Chapter 3

Governing Equations of Fluid Dynamics

The starting point of any numerical simulation are the governing equations of the physics of the problem to be solved. In this chapter, we first present the governing equations of fluid dynamics and their nondimensionalization. Then, we describe their transformation to generalized curvilinear coordinates. And finally, we close this chapter by presenting the governing equations for the case of an incompressible viscous flow.

3.1 Navier-Stokes System of Equations

The equations governing the motion of a fluid can be derived from the statements of the conservation of mass, momentum, and energy [5]. In the most general form, the fluid motion is governed by the time-dependent three-dimensional compressible Navier-Stokes system of equations. For a viscous Newtonian, isotropic fluid in the absence of external forces, mass diffusion, finite-rate chemical reactions, and external heat addition, the strong conservation form of the Navier-Stokes system of equations in compact differential form can be written as

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) &= -\nabla p + \nabla \cdot \tau \\
\frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) &= k \nabla \cdot \nabla T - \nabla p \cdot \mathbf{u} + (\nabla \cdot \tau) \cdot \mathbf{u}
\end{aligned}
$$

This set of equations can be rewritten in vector form as follows

$$
\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{E}_i}{\partial x} + \frac{\partial \mathbf{F}_i}{\partial y} + \frac{\partial \mathbf{G}_i}{\partial z} = \frac{\partial \mathbf{E}_v}{\partial x} + \frac{\partial \mathbf{F}_v}{\partial y} + \frac{\partial \mathbf{G}_v}{\partial z} \tag{3.1}
$$

where $\mathbf{Q}$ is the vector of the conserved flow variables given by

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix} \tag{3.2}$$

and $\mathbf{E}_i = \mathbf{E}_i(\mathbf{Q})$, $\mathbf{F}_i = \mathbf{F}_i(\mathbf{Q})$ and $\mathbf{G}_i = \mathbf{G}_i(\mathbf{Q})$ are the vectors containing the inviscid fluxes in the x, y and z directions and are given by

$$\mathbf{E}_i = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (\rho e_t + p)\, u \end{bmatrix}, \qquad \mathbf{F}_i = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vw \\ (\rho e_t + p)\, v \end{bmatrix}, \qquad \mathbf{G}_i = \begin{bmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ (\rho e_t + p)\, w \end{bmatrix} \tag{3.3}$$

where $\mathbf{u}$ is the velocity vector containing the u, v and w velocity components in the x, y and z directions and p, ρ and e_t are the pressure, density and total energy per unit mass respectively.

The vectors $\mathbf{E}_v = \mathbf{E}_v(\mathbf{Q})$, $\mathbf{F}_v = \mathbf{F}_v(\mathbf{Q})$ and $\mathbf{G}_v = \mathbf{G}_v(\mathbf{Q})$ contain the viscous fluxes in the x, y and z directions and are defined as follows

$$\begin{aligned} \mathbf{E}_v &= \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x \end{bmatrix} \\ \mathbf{F}_v &= \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - q_y \end{bmatrix} \\ \mathbf{G}_v &= \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - q_z \end{bmatrix} \end{aligned} \tag{3.4}$$

where the heat fluxes q_x, q_y and q_z are given by the Fourier's law of heat conduction as follows

$$\begin{aligned} q_x &= -k\,\frac{\partial T}{\partial x} \\ q_y &= -k\,\frac{\partial T}{\partial y} \\ q_z &= -k\,\frac{\partial T}{\partial z} \end{aligned} \tag{3.5}$$

and the viscous stresses τ_{xx}, τ_{yy}, τ_{zz}, τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz} and τ_{zy}, are given by the following relationships

$$
\begin{aligned}
\tau_{xx} &= \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)\\
\tau_{yy} &= \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)\\
\tau_{zz} &= \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\\
\tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\\
\tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\\
\tau_{yz} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\\
\tau_{yx} &= \tau_{xy}\\
\tau_{zx} &= \tau_{xz}\\
\tau_{zy} &= \tau_{yz}
\end{aligned}
\tag{3.6}
$$

where μ is the laminar viscosity.

Examining closely equations eq. 3.1, eq. 3.2, eq. 3.3 and eq. 3.4 and counting the number of equations and unknowns, we clearly see that we have five equations in terms of seven unknown flow field variables u, v, w, ρ, p, T, and e_t. It is obvious that two additional equations are required to close the system. These two additional equations can be obtained by determining relationships that exist between the thermodynamic variables (p, ρ, T, e_i) through the assumption of thermodynamic equilibrium. Relations of this type are known as equations of state, and they provide a mathematical relationship between two or more state functions (thermodynamic variables). Choosing the specific internal energy e_i and the density ρ as the two independent thermodynamic variables, then equations of state of the form

$$
p = p\left(e_i, \rho\right), \qquad T = T\left(e_i, \rho\right) \tag{3.7}
$$

are required.

For most problems in aerodynamics and gasdynamics, it is generally reasonable to assume that the gas behaves as a perfect gas (a perfect gas is defined as a gas whose intermolecular forces are negligible), *i.e.*,

$$
p = \rho R_g T \tag{3.8}
$$

where R_g is the specific gas constant and is equal to $287\frac{m^2}{s^2K}$ for air. Assuming also that the working gas behaves as a calorically perfect gas (a calorically perfect gas is defined as a perfect gas with constant specific heats), then the following relations hold

$$
e_i = c_v T, \qquad h = c_p T, \qquad \gamma = \frac{c_p}{c_v}, \qquad c_v = \frac{R_g}{\gamma - 1}, \qquad c_p = \frac{\gamma R_g}{\gamma - 1} \tag{3.9}
$$

where γ is the ratio of specific heats and is equal to 1.4 for air, c_v the specific heat at constant volume, c_p the specific heat at constant pressure and h is the enthalpy. By using eq. 3.8 and eq. 3.9, we obtain the following relations for pressure p and temperature T in the form of eq. 3.7

$$p = (\gamma - 1)\,\rho e_i, \qquad T = \frac{p}{\rho R_g} = \frac{(\gamma - 1)\,e_i}{R_g} \tag{3.10}$$

where the specific internal energy per unit mass $e_i = p/(\gamma - 1)\rho$ is related to the total energy per unit mass e_t by the following relationship,

$$e_t = e_i + \frac{1}{2}\left(u^2 + v^2 + w^2\right) \tag{3.11}$$

In our discussion, it is also necessary to relate the transport properties (μ, k) to the thermodynamic variables. Then, the laminar viscosity μ is computed using Sutherland's formula

$$\mu = \frac{C_1 T^{\frac{3}{2}}}{(T + C_2)} \tag{3.12}$$

where for the case of the air, the constants are $C_1 = 1.458 \times 10^{-6} \frac{kg}{ms\sqrt{K}}$ and $C_2 = 110.4K$.

The thermal conductivity, k, of the fluid is determined from the Prandtl number ($Pr = 0.72$ for air) which in general is assumed to be constant and is equal to

$$k = \frac{c_p \mu}{Pr} \tag{3.13}$$

where c_p and μ are given by equations eq. 3.9 and eq. 3.12 respectively.

The first row in eq. 3.1 corresponds to the continuity equation. Likewise, the second, third and fourth rows are the momentum equations, while the fifth row is the energy equation in terms of total energy per unit mass.

The Navier-Stokes system of equations eq. 3.1, eq. 3.2, eq. 3.3 and eq. 3.4, is a coupled system of nonlinear partial differential equations (PDE), and hence is very difficult to solve analytically. There is no general closed-form solution to this system of equations; hence we look for an approximate solution of this system of equation in a given domain $\mathcal{D}$ with prescribed boundary conditions $\partial\mathcal{D}$ and given initial conditions $\mathcal{D}\mathring{\mathbf{U}}$.

If in eq. 3.1 we set the viscous fluxes $\mathbf{E}_v = 0$, $\mathbf{F}_v = 0$ and $\mathbf{G}_v = 0$, we get the Euler system of equations, which governs inviscid fluid flow. The Euler system of equations is a set of hyperbolic equations while the Navier-Stokes system of equations is a mixed set of hyperbolic (in the inviscid region) and parabolic (in the viscous region) equations. Therefore, time marching algorithms are used to advance the solution in time using discrete time steps.

3.2 Nondimensionalization of the Governing Equations

The governing fluid dynamic equations shown previously may be nondimensionalized to achieve certain objectives. The advantage in doing this is that, firstly, it will provide conditions upon which dynamic and energetic similarity may be obtained for geometrically similar situations. Secondly, by nondimensionalizing the equations appropriately, the flow variables are normalized so that their values fall between certain prescribed limits such as zero and one. Thirdly, the procedure of nondimensionalization, also allows the solution to be independent of any system of units and helps to reduce the sensitivity of the numerical algorithm to round-off-errors. And finally, by nondimensionalizing the governing equations, characteristic parameters such as Mach number, Reynolds number and Prandtl number can be varied independently. Among many choices, in external flow aerodynamics it is reasonable to normalize with respect to the freestream parameters so that

$$\begin{aligned} \tilde{x} &= \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \tilde{z} = \frac{z}{L} \\ \tilde{u} &= \frac{u}{U_\infty}, \quad \tilde{v} = \frac{v}{U_\infty}, \quad \tilde{w} = \frac{w}{U_\infty} \\ \tilde{\rho} &= \frac{\rho}{\rho_\infty}, \quad \tilde{T} = \frac{T}{T_\infty}, \quad \tilde{p} = \frac{p}{\rho_\infty U_\infty^2} \\ \tilde{t} &= \frac{tU_\infty}{L}, \quad \tilde{e_t} = \frac{e_t}{U_\infty^2}, \quad \tilde{\mu} = \frac{\mu}{\mu_\infty} \end{aligned} \tag{3.14}$$

where $\tilde{}$ denotes nondimensional quantities, the subscript $_\infty$ denotes freestream conditions, L is some dimensional reference length (such as the chord of an airfoil or the length of a vehicle), and U_∞ is the magnitude of the freestream velocity. The reference length L is used in defining the nondimensional Reynold's number, this parameter represents the ratio of inertia forces to viscous forces, and is given by

$$Re_L = \frac{\rho_\infty U_\infty L}{\mu_\infty} \tag{3.15}$$

where the freestream laminar viscosity μ_∞ is computed using the freestream temperature T_∞ according to eq. 3.12.

When dealing with high speed compressible flow, it is also useful to introduce the Mach number. The Mach number is a nondimensional parameter that measures the speed of the gas motion in relation to the speed of sound a,

$$a = \left[\left(\frac{\partial p}{\partial \rho}\right)_s\right]^{\frac{1}{2}} = \sqrt{\gamma\frac{p}{\rho}} = \sqrt{\gamma R_g T} \tag{3.16}$$

Then the Mach number M_∞ is given by,

$$M_\infty = \frac{U_\infty}{a} = \frac{U_\infty}{\sqrt{\gamma\,(p/\rho)}} = \frac{U_\infty}{\sqrt{\gamma R_g T}} \tag{3.17}$$

Finally, the remaining nondimensional quantities are defined as follows

$$\begin{aligned}\tilde{R}_g &= \frac{R_g}{U_\infty^2/T_\infty} = \frac{1}{\gamma M_\infty^2} \\ \tilde{c}_p &= \frac{1}{(\gamma-1)\,M_\infty^2} \\ \tilde{C}_1 &= C_1\frac{T_\infty^{1/2}}{\mu_\infty} \\ \tilde{C}_2 &= \frac{C_2}{T_\infty}\end{aligned} \tag{3.18}$$

Now, by simple replacing into the governing equations eq. 3.1 the dimensional quantities by their corresponding nondimensional equivalent, the following nondimensional equations are obtained

$$\frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{t}} + \frac{\partial \tilde{\mathbf{E}}_{\boldsymbol{i}}}{\partial \tilde{x}} + \frac{\partial \tilde{\mathbf{F}}_{\boldsymbol{i}}}{\partial \tilde{y}} + \frac{\partial \tilde{\mathbf{G}}_{\boldsymbol{i}}}{\partial \tilde{z}} = \frac{\partial \tilde{\mathbf{E}}_{\boldsymbol{v}}}{\partial \tilde{x}} + \frac{\partial \tilde{\mathbf{F}}_{\boldsymbol{v}}}{\partial \tilde{y}} + \frac{\partial \tilde{\mathbf{G}}_{\boldsymbol{v}}}{\partial \tilde{z}} \tag{3.19}$$

where $\tilde{\mathbf{Q}}$ is the vector of the nondimensional conserved flow variables given by

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \tilde{\rho} \\ \tilde{\rho}\tilde{u} \\ \tilde{\rho}\tilde{v} \\ \tilde{\rho}\tilde{w} \\ \tilde{\rho}\tilde{e}_t \end{bmatrix} \tag{3.20}$$

and $\tilde{\mathbf{E}}_{\boldsymbol{i}} = \tilde{\mathbf{E}}_{\boldsymbol{i}}(\tilde{\mathbf{Q}})$, $\tilde{\mathbf{F}}_{\boldsymbol{i}} = \tilde{\mathbf{F}}_{\boldsymbol{i}}(\tilde{\mathbf{Q}})$ and $\tilde{\mathbf{G}}_{\boldsymbol{i}} = \tilde{\mathbf{G}}_{\boldsymbol{i}}(\tilde{\mathbf{Q}})$ are the vectors containing the nondimensional inviscid fluxes in the $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ directions and are given by

$$\tilde{\mathbf{E}}_{\boldsymbol{i}} = \begin{bmatrix} \tilde{\rho}\tilde{u} \\ \tilde{\rho}\tilde{u}^2 + \tilde{p} \\ \tilde{\rho}\tilde{u}\tilde{v} \\ \tilde{\rho}\tilde{u}\tilde{w} \\ (\tilde{\rho}\tilde{e}_t + \tilde{p})\,\tilde{u}, \end{bmatrix}, \qquad \tilde{\mathbf{F}}_{\boldsymbol{i}} = \begin{bmatrix} \tilde{\rho}\tilde{v} \\ \tilde{\rho}\tilde{v}\tilde{u} \\ \tilde{\rho}\tilde{v}^2 + \tilde{p} \\ \tilde{\rho}\tilde{v}\tilde{w} \\ (\tilde{\rho}\tilde{e}_t + \tilde{p})\,\tilde{v}, \end{bmatrix}, \qquad \tilde{\mathbf{G}}_{\boldsymbol{i}} = \begin{bmatrix} \tilde{\rho}\tilde{w} \\ \tilde{\rho}\tilde{w}\tilde{u} \\ \tilde{\rho}\tilde{w}\tilde{v} \\ \tilde{\rho}\tilde{w}^2 + \tilde{p} \\ (\tilde{\rho}\tilde{e}_t + \tilde{p})\,\tilde{w} \end{bmatrix} \tag{3.21}$$

and $\tilde{\mathbf{E}}_{\boldsymbol{v}} = \tilde{\mathbf{E}}_{\boldsymbol{v}}(\tilde{\mathbf{Q}})$, $\tilde{\mathbf{F}}_{\boldsymbol{v}} = \tilde{\mathbf{F}}_{\boldsymbol{v}}(\tilde{\mathbf{Q}})$ and $\tilde{\mathbf{G}}_{\boldsymbol{v}} = \tilde{\mathbf{G}}_{\boldsymbol{v}}(\tilde{\mathbf{Q}})$ are the vectors containing the nondimensional viscous fluxes in the $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ directions and are given by

$$
\begin{aligned}
\tilde{\mathbf{E}}_v &= \begin{bmatrix} 0 \\ \tilde{\tau}_{xx} \\ \tilde{\tau}_{xy} \\ \tilde{\tau}_{xz} \\ \tilde{u}\tilde{\tau}_{xx} + \tilde{v}\tilde{\tau}_{xy} + \tilde{w}\tilde{\tau}_{xz} - \tilde{q}_x \end{bmatrix} \\
\tilde{\mathbf{F}}_v &= \begin{bmatrix} 0 \\ \tilde{\tau}_{yx} \\ \tilde{\tau}_{yy} \\ \tilde{\tau}_{yz} \\ \tilde{u}\tilde{\tau}_{yx} + \tilde{v}\tilde{\tau}_{yy} + \tilde{w}\tilde{\tau}_{yz} - \tilde{q}_y \end{bmatrix} \\
\tilde{\mathbf{G}}_v &= \begin{bmatrix} 0 \\ \tilde{\tau}_{zx} \\ \tilde{\tau}_{zy} \\ \tilde{\tau}_{zz} \\ \tilde{u}\tilde{\tau}_{zx} + \tilde{v}\tilde{\tau}_{zy} + \tilde{w}\tilde{\tau}_{zz} - \tilde{q}_z \end{bmatrix}
\end{aligned}
\tag{3.22}
$$

However, in the process of nondimensionalizing the equations, the terms M_∞ and Re_L arises from the nondimensional viscous flux vectors. Therefore, the definition of the heat flux components and the viscous stresses may be modified as follows

$$
\begin{aligned}
\tilde{q}_x &= -\frac{\tilde{\mu}}{(\gamma-1)\, M_\infty^2 Re_L Pr} \frac{\partial \tilde{T}}{\partial \tilde{x}} \\
\tilde{q}_y &= -\frac{\tilde{\mu}}{(\gamma-1)\, M_\infty^2 Re_L Pr} \frac{\partial \tilde{T}}{\partial \tilde{y}} \\
\tilde{q}_z &= -\frac{\tilde{\mu}}{(\gamma-1)\, M_\infty^2 Re_L Pr} \frac{\partial \tilde{T}}{\partial \tilde{z}}
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
\tilde{\tau}_{xx} &= \frac{2}{3}\frac{\tilde{\mu}}{Re_L}\left(2\frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\partial \tilde{w}}{\partial \tilde{z}}\right)\\
\tilde{\tau}_{yy} &= \frac{2}{3}\frac{\tilde{\mu}}{Re_L}\left(2\frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{w}}{\partial \tilde{z}}\right)\\
\tilde{\tau}_{zz} &= \frac{2}{3}\frac{\tilde{\mu}}{Re_L}\left(2\frac{\partial \tilde{w}}{\partial \tilde{z}} - \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{v}}{\partial \tilde{y}}\right)\\
\tilde{\tau}_{xy} &= \frac{\tilde{\mu}}{Re_L}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{x}}\right)\\
\tilde{\tau}_{xz} &= \frac{\tilde{\mu}}{Re_L}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\right)\\
\tilde{\tau}_{yz} &= \frac{\tilde{\mu}}{Re_L}\left(\frac{\partial \tilde{v}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{y}}\right)\\
\tilde{\tau}_{yx} &= \tilde{\tau}_{xy}\\
\tilde{\tau}_{zx} &= \tilde{\tau}_{xz}\\
\tilde{\tau}_{zy} &= \tilde{\tau}_{yz}
\end{aligned}
\tag{3.24}
$$

Finally, by nondimensionalizing the equations of state eq. 3.10, we obtain

$$
\tilde{p} = (\gamma - 1)\,\tilde{\rho}\tilde{e}_i, \qquad \tilde{T} = \frac{\tilde{p}}{\tilde{\rho}\tilde{R}_g} = \frac{(\gamma - 1)\,\tilde{e}_i}{\tilde{R}_g} \tag{3.25}
$$

where the nondimensional specific internal energy per unit mass $\tilde{e}_i = \tilde{p}/(\gamma - 1)\tilde{\rho}$ is related to the nondimensional total energy per unit mass $\tilde{e}_t$ by the following relationship,

$$
\tilde{e}_t = \tilde{e}_i + \frac{1}{2}\left(\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2\right) \tag{3.26}
$$

Note that the nondimensional form of the equations given by eq. 3.19, eq. 3.20, eq. 3.21 and eq. 3.22 are identical (except for the $\tilde{}$) to the dimensional form given by equations eq. 3.1, eq. 3.2, eq. 3.3 and eq. 3.4. For the sake of simplicity, the notation $\tilde{}$ will be dropped for the remainder of this dissertation. Thus, all the equations will be given in nondimensional form unless otherwise specified.

3.3 Transformation of the Governing Equations to Generalized Curvilinear Coordinates

The Navier-Stokes system of equation (eq. 3.1, eq. 3.2, eq. 3.3 and eq. 3.4) are valid for any coordinate system. We have previously expressed these equations in terms of a Cartesian coordinate system. For many applications it is more convenient to use a generalized curvilinear coordinate system. The use of generalized curvilinear coordinates implies that a distorted region in physical space is mapped into a rectangular region in the generalized curvilinear coordinate space (figure 3.1). Often, the transformation is chosen so that the discretized equations are solved in a uniform logically rectangular domain for 2$\mathbb{D}$ applications and an equivalent uniform logically hexahedral domain for 3$\mathbb{D}$ applications. The transformation shall be such that there is a one-to-one correspondence of the grid points from the physical space (Cartesian coordinates) to computational space (generalized curvilinear coordinates).

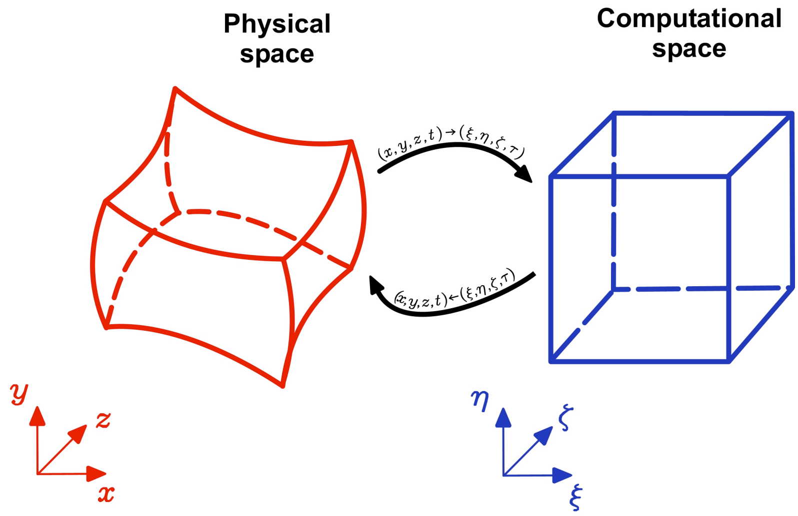

Figure 3.1: *Correspondence between the physical space (Cartesian coordinates) and the computational space (generalized curvilinear coordinates).*

Hereafter, we will describe the general transformation of the nondimensional Navier-Stokes system of equations (eq. 3.19, eq. 3.20, eq. 3.21 and eq. 3.22) given in the previous section between the physical space (Cartesian coordinates) and the computational space (generalized curvilinear coordinates). The governing equations are written in strong conservation form and expressed in terms of the generalized curvilinear coordinates as independent variables, thus the computations are performed in the generalized curvilinear coordinate space.

The governing equations of fluid dynamics are transformed from the physical space $\mathcal{P} = \mathcal{P}(x, y, z, t)$ to the computational space $\mathcal{C} = \mathcal{C}(\xi, \eta, \zeta, \tau)$ by using the following transformations

$$\begin{aligned} \tau &= \tau\left(t\right) = t \\ \xi &= \xi\left(x, y, z, t\right) \\ \eta &= \eta\left(x, y, z, t\right) \\ \zeta &= \zeta\left(x, y, z, t\right) \end{aligned} \tag{3.27}$$

where τ is considered to be equal to t and thus the transformation with respect to time is simple defined as $\tau = t$ as shown in eq. 3.27.

Applying the chain rule, the partial derivatives of any quantity $\phi = \phi(x, y, z, t)$ with respect to the Cartesian coordinates can be written as

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{\partial \phi}{\partial \tau} + \xi_t \frac{\partial \phi}{\partial \xi} + \eta_t \frac{\partial \phi}{\partial \eta} + \zeta_t \frac{\partial \phi}{\partial \zeta} \\
\frac{\partial \phi}{\partial x} &= \xi_x \frac{\partial \phi}{\partial \xi} + \eta_x \frac{\partial \phi}{\partial \eta} + \zeta_x \frac{\partial \phi}{\partial \zeta} \\
\frac{\partial \phi}{\partial y} &= \xi_y \frac{\partial \phi}{\partial \xi} + \eta_y \frac{\partial \phi}{\partial \eta} + \zeta_y \frac{\partial \phi}{\partial \zeta} \\
\frac{\partial \phi}{\partial z} &= \xi_z \frac{\partial \phi}{\partial \xi} + \eta_z \frac{\partial \phi}{\partial \eta} + \zeta_z \frac{\partial \phi}{\partial \zeta}
\end{aligned}
\tag{3.28}
$$

Then the governing equations may be transformed from physical space $\mathcal{P}$ to computational space $\mathcal{C}$ by replacing the Cartesian derivatives by the partial derivatives given in eq. 3.28, where the terms $\xi_x, \eta_x, \zeta_x, \xi_y, \eta_y, \zeta_y, \xi_z, \eta_z, \zeta_z, \xi_t, \eta_t$ and ζ_t are called metrics (they represents the ratio of arc lengths in the computational space $\mathcal{C}$ to that of the physical space $\mathcal{P}$) and where ξ_x represents the partial derivative of ξ with respect to x, *i.e.* $\partial\xi/\partial x$, and so forth.

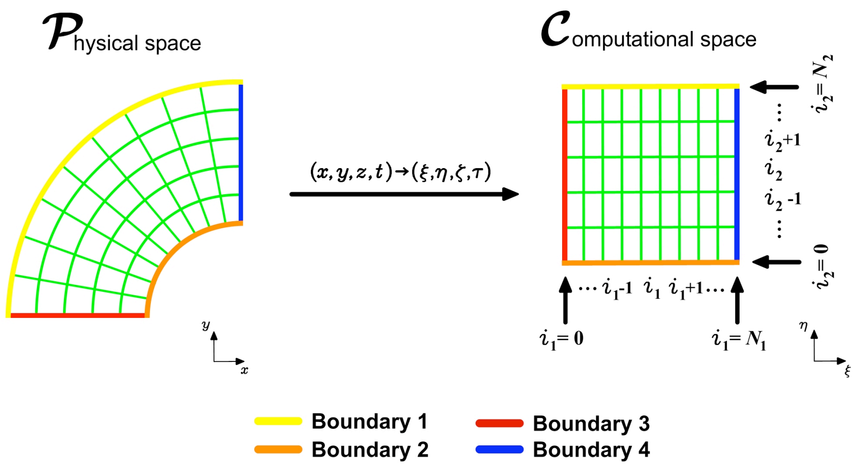

Figure 3.2: *Transformation from physical space to computational space. Left: structured grid in physical space. Right: logically uniform grid in computational space.*

In most cases, the transformation eq. 3.27 from physical space $\mathcal{P}$ to computational space $\mathcal{C}$ is not known analytically, rather it is generated numerically by a grid generation scheme. That is, we usually are provided with just the x, y and z coordinates of the grid points and we numerically generate the metrics using finite differences. The metrics $\xi_x, \eta_x, \zeta_x, \xi_y, \eta_y, \zeta_y, \xi_z, \eta_z, \zeta_z, \xi_t, \eta_t$ and ζ_t appearing in eq. 3.28 can be determined in the following manner. First, we write down the differential expressions of the inverse of the transformation eq. 3.27,

$$
\begin{aligned}
dt &= t_\tau d\tau + t_\xi d\xi + t_\eta d\eta + t_\zeta d\zeta \\
dx &= x_\tau d\tau + x_\xi d\xi + x_\eta d\eta + x_\zeta d\zeta \\
dy &= y_\tau d\tau + y_\xi d\xi + y_\eta d\eta + y_\zeta d\zeta \\
dz &= z_\tau d\tau + z_\xi d\xi + z_\eta d\eta + z_\zeta d\zeta
\end{aligned}
\tag{3.29}
$$

where the inverse of the transformation eq. 3.27 is

$$
\begin{aligned}
t &= t(\tau) = \tau \\
x &= x(\xi, \eta, \zeta, \tau) \\
y &= y(\xi, \eta, \zeta, \tau) \\
z &= z(\xi, \eta, \zeta, \tau)
\end{aligned} \tag{3.30}
$$

and recalling that for a grid that is not changing (moving, adapting or deforming)

$$
\begin{aligned}
&\frac{\partial t}{\partial \tau} = 1 \quad \text{and} \\
&\frac{\partial t}{\partial \xi} = \frac{\partial t}{\partial \eta} = \frac{\partial t}{\partial \zeta} = 0 \quad \text{thus} \\
&dt = d\tau
\end{aligned}
$$

Expressing eq. 3.29 in matrix form, we obtain

$$
\begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_\tau & x_\xi & x_\eta & x_\zeta \\ y_\tau & y_\xi & y_\eta & y_\zeta \\ z_\tau & z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} d\tau \\ d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{3.31}
$$

In a like manner, we proceed with the transformation eq. 3.27, and we obtain the following differential expressions

$$
\begin{aligned}
d\tau &= dt \\
d\xi &= \xi_t dt + \xi_x dx + \xi_y dy + \xi_z dz \\
d\eta &= \eta_t dt + \eta_x dx + \eta_y dy + \eta_z dz \\
d\zeta &= \zeta_t dt + \zeta_x dx + \zeta_y dy + \zeta_z dz
\end{aligned} \tag{3.32}
$$

which can be written in matrix form as

$$
\begin{bmatrix} d\tau \\ d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \xi_t & \xi_x & \xi_y & \xi_z \\ \eta_t & \eta_x & \eta_y & \eta_z \\ \zeta_t & \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} \tag{3.33}
$$

By relating the differential expressions eq. 3.33 of the transformation eq. 3.27 to the differential expressions eq. 3.31 of the transformation eq. 3.30, so that the metrics

$$
\xi_x, \eta_x, \zeta_x, \xi_y, \eta_y, \zeta_y, \xi_z, \eta_z, \zeta_z, \xi_t, \eta_t, \zeta_t
$$

can be found, we conclude that

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ \xi_t & \xi_x & \xi_y & \xi_z \\ \eta_t & \eta_x & \eta_y & \eta_z \\ \zeta_t & \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_\tau & x_\xi & x_\eta & x_\zeta \\ y_\tau & y_\xi & y_\eta & y_\zeta \\ z_\tau & z_\xi & z_\eta & z_\zeta \end{bmatrix}^{-1} \tag{3.34}
$$

This yields the following metrics relationships

$$
\begin{aligned}
\xi_x &= J_x \left(y_\eta z_\zeta - y_\zeta z_\eta \right) \\
\xi_y &= J_x \left(x_\zeta z_\eta - x_\eta z_\zeta \right) \\
\xi_z &= J_x \left(x_\eta y_\zeta - x_\zeta y_\eta \right) \\
\xi_t &= - \left(\tau_t x_\tau \xi_x + \tau_t y_\tau \xi_y + \tau_t z_\tau \xi_z \right) \\
\eta_x &= J_x \left(y_\zeta z_\xi - y_\xi z_\zeta \right) \\
\eta_y &= J_x \left(x_\xi z_\zeta - x_\zeta z_\xi \right) \\
\eta_z &= J_x \left(x_\zeta y_\xi - x_\xi y_\zeta \right) \\
\eta_t &= - \left(\tau_t x_\tau \eta_x + \tau_t y_\tau \eta_y + \tau_t z_\tau \eta_z \right) \\
\zeta_x &= J_x \left(y_\xi z_\eta - y_\eta z_\xi \right) \\
\zeta_y &= J_x \left(x_\eta z_\xi - x_\xi z_\eta \right) \\
\zeta_z &= J_x \left(x_\xi y_\eta - x_\eta y_\xi \right) \\
\zeta_t &= - \left(\tau_t x_\tau \zeta_x + \tau_t y_\tau \zeta_y + \tau_t z_\tau \zeta_z \right)
\end{aligned}
\tag{3.35}
$$

For ξ_t, η_t and ζ_t the following values are obtained after some manipulation

$$
\begin{aligned}
\xi_t &= J_x \left[x_\tau \left(y_\zeta z_\eta - y_\eta z_\zeta \right) + y_\tau \left(x_\eta z_\zeta - x_\zeta z_\eta \right) + z_\tau \left(x_\zeta y_\eta - x_\eta y_\zeta \right) \right] \\
\eta_t &= J_x \left[x_\tau \left(y_\xi z_\zeta - y_\zeta z_\xi \right) + y_\tau \left(x_\zeta z_\xi - x_\xi z_\zeta \right) + z_\tau \left(x_\xi y_\zeta - x_\zeta y_\xi \right) \right] \\
\zeta_t &= J_x \left[x_\tau \left(y_\eta z_\xi - y_\xi z_\eta \right) + y_\tau \left(x_\xi z_\eta - x_\eta z_\xi \right) + z_\tau \left(x_\eta y_\xi - x_\xi y_\eta \right) \right]
\end{aligned}
\tag{3.36}
$$

In eq. 3.35 and eq. 3.36, J_x is the determinant of the Jacobian matrix of the transformation defined by

$$
J_x = \left| \frac{\partial \left(\xi, \eta, \zeta \right)}{\partial \left(x, y, z \right)} \right|
$$

or

$$
J_x = \frac{1}{x_\xi \left(y_\eta z_\zeta - y_\zeta z_\eta \right) - x_\eta \left(y_\xi z_\zeta - y_\zeta z_\xi \right) + x_\zeta \left(y_\xi z_\eta - y_\eta z_\xi \right)}
\tag{3.37}
$$

which can be interpreted as the ratio of the areas (volumes in $3\mathbb{D}$) in the computational space $\mathcal{C}$ to that of the physical space $\mathcal{P}$.

Once relations for the metrics and for the Jacobian of the transformation are determined, the governing equations eq. 3.19 are then written in strong conservation form as

$$
\frac{\partial \hat{\mathbf{Q}}}{\partial t} + \frac{\partial \hat{\mathbf{E}}_{\boldsymbol{i}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}_{\boldsymbol{i}}}{\partial \eta} + \frac{\partial \hat{\mathbf{G}}_{\boldsymbol{i}}}{\partial \zeta} = \frac{\partial \hat{\mathbf{E}}_{\boldsymbol{v}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}_{\boldsymbol{v}}}{\partial \eta} + \frac{\partial \hat{\mathbf{G}}_{\boldsymbol{v}}}{\partial \zeta}
\tag{3.38}
$$

where

$$
\begin{aligned}
\hat{\mathbf{Q}} &= \frac{\mathbf{Q}}{J_x} \\
\hat{\mathbf{E}}_i &= \frac{1}{J_x}\left(\xi_t \mathbf{Q} + \xi_x \mathbf{E}_i + \xi_y \mathbf{F}_i + \xi_z \mathbf{G}_i\right) \\
\hat{\mathbf{F}}_i &= \frac{1}{J_x}\left(\eta_t \mathbf{Q} + \eta_x \mathbf{E}_i + \eta_y \mathbf{F}_i + \eta_z \mathbf{G}_i\right) \\
\hat{\mathbf{G}}_i &= \frac{1}{J_x}\left(\zeta_t \mathbf{Q} + \zeta_x \mathbf{E}_i + \zeta_y \mathbf{F}_i + \zeta_z \mathbf{G}_i\right) \\
\hat{\mathbf{E}}_v &= \frac{1}{J_x}\left(\xi_x \mathbf{E}_v + \xi_y \mathbf{F}_v + \xi_z \mathbf{G}_v\right) \\
\hat{\mathbf{E}}_v &= \frac{1}{J_x}\left(\eta_x \mathbf{E}_v + \eta_y \mathbf{F}_v + \eta_z \mathbf{G}_v\right) \\
\hat{\mathbf{E}}_v &= \frac{1}{J_x}\left(\zeta_x \mathbf{E}_v + \zeta_y \mathbf{F}_v + \zeta_z \mathbf{G}_v\right)
\end{aligned}
\tag{3.39}
$$

The viscous stresses given by eq. 3.24 in the transformed computational space are

$$
\begin{aligned}
\hat{\tau}_{xx} &= \frac{2}{3}\frac{\mu}{Re_L}[2\left(\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta\right) - \left(\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta\right)\dots \\
&\qquad \dots - \left(\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta\right)] \\
\hat{\tau}_{yy} &= \frac{2}{3}\frac{\mu}{Re_L}[2\left(\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta\right) - \left(\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta\right)\dots \\
&\qquad \dots - \left(\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta\right)] \\
\hat{\tau}_{zz} &= \frac{2}{3}\frac{\mu}{Re_L}[2\left(\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta\right) - \left(\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta\right)\dots \\
&\qquad \dots - \left(\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta\right)] \\
\hat{\tau}_{xy} &= \hat{\tau}_{yx} = \frac{\mu}{Re_L}\left(\xi_y u_\xi + \eta_y u_\eta + \zeta_y u_\zeta + \xi_x v_\xi + \eta_x v_\eta + \zeta_x v_\zeta\right) \\
\hat{\tau}_{xz} &= \hat{\tau}_{zx} = \frac{\mu}{Re_L}\left(\xi_z u_\xi + \eta_z u_\eta + \zeta_z u_\zeta + \xi_x w_\xi + \eta_x w_\eta + \zeta_x w_\zeta\right) \\
\hat{\tau}_{yz} &= \hat{\tau}_{zy} = \frac{\mu}{Re_L}\left(\xi_z v_\xi + \eta_z v_\eta + \zeta_z v_\zeta + \xi_y w_\xi + \eta_y w_\eta + \zeta_y w_\zeta\right)
\end{aligned}
\tag{3.40}
$$

and the heat flux components given by eq. 3.23 in the computational space are

$$
\begin{aligned}
\hat{q}_x &= -\frac{\mu}{(\gamma-1)\,M_\infty^2 Re_L Pr}\left(\xi_x T_\xi + \eta_x T_\eta + \zeta_x T_\zeta\right) \\
\hat{q}_y &= -\frac{\mu}{(\gamma-1)\,M_\infty^2 Re_L Pr}\left(\xi_y T_\xi + \eta_y T_\eta + \zeta_y T_\zeta\right) \\
\hat{q}_z &= -\frac{\mu}{(\gamma-1)\,M_\infty^2 Re_L Pr}\left(\xi_z T_\xi + \eta_z T_\eta + \zeta_z T_\zeta\right)
\end{aligned}
\tag{3.41}
$$

Equations eq. 3.38 and eq. 3.39 are the generic form of the governing equations written in strong conservation form in the transformed computational space $\mathcal{C}$ (see [14], [85] and [181] for a detailed derivation). The coordinate transformation presented in this section, follows the same development proposed by Viviand [202] and Vinokur [201], where they show that the governing equations of fluid dynamics can be put back into strong conservation form after a coordinate transformation has been applied.

Comparing the original governing equations eq. 3.19, eq. 3.20, eq. 3.21 and eq. 3.22 and the transformed equations eq. 3.38 and eq. 3.39, it is obvious that the transformed equations are more complicated than the original equations. Thus, a trade-off is introduced whereby advantages gained by using the generalized curvilinear coordinates are somehow counterbalanced by the resultant complexity of the equations. However, the advantages (such as the capability of using standard finite differences schemes and solving the equations in a uniform rectangular logically grid) by far outweigh the complexity of the transformed governing equations.

One final word of caution. The strong conservation form of the governing equations in the transformed computational space $\mathcal{C}$ is a convenient form for applying finite difference schemes. However, when using this form of the equations, extreme care must be exercised if the grid is changing (that is moving, adapting or deforming). In this case, a constraint on the way the metrics are differenced, called the geometric conservation law or GCL (see [50], [55] and [185]), must be satisfied in order to prevent additional errors from being introduced into the solution.

3.4 Simplification of the Navier-Stokes System of Equations: Incompressible Viscous Flow Case

Equations eq. 3.1, eq. 3.2, eq. 3.3 and eq. 3.4 with an appropriate equation of state and boundary and initial conditions, governs the unsteady three-dimensional motion of a viscous Newtonian, compressible fluid. In many applications the fluid density may be assumed to be constant. This is true not only for liquids, whose compressibility may be neglected, but also for gases if the Mach number is below 0.3 [6, 53]; such flows are said to be incompressible. If the flow is also isothermal, the viscosity is also constant. In this case, the dimensional governing equations in primitive variable formulation (u, v, w, p) and written in compact conservative differential form reduce to the following set

$$\nabla \cdot (\mathbf{u}) = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

where ν is the kinematic viscosity and is equal $\nu = \mu/\rho$. The same set of equations in nondimensional form is written as follows

$$\nabla \cdot (\mathbf{u}) = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p + \frac{1}{Re_L} \nabla^2 \mathbf{u}$$

which can be also written in nonconservative form (or advective/convective form [60])

$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re_L} \nabla^2 \mathbf{u}$$

or in expanded three-dimensional Cartesian coordinates

$$
\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0\\
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=-\frac{\partial p}{\partial x}+\frac{1}{Re_L}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=-\frac{\partial p}{\partial x}+\frac{1}{Re_L}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=-\frac{\partial p}{\partial x}+\frac{1}{Re_L}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}
\tag{3.42}
$$

This form (the advective/convective form), provides the simplest form for discretization and is widely used when implementing numerical methods for solving the incompressible Navier-Stokes equations, as noted by Gresho [60].

Equation eq. 3.42 governs the unsteady three-dimensional motion of a viscous, incompressible and isothermal flow. This simplification is generally not of a great value, as the equations are hardly any simpler to solve. However, the computing effort may be much smaller than for the full equations (due to the reduction of the unknowns and the fact that the energy equation is decoupled from the system of equation), which is a justification for such a simplification. The set of equations eq. 3.42 can be rewritten in vector form as follow

$$
\frac{\partial \mathbf{Q}}{\partial t}+\frac{\partial \mathbf{E}_i}{\partial x}+\frac{\partial \mathbf{F}_i}{\partial y}+\frac{\partial \mathbf{G}_i}{\partial z}=\frac{\partial \mathbf{E}_v}{\partial x}+\frac{\partial \mathbf{F}_v}{\partial y}+\frac{\partial \mathbf{G}_v}{\partial z}
\tag{3.43}
$$

where $\mathbf{Q}$ is the vector containing the primitive variables and is given by

$$
\mathbf{Q}=\begin{bmatrix}0\\u\\v\\w\end{bmatrix}
\tag{3.44}
$$

and $\mathbf{E}_i$, $\mathbf{F}_i$ and $\mathbf{G}_i$ are the vectors containing the inviscid fluxes in the x, y and z directions and are given by

$$
\mathbf{E}_i=\begin{bmatrix}u\\u^2+p\\uv\\uw\end{bmatrix},\qquad
\mathbf{F}_i=\begin{bmatrix}v\\vu\\v^2+p\\vw\end{bmatrix},\qquad
\mathbf{G}_i=\begin{bmatrix}w\\wu\\wv\\w^2+p\end{bmatrix}
\tag{3.45}
$$

The viscous fluxes in the x, y and z directions, $\mathbf{E}_v$, $\mathbf{F}_v$ and $\mathbf{G}_v$ respectively, are defined as follows

$$
\mathbf{E}_v=\begin{bmatrix}0\\\tau_{xx}\\\tau_{xy}\\\tau_{xz}\end{bmatrix},\qquad
\mathbf{F}_v=\begin{bmatrix}0\\\tau_{yx}\\\tau_{yy}\\\tau_{yz}\end{bmatrix},\qquad
\mathbf{G}_v=\begin{bmatrix}0\\\tau_{zx}\\\tau_{zy}\\\tau_{zz}\end{bmatrix}
\tag{3.46}
$$

Since we made the assumptions of an incompressible flow, appropriate nondimensional terms and expressions for shear stresses must be used, these expressions are given as follows

$$
\begin{aligned}
\tau_{xx} &= \frac{2}{Re_L}\frac{\partial u}{\partial x} \\
\tau_{yy} &= \frac{2}{Re_L}\frac{\partial v}{\partial y} \\
\tau_{zz} &= \frac{2}{Re_L}\frac{\partial w}{\partial z} \\
\tau_{xy} &= \frac{1}{Re_L}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
\tau_{xz} &= \frac{1}{Re_L}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\
\tau_{yz} &= \frac{1}{Re_L}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\tau_{yx} &= \tau_{xy} \\
\tau_{zx} &= \tau_{xz} \\
\tau_{zy} &= \tau_{yz}
\end{aligned}
\tag{3.47}
$$

Following the procedure presented in the previous section, the nondimensional incompressible Navier-Stokes system of equations eq. 3.43 in the computational space $\mathcal{C}$ is expressed as

$$
\frac{\partial \hat{\mathbf{Q}}}{\partial t} + \frac{\partial \hat{\mathbf{E}}_{\boldsymbol{i}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}_{\boldsymbol{i}}}{\partial \eta} + \frac{\partial \hat{\mathbf{G}}_{\boldsymbol{i}}}{\partial \zeta} = \frac{\partial \hat{\mathbf{E}}_{\boldsymbol{v}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}_{\boldsymbol{v}}}{\partial \eta} + \frac{\partial \hat{\mathbf{G}}_{\boldsymbol{v}}}{\partial \zeta} \tag{3.48}
$$

where

$$
\begin{aligned}
\hat{\mathbf{Q}} &= \frac{\mathbf{Q}}{J_x} \\
\hat{\mathbf{E}}_{\boldsymbol{i}} &= \frac{1}{J_x}\left(\xi_x \mathbf{E}_{\boldsymbol{i}} + \xi_y \mathbf{F}_{\boldsymbol{i}} + \xi_z \mathbf{G}_{\boldsymbol{i}}\right) \\
\hat{\mathbf{F}}_{\boldsymbol{i}} &= \frac{1}{J_x}\left(\eta_x \mathbf{E}_{\boldsymbol{i}} + \eta_y \mathbf{F}_{\boldsymbol{i}} + \eta_z \mathbf{G}_{\boldsymbol{i}}\right) \\
\hat{\mathbf{G}}_{\boldsymbol{i}} &= \frac{1}{J_x}\left(\zeta_x \mathbf{E}_{\boldsymbol{i}} + \zeta_y \mathbf{F}_{\boldsymbol{i}} + \zeta_z \mathbf{G}_{\boldsymbol{i}}\right) \\
\hat{\mathbf{E}}_{\boldsymbol{v}} &= \frac{1}{J_x}\left(\xi_x \mathbf{E}_{\boldsymbol{v}} + \xi_y \mathbf{F}_{\boldsymbol{v}} + \xi_z \mathbf{G}_{\boldsymbol{v}}\right) \\
\hat{\mathbf{F}}_{\boldsymbol{v}} &= \frac{1}{J_x}\left(\eta_x \mathbf{E}_{\boldsymbol{v}} + \eta_y \mathbf{F}_{\boldsymbol{v}} + \eta_z \mathbf{G}_{\boldsymbol{v}}\right) \\
\hat{\mathbf{G}}_{\boldsymbol{v}} &= \frac{1}{J_x}\left(\zeta_x \mathbf{E}_{\boldsymbol{v}} + \zeta_y \mathbf{F}_{\boldsymbol{v}} + \zeta_z \mathbf{G}_{\boldsymbol{v}}\right)
\end{aligned}
\tag{3.49}
$$

In eq. 3.49, $\hat{\mathbf{Q}}$ is the vector containing the primitive variables and $\hat{\mathbf{E}}_{\boldsymbol{i}}$, $\hat{\mathbf{F}}_{\boldsymbol{i}}$ and $\hat{\mathbf{G}}_{\boldsymbol{i}}$ are the vectors containing the inviscid fluxes in the ξ, η and ζ directions respectively, and are given by

$$\hat{\mathbf{Q}} = \frac{1}{J_x}\begin{bmatrix} 0 \\ u \\ v \\ w \end{bmatrix}, \qquad \hat{\mathbf{E}}_{\boldsymbol{i}} = \frac{1}{J_x}\begin{bmatrix} U \\ uU + p\xi_x \\ vU + p\xi_y \\ wU + p\xi_z \end{bmatrix},$$
$$\hat{\mathbf{F}}_{\boldsymbol{i}} = \frac{1}{J_x}\begin{bmatrix} V \\ uV + p\eta_x \\ vV + p\eta_y \\ wV + p\eta_z \end{bmatrix}, \qquad \hat{\mathbf{G}}_{\boldsymbol{i}} = \frac{1}{J_x}\begin{bmatrix} W \\ uW + p\zeta_x \\ vW + p\zeta_y \\ wW + p\zeta_z \end{bmatrix} \tag{3.50}$$

where U, V and W are the contravariant velocities

$$U = u\xi_x + v\xi_y + w\xi_z, \quad V = u\eta_x + v\eta_y + w\eta_z, \quad W = u\zeta_x + v\zeta_y + w\zeta_z$$

The shear stresses given by eq. 3.47 expressed in the computational space $\mathcal{C}$ are as follow

$$\begin{aligned}
\tau_{xx} &= \frac{2}{Re_L}\left(\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta\right) \\
\tau_{yy} &= \frac{2}{Re_L}\left(\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta\right) \\
\tau_{zz} &= \frac{2}{Re_L}\left(\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta\right) \\
\tau_{xy} &= \frac{1}{Re_L}\left(\xi_y u_\xi + \eta_y u_\eta + \zeta_y u_\zeta + \xi_x v_\xi + \eta_x v_\eta + \zeta_x v_\zeta\right) \\
\tau_{xz} &= \frac{1}{Re_L}\left(\xi_z u_\xi + \eta_z u_\eta + \zeta_z u_\zeta + \xi_x w_\xi + \eta_x w_\eta + \zeta_x w_\zeta\right) \\
\tau_{yz} &= \frac{1}{Re_L}\left(\xi_y w_\xi + \eta_y w_\eta + \zeta_y w_\zeta + \xi_z v_\xi + \eta_z v_\eta + \zeta_z v_\zeta\right) \\
\tau_{yx} &= \tau_{xy} \\
\tau_{zx} &= \tau_{xz} \\
\tau_{zy} &= \tau_{yz}
\end{aligned} \tag{3.51}$$

Substituting the expressions for the shear stresses given by eq. 3.51 into the viscous flux vectors $\hat{\mathbf{E}}_{\boldsymbol{v}}$, $\hat{\mathbf{F}}_{\boldsymbol{v}}$ and $\hat{\mathbf{G}}_{\boldsymbol{v}}$ (given by eq. 3.49) in the ξ, η and ζ directions respectively, we obtain the following equations

$$
\begin{aligned}
\hat{\mathbf{E}}_{\boldsymbol{v}} &= \frac{1}{J_x Re_L}\begin{bmatrix} 0 \\ a_1 u_\xi + b_1 u_\eta - c_1 v_\eta + c_2 w_\eta + b_2 u_\zeta - d_1 v_\zeta + d_2 w_\zeta \\ a_1 v_\xi + c_1 u_\eta + b_1 v_\eta - c_3 w_\eta + d_1 u_\zeta + b_2 v_\zeta - d_3 w_\zeta \\ a_1 w_\xi - c_2 u_\eta + c_3 v_\eta + b_1 w_\eta - d_2 u_\zeta + d_3 v_\zeta + b_2 w_\zeta \end{bmatrix} \\
\hat{\mathbf{F}}_{\boldsymbol{v}} &= \frac{1}{J_x Re_L}\begin{bmatrix} 0 \\ a_2 u_\eta + b_1 u_\xi + c_1 v_\xi - c_2 w_\xi + b_2 u_\zeta - e_1 v_\zeta + e_2 w_\zeta \\ a_2 v_\eta - c_1 u_\xi + b_1 v_\xi + c_3 w_\xi + e_1 u_\zeta + b_3 v_\zeta - e_3 w_\zeta \\ a_2 w_\eta + c_2 u_\xi - c_3 v_\xi + b_1 w_\xi - e_2 u_\zeta + e_3 v_\zeta + b_3 w_\zeta \end{bmatrix} \\
\hat{\mathbf{G}}_{\boldsymbol{v}} &= \frac{1}{J_x Re_L}\begin{bmatrix} 0 \\ a_3 u_\zeta + b_2 u_\xi + d_1 v_\xi - d_2 w_\xi + b_3 u_\eta + e_1 v_\eta - e_2 w_\eta \\ a_3 v_\zeta - c_4 u_\xi + b_2 v_\xi + d_3 w_\xi - e_1 u_\eta + b_3 v_\eta + e_3 w_\eta \\ a_3 w_\zeta + d_2 u_\xi - d_3 v_\xi + b_2 w_\xi + c_8 u_\eta - e_3 v_\eta + b_3 w_\eta \end{bmatrix}
\end{aligned}
\tag{3.52}
$$

where

$$
\begin{aligned}
& a_1 = \xi_x^2 + \xi_y^2 + \xi_z^2, \qquad a_2 = \eta_x^2 + \eta_y^2 + \eta_z^2, \qquad a_3 = \zeta_x^2 + \zeta_y^2 + \zeta_z^2, \\
& b_1 = \xi_x \eta_x + \xi_y \eta_y + \xi_z \eta_z, \qquad b_2 = \xi_x \zeta_x + \xi_y \zeta_y + \xi_z \zeta_z, \\
& b_3 = \zeta_x \eta_x + \zeta_y \eta_y + \zeta_z \eta_z, \\
& c_1 = \xi_x \eta_y - \eta_x \xi_y, \qquad c_2 = \eta_x \xi_z - \xi_x \eta_z, \qquad c_3 = \xi_y \eta_z - \eta_y \xi_z, \\
& d_1 = \xi_x \zeta_y - \zeta_x \xi_y, \qquad d_2 = \zeta_x \xi_z - \xi_x \zeta_z, \qquad d_3 = \xi_y \zeta_z - \zeta_y \xi_z, \\
& e_1 = \eta_x \zeta_y - \zeta_x \eta_y, \qquad e_2 = \zeta_x \eta_z - \eta_x \zeta_z, \qquad e_3 = \eta_y \zeta_z - \zeta_y \eta_z
\end{aligned}
\tag{3.53}
$$

equations eq. 3.52 and eq. 3.53 written in a more compact way, can be expressed as

$$
\begin{aligned}
\hat{\mathbf{E}}_{\boldsymbol{v}} &= \frac{1}{J_x Re_L}\begin{bmatrix} 0 \\ (\nabla\xi\cdot\nabla\xi)\, u_\xi + (\nabla\xi\cdot\nabla\eta)\, u_\eta + (\nabla\xi\cdot\nabla\zeta)\, u_\zeta \\ (\nabla\xi\cdot\nabla\xi)\, v_\xi + (\nabla\xi\cdot\nabla\eta)\, v_\eta + (\nabla\xi\cdot\nabla\zeta)\, v_\zeta \\ (\nabla\xi\cdot\nabla\xi)\, w_\xi + (\nabla\xi\cdot\nabla\eta)\, w_\eta + (\nabla\xi\cdot\nabla\zeta)\, w_\zeta \end{bmatrix} \\
\hat{\mathbf{F}}_{\boldsymbol{v}} &= \frac{1}{J_x Re_L}\begin{bmatrix} 0 \\ (\nabla\eta\cdot\nabla\xi)\, u_\xi + (\nabla\eta\cdot\nabla\eta)\, u_\eta + (\nabla\eta\cdot\nabla\zeta)\, u_\zeta \\ (\nabla\eta\cdot\nabla\xi)\, v_\xi + (\nabla\eta\cdot\nabla\eta)\, v_\eta + (\nabla\eta\cdot\nabla\zeta)\, v_\zeta \\ (\nabla\eta\cdot\nabla\xi)\, w_\xi + (\nabla\eta\cdot\nabla\eta)\, w_\eta + (\nabla\eta\cdot\nabla\zeta)\, w_\zeta \end{bmatrix} \\
\hat{\mathbf{G}}_{\boldsymbol{v}} &= \frac{1}{J_x Re_L}\begin{bmatrix} 0 \\ (\nabla\zeta\cdot\nabla\xi)\, u_\xi + (\nabla\zeta\cdot\nabla\eta)\, u_\eta + (\nabla\zeta\cdot\nabla\zeta)\, u_\zeta \\ (\nabla\zeta\cdot\nabla\xi)\, v_\xi + (\nabla\zeta\cdot\nabla\eta)\, v_\eta + (\nabla\zeta\cdot\nabla\zeta)\, v_\zeta \\ (\nabla\zeta\cdot\nabla\xi)\, w_\xi + (\nabla\zeta\cdot\nabla\eta)\, w_\eta + (\nabla\zeta\cdot\nabla\zeta)\, w_\zeta \end{bmatrix}
\end{aligned}
\tag{3.54}
$$

Equation eq. 3.48, together with eq. 3.49, eq. 3.50 and eq. 3.54, are the governing equations of an incompressible viscous flow written in strong conservation form in the transformed computational space $\mathcal{C}$. Hence, we look for an approximate solution of this set of equations in a given domain $\mathcal{D}$ with prescribed boundary conditions $\partial\mathcal{D}$ and given initial conditions $\mathcal{D}\mathring{\mathbf{U}}$. So far, we have just presented the governing equations; in the following chapters the grid generation method as well as the numerical scheme for solving the governing equations will be explained.