$$

- At this point, we need to find the closure coefficients, which can be constant or a function of a variable or combination of tensors.

References:
[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.
[3] M. Leschziner. Statistical Turbulence Modeling for Fluid Dynamics. Imperial College Press, 2016.

Non-linear eddy viscosity models

- The exact two-dimensional effective-viscosity hypothesis constitutive expression only retain the first two invariants and can be written as follows,

$$\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij} = a_{ij} = C_\mu k \mathbf{S} + C_1 k \left(\mathbf{S} \cdot \Omega - \Omega \cdot \mathbf{S} \right)$$

- At this point, we need to find the closure coefficients, which can be constant or a function of a variable or combination of tensors.
- The derivation of the invariants is addressed in references [1,2,3].
- Note that if $C_1 = 0$, $C_\mu = -C_\mu$, and $\mathbf{S} = \frac{k}{\epsilon} 2 S_{ij}$, we recover the linear $k - \epsilon$ turbulence viscosity formulation,

$$\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij} = -C_\mu \frac{k^2}{\epsilon} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

References:

[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.

[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.

[3] M. Leschziner. Statistical Turbulence Modeling for Fluid Dynamics. Imperial College Press, 2016.

Non-linear eddy viscosity models

- In the previous equations, the tensor a_{ij} is known as the deviatoric anisotropic part of the Reynolds stress tensor and is defined as,

$$a_{ij} = \overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij}$$

- The normalized anisotropy tensor b_{ij}, can be written as follows,

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\overline{u_i' u_j'}}{\overline{u_k' u_k'}} - \frac{1}{3} \delta_{ij}$$

References:
[1] S. Pope. A More General Effective Viscosity Hypothesis. Journal of Fluid Mechanics, Vol. 72, 1975.
[2] S. Pope. Turbulent Flows. Cambridge University Press, 2010.
[3] M. Leschziner. Statistical Turbulence Modeling for Fluid Dynamics. Imperial College Press, 2016.