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Students projects in Optimization, Stability and Control

Work performed in the course
Advanced Fluid Dynamics, 60369, 2011/2012
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Faculty of Engineering
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Contents

CHAPTER 1

Introduction

This report contains the project papers written by the students who followed the 6 credit course *Advanced Fluid Dynamics*, Course ID 60369, SSD ING-ING/06, at the Faculty of Engineering at the University of Genoa 2011/2012. A brief outline of the course is given below.

Learning Outcomes

The course is intended to increase the knowledge about modern tools, such as sensitivity analysis, constrained optimization and nonmodal stability analysis, which are useful when analyzing linear systems which evolve in time. The different methods are not only applicable to fluid dynamics problems. A common ingredient to all methods is the efficient computation of sensitivities using so called adjoint equations. The mini projects introduced during the course give the students a chance to develop, test and document tools which might be useful for future studies.

Course Organisation Details

The course is roughly divided into three parts; sensitivity analysis, constrained optimization and nonmodal stability analysis. The different lectures include both a theoretical part and practical numerical examples in which the students put into practice what they learn. In order to facilitate the practical part regarding numerical examples, the initial lectures of the course comprise a short repetition regarding basic numerical analysis. At the beginning of the course the students choose, together with the lecturer, a topic related to the content of the course which they shall study both theoretically and numerically. This "mini" project shall be summarized in a report and finally presented at the end of the course. A sample document regarding the report style is handed out and discussed in the beginning of the course. A rough outline of the course is given below

- Short introduction about report writing, discussion about student projects
- Repetition of basic concepts in numerical analysis
- Part 1 : Gradient computations using adjoints & structural sensitivity
- Part 2 : Constrained optimization
- Part 3 : Nonmodal stability analysis
- Presentations of student projects

Organization and examinations

The course is based on a series of conventional lectures, and numerical examples in relation to respective lecture, for the students to set in practice what they learn. Each student does, for a topic chosen at the beginning of the course, prepare a report including an introduction, theoretical problem formulation and numerical calculations. This project is presented by the student at the end of the course and is part of the final mark. A written examination is given at two occasions during the course. The final mark is based on both the project and the two written exams. There is also the possibility to choose one written exam at the end of the course without doing the project. In this case the exam is more comprehensive.

References

Books

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Scientific articles

- Schmid, P.J., 2007, "Nonmodal stability theory", *Annu. Rev. Fluid Mech.*, **39**, p. 129-162
- Trefethen, L. N., Trefethen, A. E., Reddy, S. C. & Driscoll, T. A., 1993, "Hydrodynamic stability without eigenvalues", *Science*, **261**, p. 578-584
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Lecture notes

- Hand outs by J. Pralits
- Gunzburger, M.D., "Abstract description of an optimization problem".

CHAPTER 2

Summary of students projects

1. Optimal Boundary layer control

Students	Dario Barsi & Gianluca Ricci
Title	Minimization of boundary layer kinetic energy by means of lagrangian multipliers approach

Abstract

The aim of this work is to develop a control model able to minimize the kinetic energy of a viscous boundary layer bounded between a steady and a moving wall. The physic of the problem is based upon the one dimensional homogenous viscous Burgers equation, which represents the motion of the fluid. The proposed method to evaluate the upper moving wall velocity able to minimize the kinetic energy is based on the Lagrangian multipliers approach. At first the problem is solved in an approximate way, that is to say solving the linearized Burgers' Equation. Successively the problem is solved by employing the non-linear homogenous Burgers Equation. Numerical examples are presented to illustrate the effectiveness of the method.

2. Optimal control of pollutants dispersion

Student	Stefano Olivieri
Title	Optimal control of advection-diffusion equation

Abstract

Advection-diffusion equation is certainly a frequently studied PDE with important applications such as, e.g., modeling the dispersion of pollutants in air or water. Moreover, the goal could be not only to simulate dispersion processes but also to control emissions with an active device and maximum efficiency. The present paper concerns with the solution of such kind of issues, which involve an optimization problem with the following features: derivation of optimality system, numerical resolution of governing PDEs and definition of a suitable algorithm to implement the optimization. Each step of the general procedure is explained in sections 2 and 3, while some numerical tests are presented in section 4. In the present study the following assumptions are made: mono dimensional and unbounded domain, unsteady state, point source and point control.

3. Transient growth of the Ginzburg-Landau model

Student	Damiano Natali
Title	Optimal perturbation and stability analysis of a spatial developing flow

Abstract

Short-term instabilities play an important role in fluid dynamical stability theory, where the most common approach is dominated by the quest for the optimal initial conidition that results in the maximum amplification of itself over a finite time span. In the present paper, both the optimal perturbation and the non-modal stability theory is applied to the one-dimensional linearized Ginzburg-Landau model, which describes the evolution of a perturbation in a spatially developing flow.

4. Optimal control of aeroelasticity

Student	Paolo Bertocchi & Marco Ferrando
Title	Reduction of the instability in an aeroelasticity problem using an optimization method

Abstract

The aim of this work is to apply an optimization process, based on the Lagrange multipliers method, to reduce the oscillations and the instabilities of a wing due to aeroelasticity effects. The problem is governed by a linear dynamical system; an objective (or cost) function is defined, taking in account both the state of the system and the control term. The goal is to find the optimal control law in order to minimize the objective function in a specified time range, where the final time is a parameter of the problem.

5. Optimized fishing using a coastal population model

Student	Alessandro Cavuoto
Title	Optimization of fishing activity and repopulation in a simulated model

Abstract

Study of a model representing the growth of a coastal population under an external forcing. The work first concerned the determination of the state equation governing the problem and the definition of the variables and parameters required to deal the problem. Then has been done the analysis of the problem using the Lagrange multipliers method in order to obtain the fundamental equations to write down the models code. Once discretized the equations and defined the fundamentals matrices has been possible to implement the code and use it to simulate different dynamic situations of a coastal population growth, with and without an external forcing. It resulted that the optimization code enables to find the optimal fishing/repopulation vector which guarantee the survival of the species.

6. Optimized hydraulic conductivity of a poroelastic model

Student	Tobias Ansaldi
Title	Validation of a poroelastic model

Abstract

This studies aim, is to verify from a mathematical point of view the validity of a model for the infusion of a drug inside a cancer tissue. Avery important parameter has been optimised in the description of the physical phenomenon the hydraulic conductivity K. The optimal value has been determined with Lagrange's approach. The function K was considered as the optimal value that better represent experimental data. The results show how the optimised function has a reasonable tendency from a physical point of view and furthermore has a singular tendency to the one obtained in the model.

7. Optimal control in a mono-dimensional resonator

Student	Matteo Bargiacchi
Title	Optimal control of a non-homogeneous convective wave equation in a mono-dimensional resonator: a variational approach

Abstract

Low level of pollutants can be achieved by a lean and premixed burning. Unfortunately, these are the conditions causing the undesirable phenomenon of self-excited thermo-acoustic oscillations, responsible for inefficient burning and structural stresses so intense that they can lead to engine and combustor failure. The phenomena is well described by the non-homogeneous convective wave equation that, in its simplest application, could be written in a one dimensional space domain. The article wants to let the reader gain sensitivity on the effect of the heat released from a source located in bounded flow. A variational analysis will be performed to show the optimal time-dependence of the heat source in order to minimize the oscillations inside the resonator.

CHAPTER 3

Project reports

Minimization of Boundary Layer Kinetic Energy by means of Lagrangian Multipliers Approach

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SUMMARY

The aim of this work is to develop a control model able to minimize the kinetic energy of a viscous boundary layer bounded between a steady and a moving wall. The physic of the problem is based upon the one dimensional homogenous viscous Burger's equation, which represents the motion of the fluid. The proposed method to evaluate the upper moving wall velocity able to minimize the kinetic energy is based on the Lagrangian multipliers approach. At first the problem is solved in an approximate way, that is to say solving the linearized Burgers' Equation. Successively the problem is solved by employing the non-linear homogenous Burgers' Equation. Numerical examples are presented to illustrate the effectiveness of the method.

NOMENCLATURE

a, b, c, d	Lagrangian multipliers
A	Matrix of the direct problem
B	Matrix of the adjoint problem
C_1, C_2	Constants of integration
J	Cost function
I	Identity matrix
$\mathcal{L}$	Lagrangian operator
L	Distance between the two flat walls
p	Search direction
q	Weight function
S_L, S_R	Slope of the left and right segment in the smoothing procedure
t	Time
T	Time of simulation
FT	Forcing term
U	Velocity W at $x = L$
u	Control function
w	Solution of the linearized direct problem
$\overline{w}_0$	Initial solution for the direct problem
$\overline{w}$	Solution for the direct problem
W	Solution of the stationary Burger's Equation
x, y	Coordinate system

Greeks

α	Exponential parameter
Δx	Spatial interval
Δt	Time interval
ε	Small perturbation coefficient
ν	Kinematic viscosity of the fluid
σ	Step length

Apex

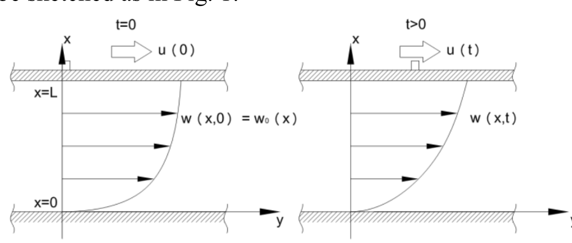
n	Time index
N	Upper bound of time index

Subscripts

i	Spatial index
M	Upper bound of spatial index

INTRODUCTION

We consider the problem of a boundary layer velocity profile bounded between two flat walls. The problem can be sketched as in Fig. 1:



(a) (b)

Figure 1 – Problem representation at initial condition (a) and at time t (b)

The lower wall is stuck, while the upper one is moving with a time dependent speed $u(t)$, which is assumed as the control function of the problem. The coordinate system x, y is setted as in Fig. 1, with the x direction bounded between 0 and L, and the y coordinate directed along the flat plane. The profile velocity is indicated with w, and it depends on both time and spatial coordinate x. The study of this kind of problem is motivated by flow control problems where the control action is located on the walls [1], [2], [3]. A similar approach, but based on the optimal feedback law derived from distributed parameters, has been developed in [4], while in this paper we will focus on the minimization of boundary layer kinetic energy through the Lagrangian multipliers approach [5].

MATHEMATICAL MODEL

In order to represent the motion inside boundary layer we consider the viscous Burgers' Equation:

$$\frac{\partial}{\partial t}\overline{w}(x,t) = \nu\frac{\partial^2}{\partial x^2}\overline{w}(x,t) - \frac{\partial}{\partial x}\frac{1}{2}\overline{w}(x,t)^2 \qquad 0<x<L \qquad 0<t\le T \tag{1}$$

with homogeneous boundary condition at $x = 0$

$$\overline{w}(x=0,t) = 0 \tag{2}$$

and the Dirichlét boundary control at $x = L$:

$$\overline{w}(x=L,t) = u(t) \tag{3}$$

The initial condition is given by:

$$\overline{w}(x,t=0) = \overline{w}_0(x) \tag{4}$$

The constant parameter ν represents the kinematic viscosity of the fluid. As a first approach, in order to simplify the problem solution and to increase the rate of convergence and stability, the homogeneous viscous Burgers' Equation has been linearized. The linearization is made by applying a small perturbation in the neighbourhood of the stationary solution of the following problem:

$$\nu\frac{\partial^2}{\partial x^2}W(x) - \frac{\partial}{\partial x}\frac{W^2(x)}{2} = 0 \qquad 0<x<L \tag{5}$$

With the boundary conditions:

$$W(x=0) = 0 \tag{6}$$

$$W(x=L) = U \tag{7}$$

The general solution of Eq.(5) is:

$$W(x) = \sqrt{2C_1}\tanh\left(-\frac{\sqrt{C_1}x}{\nu\sqrt{2}} + C_2\right) \tag{8}$$

By imposing boundary conditions of Eq.(6) and Eq.(7) the two constants C_1 and C_2 are:

$$\tanh\left(-\frac{\sqrt{C_1}}{\sqrt{2}\nu}\right)\sqrt{2C_1} - U = 0 \tag{9}$$

$$C_2 = 0 \tag{10}$$

If we consider that:

$$\overline{w}(x,t) = W(x) + \varepsilon w(x,t) \qquad \varepsilon \ll 1 \tag{11}$$

the linearization of problem (1) can be obtained as:

$$\frac{\partial}{\partial t}[W(x)+\varepsilon w(x,t)] = \nu\frac{\partial^2}{\partial x^2}[W(x)+\varepsilon w(x,t)] - \frac{\partial}{\partial x}\frac{1}{2}[W(x)+\varepsilon w(x,t)]^2 \tag{12}$$

And thus, neglecting the higher order terms, we obtain:

$$\frac{\partial}{\partial t}w(x,t) = \nu\frac{\partial^2}{\partial x^2}w(x,t) - \frac{\partial}{\partial x}[W(x)w(x,t)] \tag{13}$$

With the boundary conditions:

$$w(x=0,t) = 0 \tag{14}$$

$$w(x=L,t) = u(t) \tag{15}$$

And the following initial condition:

$$w(x,t=0) = w_0(x) \tag{16}$$

In order to minimize the kinetic energy of the boundary layer, the following cost function J is chosen [4]:

$$J = \int_0^T e^{\alpha t}\left[\int_0^L q(x)\,\overline{w}(x,t)^2 dx + u(t)^2\right]dt \tag{17}$$

That for the linearized equation becomes:

$$J = \int_0^T e^{\alpha t}\left[\int_0^L q(x)\,w(x,t)^2 dx + u(t)^2\right]dt \tag{18}$$

The constant α is a positive number, and q(x) is a user defined weight function. When α is a positive number, there is an additional performance requirement [6], [7], [8].

LAGRANGIAN APPROACH

If we consider Eq. (13), the Lagrangian approach leads to the expression:

$$\begin{aligned}\mathcal{L}(w,u,a,b,c,d) = \\ = J - \int_0^T\int_0^L a\left\{\frac{\partial}{\partial t}w(x,t) - \nu\frac{\partial^2}{\partial x^2}w(x,t) + \right. \\ \left. + \frac{\partial}{\partial x}[W(x)w(x,t)]\right\}dxdt + \\ - \int_0^L b[w(x,0) - w_0(x)]\,dx + \\ - \int_0^T c[w(0,t) - 0]\,dt + \\ - \int_0^T d[w(L,t) - u(t)]\,dt\end{aligned} \tag{19}$$

where a, b, c, d are the Lagrangian multipliers. Thus, setting the gradient of $\mathcal{L}$ equal to zero, we obtain the necessary conditions for the problem resolution:

■ $\dfrac{\partial\mathcal{L}}{\partial a} = 0$

$$\frac{\partial}{\partial t}w(x,t) = \nu\frac{\partial^2}{\partial x^2}w(x,t) - \frac{\partial}{\partial x}[W(x)w(x,t)] \tag{20}$$

■ $\dfrac{\partial\mathcal{L}}{\partial b} = 0$

$$w(x,0) = w_0(x) \tag{21}$$

■ $\dfrac{\partial\mathcal{L}}{\partial c} = 0$

$$w(0,t) = 0 \tag{22}$$

■ $\dfrac{\partial \mathcal{L}}{\partial d} = 0$

$$w(L,t) = u(t) \tag{23}$$

■ $\dfrac{\partial \mathcal{L}}{\partial w} = 0$

$$\begin{gathered}-\frac{\partial}{\partial t}a(x,t) - W(x)\frac{\partial}{\partial x}a(x,t) + \\ -\nu\frac{\partial^2}{\partial x^2}a(x,t) - e^{\alpha t}[2q(x)w(x,t)] = 0\end{gathered} \tag{24}$$

$$a(0,t) = 0 \tag{25}$$

$$a(L,t) = 0 \tag{26}$$

$$a(x,T) = 0 \tag{27}$$

$$b(x) = a(x,0) \tag{28}$$

$$c(t) = \nu\frac{\partial}{\partial x}a(x,t)\bigg|_{x=0} \tag{29}$$

$$d(t) = -\nu\frac{\partial}{\partial x}a(x,t)\bigg|_{x=L} \tag{30}$$

■ $\dfrac{\partial \mathcal{L}}{\partial u} = 0$

$$u(t) = \frac{\nu}{2e^{\alpha t}}\frac{\partial}{\partial x}a(x,t)\bigg|_{x=L} \tag{31}$$

As one can note:

- the derivative with respect to a leads to the definition of the direct problem;
- the ones with respect to b, c and d leads to the definition of the direct problem boundary and initial conditions respectively;
- the one with respect to w leads to the definition of the adjoint problem [9], of its boundary and initial conditions and of the Lagrangian operator b, c and d;
- the one with respect to u leads to the definition of the optimal condition for control $u(t)$.

Furthermore, the adjoint equation Eq.(24) has to be integrated backward in time, because of the negative sign of the time derivative. Thus, the "initial" condition of the adjoint problem is intended at time $\mathrm{t} = \mathrm{T}$.

DISCRETIZATION METHOD

In this paragraph the discretization methods for the analysis of both original and linearized model are presented. The discretization approach employes for both cases an implicit finite difference scheme.

Non linearized model discretization

Starting from Eq. (1), if we choose an implicit finite difference scheme, with first order approach in time and second order approach in space, we have for the direct problem:

$$\begin{gathered}\frac{\overline{w}_i^{n+1} - \overline{w}_i^n}{\Delta t} = \nu\frac{\overline{w}_{i+1}^{n+1} - 2\overline{w}_i^{n+1} + \overline{w}_{i-1}^{n+1}}{\Delta x^2} + \\ -\frac{\left(\overline{w}_{i+1}^{n+1}\right)^2 - \left(\overline{w}_{i-1}^{n+1}\right)^2}{4\Delta x}\end{gathered} \tag{32}$$

This scheme leads to the system:

$$\overline{w}_1^{n+1} - 0 = 0$$

$$\begin{gathered}\overline{w}_i^{n+1} - \frac{\Delta t\,\nu}{\Delta x^2}\left(\overline{w}_{i+1}^{n+1} - 2\overline{w}_i^{n+1} + \overline{w}_{i-1}^{n+1}\right) + \\ + \frac{\Delta t}{4\Delta x}\left[\left(\overline{w}_{i+1}^{n+1}\right)^2 - \left(\overline{w}_{i-1}^{n+1}\right)^2\right] - \overline{w}_i^n = 0 \\ i = 2, \dots M\end{gathered} \tag{33}$$

$$\overline{w}_{M+1}^{n+1} - u^{n+1} = 0$$

Where:

$$n = 1, \dots N$$

This non-linear system can be solved for example through a Levenberg-Marquard method [10].

Linearized model discretization

Starting from Eq.(20), if we choose, as for the non-linearized model, an implicit finite difference scheme, with first order approach in time and second order approach in space, we have for the direct problem:

$$\begin{gathered}\frac{w_i^{n+1} - w_i^n}{\Delta t} = \nu\frac{w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1}}{\Delta x^2} + \\ + \frac{W_{i+1}w_{i+1}^{n+1} - W_{i-1}w_{i-1}^{n+1}}{2\Delta x}\end{gathered} \tag{34}$$

This scheme leads to the system:

$$\frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} + \underline{\underline{A}}\underline{w}^{n+1} - \underline{FT} = \underline{0} \tag{35}$$

where:

$$\underline{FT} = \left[w_1^{n+1}\left(\frac{\nu}{\Delta x^2} + \frac{W_1}{2\Delta x}\right) \quad 0 \dots 0 \quad w_{M+1}^{n+1}\left(\frac{\nu}{\Delta x^2} + \frac{W_{M+1}}{2\Delta x}\right)\right]^T$$

and the matrix $\underline{\underline{A}}$ is a three-diagonal matrix with the elements:

$$A_{i,j} = \frac{2\nu}{\Delta x^2} \qquad if\ i = j$$

$$A_{i,j} = -\frac{\nu}{\Delta x^2} - \frac{W_i}{2\Delta x} \qquad if\ i = j + 1$$

$$A_{i,j} = -\frac{\nu}{\Delta x^2} + \frac{W_{i+2}}{2\Delta x} \qquad if\ i = j - 1$$

where we have indicated:

$$i = 2, \dots, M$$

$$n = 1, \dots, N$$

$$\Delta x = \frac{L}{M}$$

$$\Delta t = \frac{T}{N}$$

Since the scheme is implicit, its resolution leads to:

$$\underline{w}^{n+1} = \left(\underline{\underline{I}} + \Delta t\underline{\underline{A}}\right)^{-1}\left(\underline{w}^n + \Delta t\underline{FT}\right)\ n = 1, \dots, N \tag{36}$$

For the adjoint problem resolution, starting from Eq.(24), if we choose a coherent implicit finite difference scheme, with first order approach in time and second order approach in space, we have:

$$-\frac{a_i^{n+1}-a_i^n}{\Delta t}=\nu\frac{a_{i+1}^n-2a_i^n+a_{i-1}^n}{\Delta x^2}+ +W_i\frac{a_{i+1}^n-a_{i-1}^n}{2\Delta x}-\underline{FT}_{adj} \quad (37)$$

This scheme leads to the system:

$$-\frac{\underline{a}^{n+1}-\underline{a}^n}{\Delta t}+\underline{\underline{B}}\,\underline{a}^n-\underline{FT}_{adj}=\underline{0} \quad (38)$$

where:

$$\underline{FT}_{adj}= =\frac{\left[2q_2w_2^n \quad 2q_3w_3^n \quad \cdots \quad 2q_{M-2}w_{M-2}^n \quad 2q_{M-1}w_{M-1}^n\right]^T}{e^{-\alpha(n-1)\Delta t}}$$

and the matrix $\underline{\underline{B}}$ is a three-diagonal matrix with the elements:

$$B_{i,j}=\frac{2\nu}{\Delta x^2} \qquad if\ i=j$$

$$B_{i,j}=-\frac{\nu}{\Delta x^2}-\frac{W_i}{2\Delta x} \qquad if\ i=j+1$$

$$B_{i,j}=-\frac{\nu}{\Delta x^2}+\frac{W_i}{2\Delta x} \qquad if\ i=j-1$$

with the same spatial and time discretization of the direct problem. Since the scheme is implicit, its resolution leads to:

$$\underline{a}^n=\left(\underline{\underline{I}}+\Delta t\underline{\underline{B}}\right)^{-1}\left(\underline{a}^{n+1}+\Delta t\underline{FT}_{adj}\right) \quad n=N,\ldots,1 \quad (39)$$

APPLICATIONS

The method previously shown has been written in an automatic calculation procedure in Matlab® environment [11]. It has been applied a smoothing procedure on the optimal condition of the control in order to keep the compatibility condition between the initial condition and the upper boundary condition. Such a algorithm is presented for case in which the first eight time step of the control are not calculated by the optimal condition. Since the first time step is given by the compatibility condition if one takes as unknowns the control points between the second and the eighth time step one can proceed as indicated in the following expressions.

$$u_5=-\frac{S_{R9}+S_{L9}}{2}4\Delta t+u_9$$

$$u_7=-\frac{S_{R5}+S_{R9}}{2}2\Delta t+u_9$$

$$u_3=-\frac{S_{L7}+S_{L5}}{2}2\Delta t+u_5$$

$$u_8=-\frac{S_{R9}+S_{R7}}{2}\Delta t+u_9$$

$$u_6=-\frac{S_{L8}+S_{L7}}{2}\Delta t+u_7$$

$$u_4=-\frac{S_{R3}+S_{L6}}{2}\Delta t+u_5$$

$$u_2=-\frac{S_{L4}+S_{L3}}{2}\Delta t+u_3$$

In Fig. 2 a graphical representation of the smoothing procedure is shown.

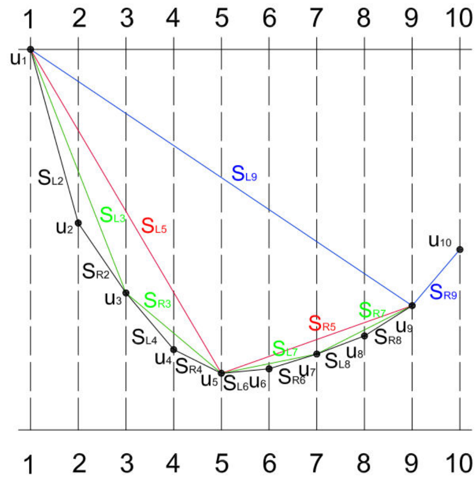


Figure 2 - Graphical representation of the smoothing procedure for the control determination

As a first application the linearized model has been employed to simulate the problem.

The values of α and $q(x)$ have been chosen as variable parameters. In particular:

$$\alpha=0,0.4,\ldots,1.2$$
$$q(x)=k \ \text{ where } k=1,11,\ldots,31 \quad (40)$$

As a first result the stability of the method has been investigated in order to define the proper discretization method (implicit or explicit). The Euler circle [12], which represents the stability zone of the scheme, is:

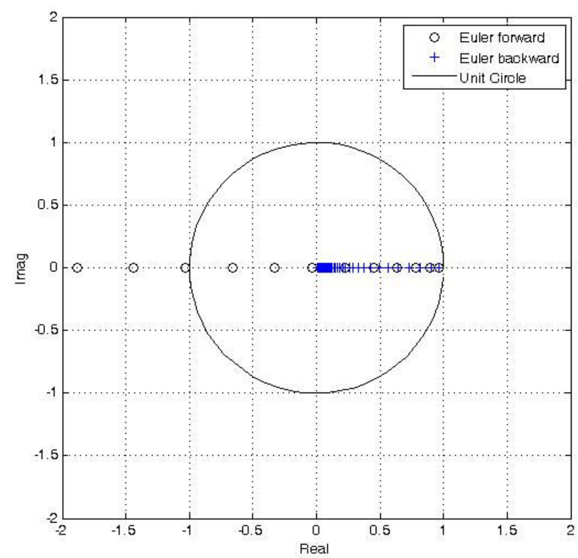
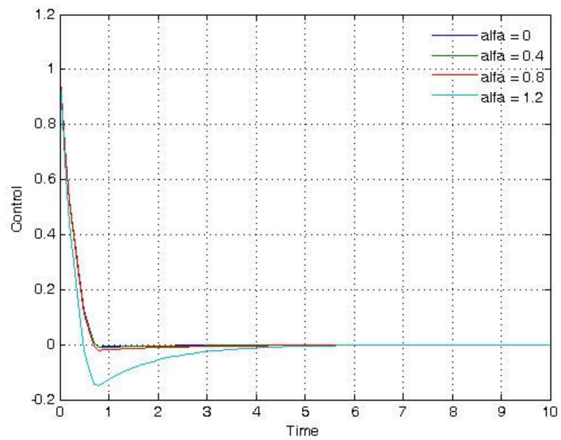
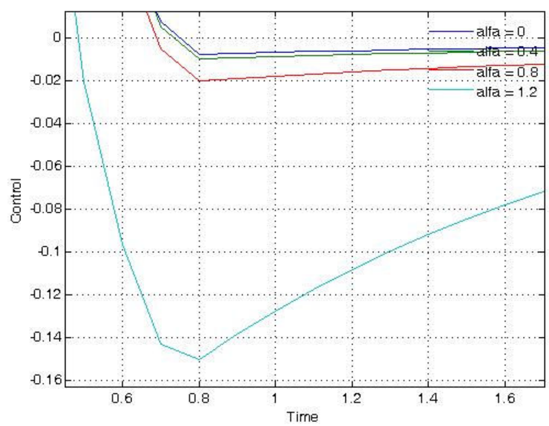


Figure 3 - Euler circle for the linearized model

Thus, an implicit scheme has been adopted. The calculated control law is presented. Firstly fixing $q(x)$ to one and letting α to vary as indicated in Eq.(40) [Figure 4 (a) and (b)], and then fixing α to zero and letting q(x) to vary as indicated in Eq.(40) [Figure 5 (a) and (b)].

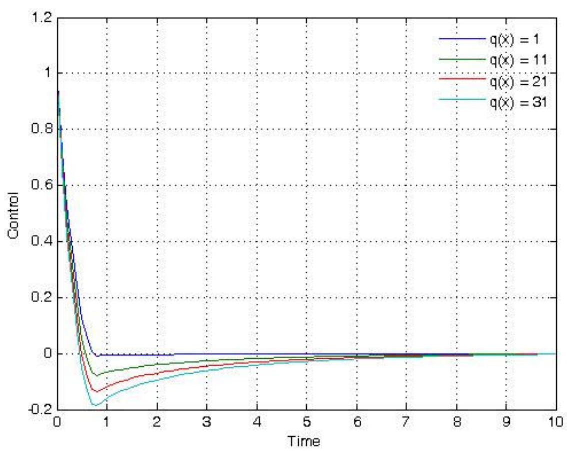


(a)

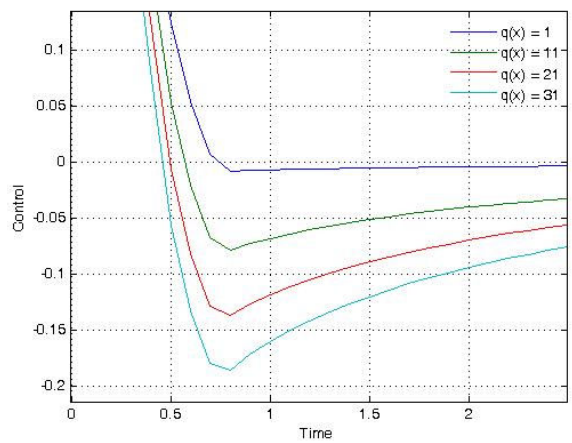


(b)

Figure 4 – Control law for the linearized problem letting α to vary: (a) global view, (b) zoomed view



(a)



(b)

Figure 5 – Control law for the linearized problem letting $q(x)$ to vary: (a) global view, (b) zoomed view

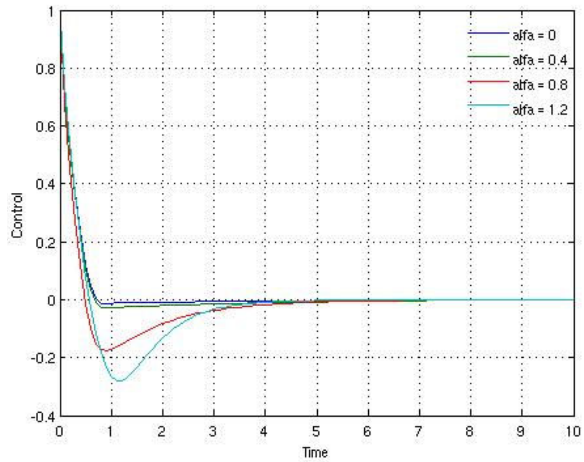
As a second application the non linearized model has been simulated. The adjoint problem resolution is determined using Eq. (41) instead of Eq. (37).

$$-\frac{a_i^{n+1}-a_i^n}{\Delta t}=\nu\frac{a_{i+1}^n-2a_i^n+a_{i-1}^n}{\Delta x^2}+ \\ +\overline{w}_i^n\frac{a_{i+1}^n-a_{i-1}^n}{2\Delta x}-\underline{FT}_{adj} \quad (41)$$

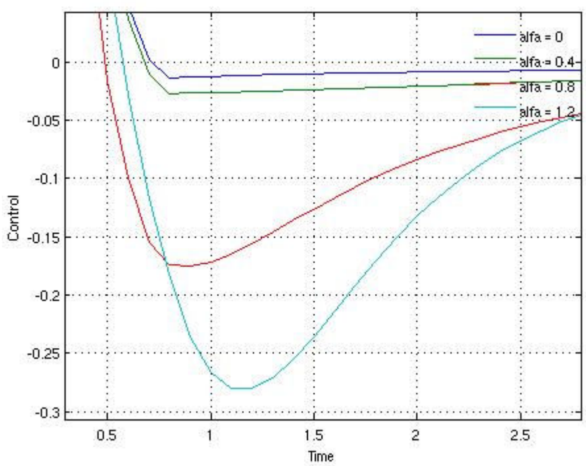
Eq.(41) is rigorous for the linearization of Eq. (32) in the neighborhood of the solution $\overline{\mathrm{w}}_{\mathrm{i}}^{\mathrm{n}}$ and not for the adopted Eq. (32). For this reason the optimal control $u_{opt}(t)$ from Eq. (31) is not used directly, but only to find out the descent direction and to calculate, through a proper choose of the step length [13], the new value of the control law in the iterative process.

$$u_{new}(t)=u_{old}(t)+\sigma p \\ u_{new}(t)=u_{old}(t)+\sigma(u_{opt}(t)-u_{old}(t)) \quad (42)$$

The values of $\boldsymbol{\alpha}$ and $\boldsymbol{q(x)}$ have been chosen as variable parameters taken as just written in Eq. (40). The calculated control law is presented. Firstly fixing $\boldsymbol{q(x)}$ to unity and letting $\boldsymbol{\alpha}$ to vary as indicated in Eq.(40), then fixing $\boldsymbol{\alpha}$ to zero and letting $\mathbf{q(x)}$ to vary as indicated in Eq.(40).

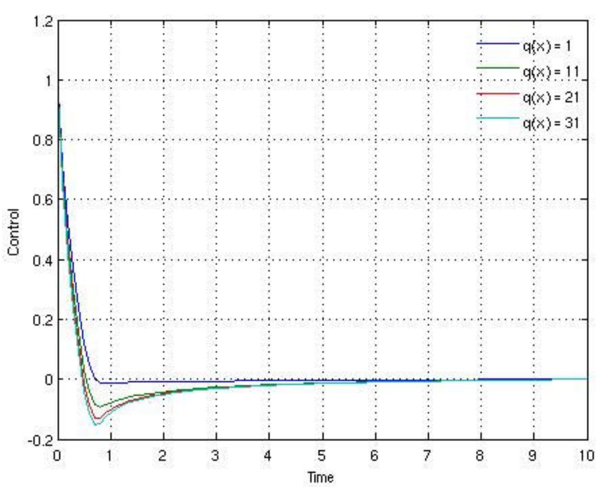


(a)

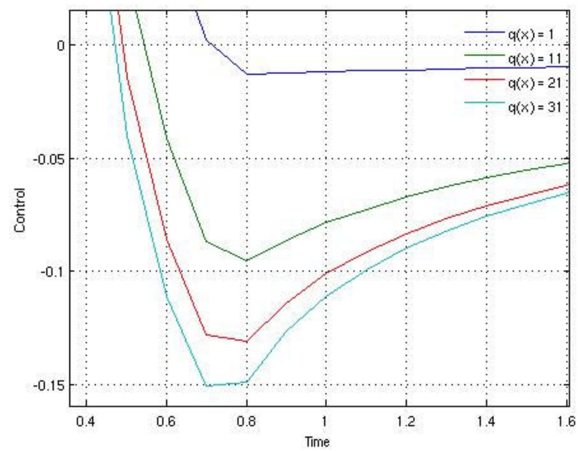


(b)

Figure 6 – Control law for the non-linearized problem letting α to vary: (a) global view, (b) zoomed view



(a)



(b)

Figure 7 – Control law for the non-linearized problem letting $q(x)$ to vary: (a) global view, (b) zoomed view

In the last two applications the step length has been chosen respectively equal to 0.03 and 0.2. Finally the solution $\overline{w}(x,t)$ resulting from the found control law is shown.

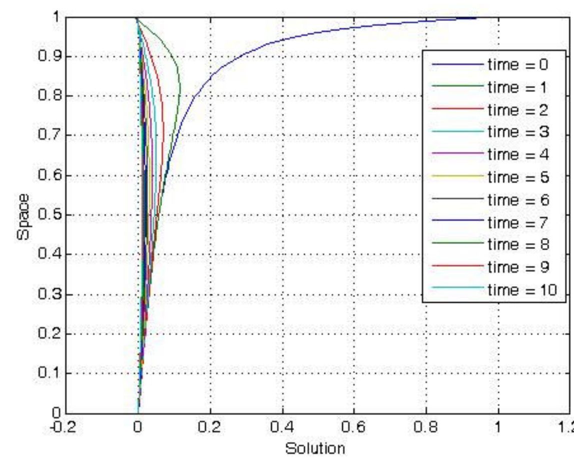


Figure 8 - Solution $\overline{w}(x,t)$: q(x) equal to 1 and α equal to 0

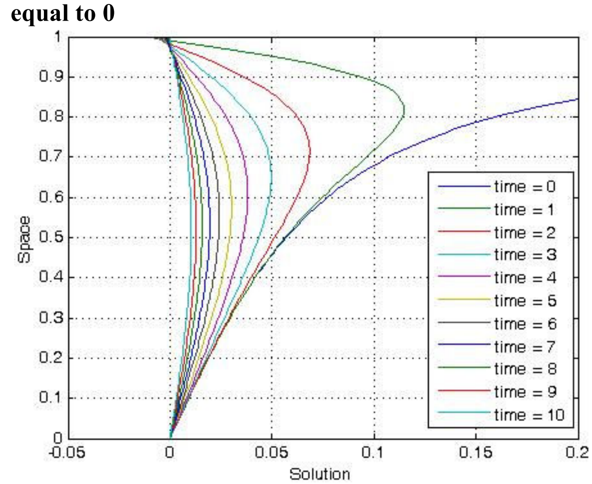


Figure 9 - Solution $\overline{w}(x,t)$: q(x) equal to 1 and α equal to 0, detailed view

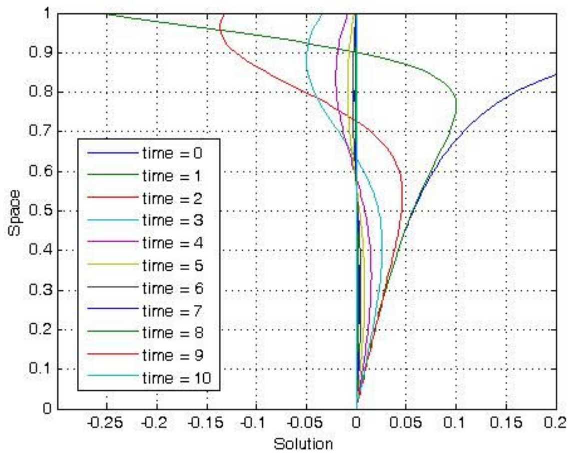


Figure 10 - Solution $\bar{w}(x,t)$: q(x) equal to 1 and α equal to 1.2, detailed view

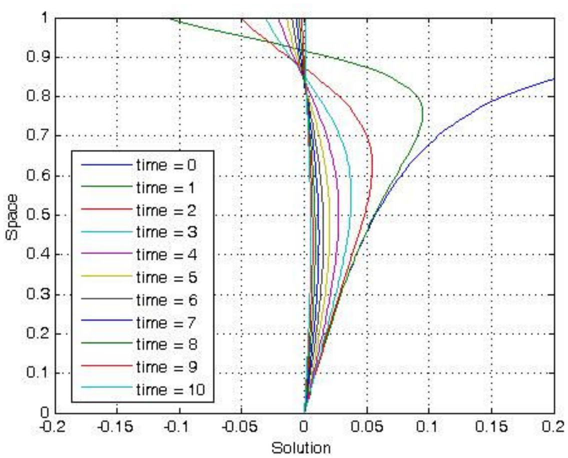


Figure 11 - Solution $\bar{w}(x,t)$: q(x) equal to 31 and α equal to 0, detailed view

CONCLUSIONS

A control model, able to minimize the kinetic energy of a viscous boundary layer, has been developed. The mathematical problem, based on the control of Burgers' Equation, has been solved through the employ of Lagrangian multipliers approach. The model, developed in Matlab environment, has been applied to the study of a boundary layer bounded between two walls. As a first application the Burgers' Equation has been linearized by applying a little perturbation in a neighborhood of a stationary solution in order to simplify the problem of minimizing the kinetic energy. In fact, for this case the kinetic energy is due to only the speed associated to the small perturbation. As one can notes the control begins to modify the solution for low values of the weight function $q(x)$, while the exponential coefficient α contribution is relevant on the control only for values greater then unity. As a second case the control model has been applied to the control of the non linear Burgers' Equation. For this case the sensitivity of the control with respect to the weight function is higher for lower values of $q(x)$, while the influence of coefficient α begin for lower values with respect to the first application. The smoothing procedure on control law (Fig. 2) allows to respect the compatibility condition, which is not a strictly imposed condition in the Lagrangian approach. Furthermore, the values of control law, as indicated in Figures from 4 to 7, strongly decreases from the initial condition till a negative peak, which amplitude depends on both exponential coefficient and weight function. This kind of behavior, unexpected in a first analysis of the problem, means that in order to strongly reduce the kinetic energy associated to the boundary layer velocity profile, it is necessary to have in the first instants of control, a reverse motion of the upper bounding wall, that successively start to increase till the value of zero. Thus, the only decrease of upper bounding wall speed till the value of zero should not be sufficient to minimize the kinetic energy of the problem.

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OPTIMAL CONTROL OF ADVECTION-DIFFUSION EQUATION

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1 Introduction

Advection-diffusion equation is certainly a frequently studied PDE with important applications such as, e.g., modeling the dispersion of pollutants in air or water. Moreover, the goal could be not only to simulate dispersion processes but also to control emissions with an active device and maximum efficiency [1, 2]. The present paper concerns with the solution of such kind of issues, which involve an optimization problem with the following features: derivation of optimality system, numerical resolution of governing PDEs and definition of a suitable algorithm to implement the optimization. Each step of the general procedure is explained in section 2, while some numerical tests are presented in section 3.

In the present study the following assumptions are made: monodimensional and unbounded domain, unsteady state. Moreover, the particular case of source and control concentrated at single points is considered.

2 Method

2.1 Theoretical framework

2.1.1 State equation

The governing PDE of the model, which acts like a constraint in the optimization problem written as the state equation, is:

$$F(C,q) = \frac{\partial C}{\partial t} + \frac{\partial (U \cdot C)}{\partial x} - D\frac{\partial^2 C}{\partial x^2} - g - q = 0 \tag{2.1}$$

with $0 < x < L$ and $0 < t < T$ and where $C = C(x,t)$ is the concentration rate of a scalar quantity of interest, $U = U(x,t)$ is the velocity of the advective flow, D is the diffusivity coefficient, $g = g(x,t)$ is the source term and $q = q(x,t)$ is the control one. U and g are known distributions, while C is obtained by solving the PDE; optimal q (which minimizes the objective function, defined later) is computed with an iterative optimization procedure. Boundary conditions are $C(0,t) = C(L,t) = 0$ in order to simulate an unbounded domain, while initial condition is $C(x,0) = C_0(x)$.

2.1.2 Objective function

In the present study the goal is chosen to minimize the concentration rate at a point P while minimizing the cost of the control, which is located at a point C. Hence, the objective function, using Dirac deltas, is defined as:

$$J(C,q) = \gamma_1 \int_{-\infty}^{\infty} C(x,T)\delta(x-x_P)\mathrm{d}x + \frac{1}{2}\gamma_2 \int_0^T \int_{-\infty}^{\infty} [q(x,t)]^2 \mathrm{d}x\,\mathrm{d}t$$

with the choice of $q(x,t) = Q(t)\delta(x-x_C)$, as mentioned in the introduction, and where $\delta(x-\bar{x}) = \begin{cases} 1 & x = \bar{x} \\ 0 & x \neq \bar{x} \end{cases}$, $\int_{-\infty}^{\infty} f(x)\delta(x-\bar{x})\mathrm{d}x = f(\bar{x})$; thus we can rewrite the expression above as:

$$J(C,q) = \gamma_1 C(x_P,T) + \frac{1}{2}\gamma_2 \int_0^T [Q(t)]^2 \mathrm{d}t$$

2.1.3 Optimality System

The Lagrangian is defined as:

$$\begin{aligned}
\mathcal{L}(C,q,a) &= J \quad - \langle a, F\rangle - \langle b, C(x,0) - C_0\rangle - \langle c, C(0,t)\rangle - \langle d, C(L,t)\rangle \\
&= \quad \gamma_1 C(x_P,T) + \frac{1}{2}\gamma_2 \int_0^T [Q(t)]^2 \mathrm{d}t - \int_0^T \int_{-\infty}^{\infty} a\left(\frac{\partial C}{\partial t} + \frac{\partial (U\cdot C)}{\partial x} - D\frac{\partial^2 C}{\partial x^2} - g - q\right)\mathrm{d}x\,\mathrm{d}t + \\
&\quad - \int_{-\infty}^{\infty} b(C(x,0) - C_0)\mathrm{d}x - \int_0^T c\, C(0,t)\mathrm{d}t - \int_0^T d\, C(L,t)\mathrm{d}t
\end{aligned}$$

where $a = a(x,t)$, $b = b(x)$,$c = c(t)$,$d = d(t)$ are the Lagrange multipliers.

Next step is to apply the stationarity condition on $\mathcal{L}$. Thus, the following system is obtained:

$$\begin{cases}
\frac{\delta\mathcal{L}}{\delta a} = 0 & \Rightarrow & F = 0 & \text{(state equation)} \\
\frac{\delta\mathcal{L}}{\delta b} = 0 & \Rightarrow & C(x,0) = C_0 & \text{(initial condition)} \\
\frac{\delta\mathcal{L}}{\delta c} = 0 & \Rightarrow & C(0,t) = 0 & \text{(boundary conditions)} \\
\frac{\delta\mathcal{L}}{\delta d} = 0 & \Rightarrow & C(L,t) = 0 & \\
\frac{\delta\mathcal{L}}{\delta C} = 0 & \Rightarrow & \frac{\partial a}{\partial t} + U\frac{\partial a}{\partial x} + D\frac{\partial^2 a}{\partial x^2} = 0 & \text{(adjoint equation)} \\
 & & a(x,T) = \gamma_1 & \text{(terminal condition)} \\
\frac{\delta\mathcal{L}}{\delta q} = 0 & \Rightarrow & q(x_C,t) = -\gamma_2\, a(x_C,t) & \text{(optimality condition)}
\end{cases}$$

2.2 Numerical methods and implementation

2.2.1 Discretization

State and adjoint equations are solved by discretization with finite difference method using an explicit, first order in time and second order in space scheme (using central difference for advection term). Hence, for the state equation it turns: $\frac{\partial C}{\partial t} \approx \frac{C_i^{n+1} - C_i^n}{\Delta t}$,$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{i+1} - 2C_i + C_{i-1}}{(\Delta x)^2}$, $\frac{\partial (U\cdot C)}{\partial x} \approx \frac{(U\cdot C)_{i+1} - (U\cdot C)_{i-1}}{2\Delta x}$. Approximation of 2.1 is then:

$$\frac{C_i^{n+1} - C_i^n}{\Delta t} + \frac{(U\cdot C)_{i+1}^n - (U\cdot C)_{i-1}^n}{2\Delta x} - D\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2} = g_i^n + q_i^n$$

Hence, the scheme is:

$$
\begin{aligned}
C_i^{n+1} &= C_i^n - \Delta t\,\frac{(U\cdot C)_{i+1}^n - (U\cdot C)_{i-1}^n}{2\Delta x} + D\Delta t\,\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2} + \Delta t\, g_i^n + \Delta t\, q_i^n \\
&= (\beta - \frac{\lambda}{2}U_{i+1}^n)\, C_{i+1}^n + (1-2\beta)\, C_i^n + (\beta + \frac{\lambda}{2}U_{i-1}^n)\, C_{i-1}^n + \Delta t\, g_i^n + \Delta t\, q_i^n
\end{aligned}
\tag{2.2}
$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\beta = D\frac{\Delta t}{(\Delta x)^2}$, and for $i = 2, \ldots, M-1$, $n = 1, \ldots, N$, where M is the number of nodes in space and N is the number of nodes in time.

Equation 2.2 has to satisfy two conditions in order to have numerical stability, according to [3]:

$$
\sigma^2 \leq 2\beta \leq 1 \tag{2.3}
$$

where $\sigma = max(U)\cdot\lambda$.

2.3 involves a limitation on the time step:

$$
\Delta t \leq min\left(\frac{2D}{U_{max}^2}, \frac{(\Delta x)^2}{2D}\right)
$$

Furthermore, stability check by calculation of eigenvalues is performed.

Each time step is solved looking at a vectorial form of 2.2, introducing the matrix $\boldsymbol{V}$:

$$
\boldsymbol{C}^{n+1} = \boldsymbol{V}\,\boldsymbol{C}^n + \Delta t\,\boldsymbol{g}^n + \Delta t\,\boldsymbol{q}^n
$$

$$
\boldsymbol{V} = \begin{pmatrix}
1-2\beta & \beta - \frac{\lambda}{2}U_{i+1}^n & 0 & 0 & 0 \\
\beta + \frac{\lambda}{2}U_{i-1}^n & \ddots & \ddots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \beta - \frac{\lambda}{2}U_{i+1}^n \\
0 & 0 & 0 & \beta + \frac{\lambda}{2}U_{i-1}^n & 1-2\beta
\end{pmatrix}
$$

The same approach has been chosen for solving the adjoint equation and the resulting scheme is similar:

$$
\boldsymbol{a}^n = \boldsymbol{V}^\dagger\,\boldsymbol{a}^{n+1}
$$

$$
\boldsymbol{V}^\dagger = \begin{pmatrix}
1-2\beta & \beta + \frac{\lambda}{2}U_i^n & 0 & 0 & 0 \\
\beta - \frac{\lambda}{2}U_i^n & \ddots & \ddots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \beta + \frac{\lambda}{2}U_i^n \\
0 & 0 & 0 & \beta - \frac{\lambda}{2}U_i^n & 1-2\beta
\end{pmatrix}
$$

2.2.2 Accuracy of the adjoint

Implementation is checked with the computation of the following error:

$$
\epsilon_{adj} = |\boldsymbol{a}^N\cdot\boldsymbol{C}^N - \boldsymbol{a}^1\cdot\boldsymbol{C}^1 - \sum_{n=1}^{N-1}\Delta t\,\boldsymbol{a}^{n+1}(\boldsymbol{g}^n + \boldsymbol{q}^n)| \tag{2.4}
$$

Comforting values of ϵ_{adj} (less than about 10^{-15}) are found during computations.

2.2.3 Iterative algorithm

The optimization technique is implemented with the following iterative scheme:

1. Initialization of all variables
2. Solution of state equation
3. Computation of objective function and relative error
4. Solution of adjoint equation and accuracy check of the adjoint
5. Computation of control (optimality condition)
6. If the desired tolerance has not been reached, repetition from step 2 with updated variables.

3 Numerical tests

Two examples are presented to test the developed method and to view at significant results.
It has to be recalled that for advection-diffusion model, the Peclét number has great importance, being defined as $Pe = \frac{UL}{D}$. Large Pe indicates dominant advection.
For both cases, the assumed domain is $x \in [0, 1]$ and $D = 0.005$, $\gamma_2 = 1$ and a parametrization of the weight coefficient γ_1 is done. For simplicity, a constant advective field $U(x, t) = U_0$, with $U_0 = 0.5$ (and $Pe = 100$), is assumed, even if the method works with any velocity distribution. Firstly, initial situation and stability check are presented, then results of optimization are given.
Computations are made using *Gnu Octave* [4].

3.1 Dispersion of a spot

This case could be representative of the accidental dumping of a certain substance into air or water and a consequent operating procedure aimed to minimize the concentration rate of the dispersed substance at a specific point (e.g. a point that must be particularly preserved). For simulating this situation, $g = 0$ and an initial profile of concentration is defined (see at fig. 3.1(a)).

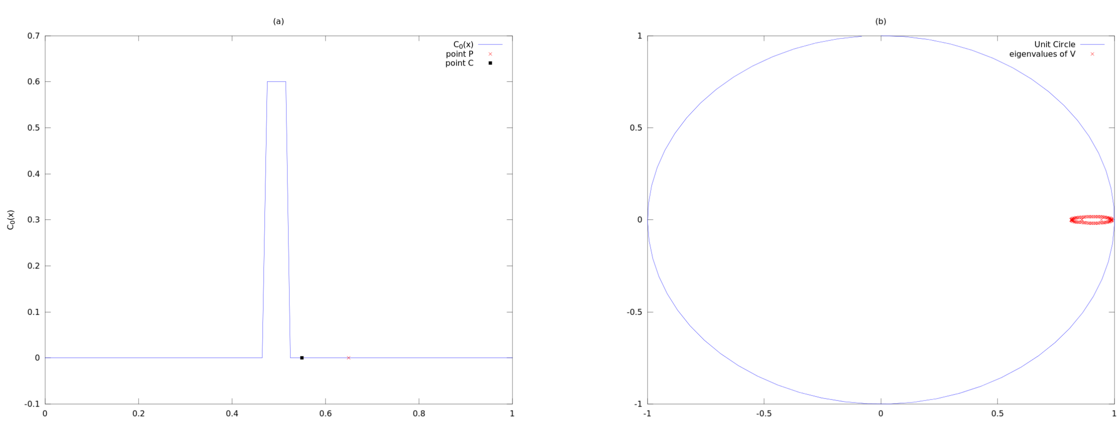

Figure 3.1: *Case 1 - dispersion of a spot: (a)Initial situation and location of points P and C. (b)Plot of eigenvalues of companion matrix for stability check.*

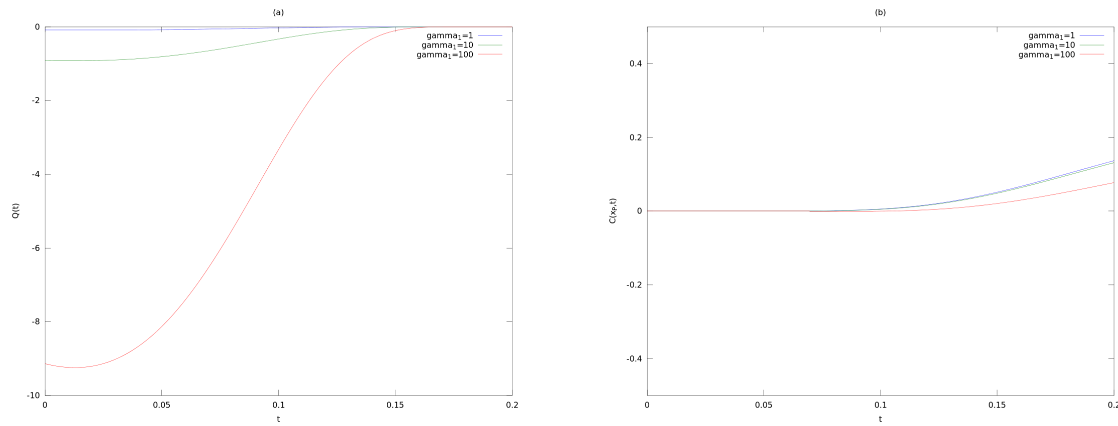

Figure 3.2: *Case 1 - dispersion of a spot: (a)Optimal control laws. (b)Concentration at point P.*

3.2 Constant point source

In the second case we consider a point source ($g(x,t) = G(t)\delta(x - x_S)$), e.g. an industrial chimney, starting to blow at the initial time and at a constant rate. Optimal control could be needed to limit the emissions of the plant. Point C and point S are set as coincident. Actually, this case deals with a transient situation. No initial concentration and constant advective field (with $U_0 = 0.5$ and $Pe = 100$) are assumed for the test.

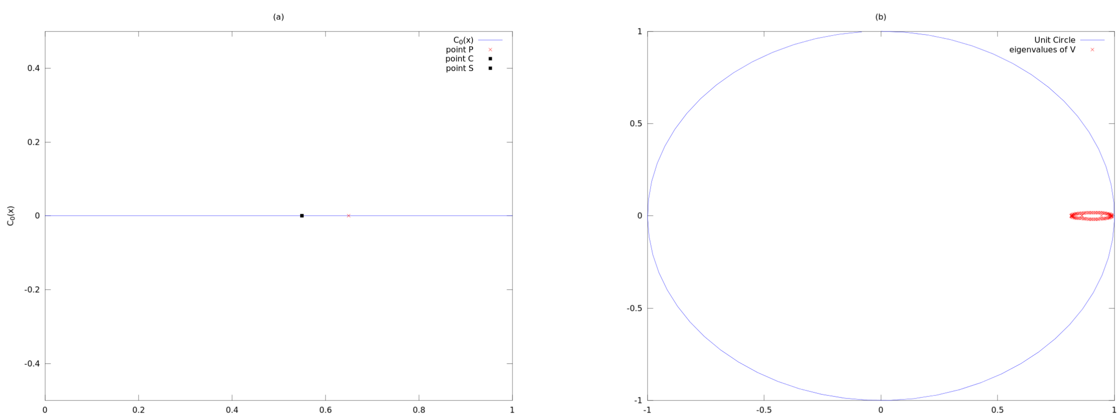

Figure 3.3: *Case 2 - point source: (a)Initial situation and location of points P,C and S. (b)Plot of eigenvalues of companion matrix for stability check.*

4 Discussion

Optimality system for monodimensional unsteady advection-diffusion control problem, in the case of point source and point control, has been derived and a suitable algorithm, numerical methods and relative stability test have been presented. Computations have been also verified by checking errors on the adjoint identity 2.4. Two numerical examples have been presented, besides applicability is certainly wider. The resulting method could be useful for forecasting particular situations in environmental scenarios like dispersion of pollutants.
A remark must be done on the convergence of iterations, which has been seen, in some not presented cases, not always successful; further analyses could be therefore developed for this particular issue. Anyway, the

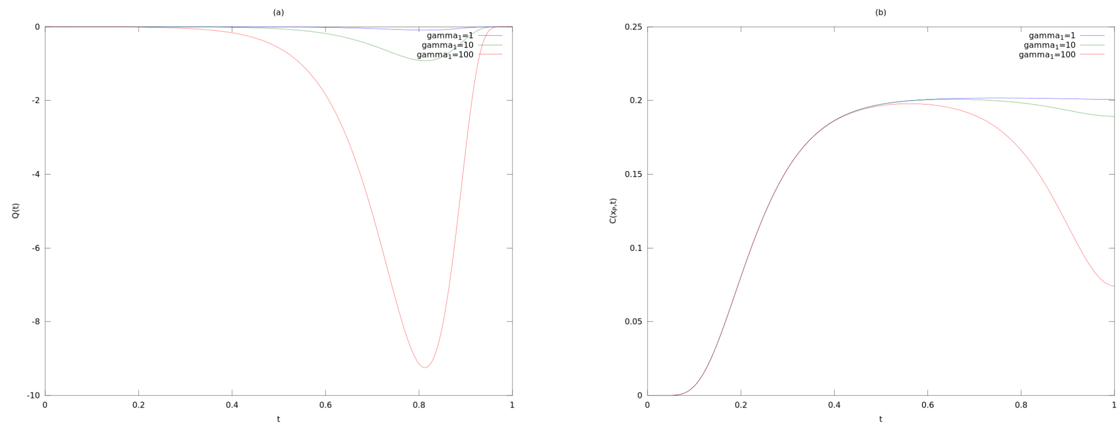

Figure 3.4: *Case 2 - point source: (a)Optimal control laws. (b)Concentration at point P.*

developed code can be applied to different situations by adjusting variables, especially the weight coefficients in the objective function.

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Optimal perturbation and stability analysis of a spatial developing flow

Damiano Natali

Abstract

Short-term instabilities play an important role in fluid dynamical stability theory, where the most common approach is dominated by the quest for the optimal initial condition that results in the maximum amplification of itself over a finite time span. In the present paper, both the optimal perturbation and the non-modal stability theory is applied to the one-dimensional linearized Ginzburg-Landau model, which describes the evolution of a perturbation in a spatially developing flow.

1 Introduction

The aim of the present paper is dual. First, we look for the optimal perturbation in a spatially developing flow, then the stability of the flow is determined applying tools from both the modal and non-modal stability analysis. The fluid dynamical system in object is described by the Ginzburg-Landau model, which is used to describe a wide variety of phenomena, from phase transition in thermodynamic systems to superconductivity. However, in our case, the Ginzburg-Landau model will be used to describe the wave amplitude in a bifurcating spatially developing flow.

After the declaration of all the quantities of the problem, both the adjoint equation and the optimality system is derived for the Ginzburg-Landau model in Section 2 and numerically discretized along with the direct equation in Section 3. Moreover, some optimal perturbations for different sets of parameters are shown in Section 4. The stability analysis will be discussed in Section 5 and 6 with respectively modal and non-modal theory. Finally, conclusions and future improvements are depicted in Section 7.

The linearized equation for the amplitude of a perturbation about the basic state is governed by the Ginzburg-Landau model:

$$\frac{\partial \phi}{\partial t} = (-U\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x^2} + \sigma)\ \phi \tag{1}$$

where

- $\phi = \phi(x,t)$ is the wave amplitude of the perturbation,
- U is the velocity of the mean flow,
- γ is the diffusion coefficient,
- $\sigma(x)$ is the local bifurcation parameter ($\sigma(x) = \sigma_0 - \sigma_2\frac{x^2}{2}, \sigma_2 \geq 0$),
- $g = g(x)$ is the initial condition.

The above-written equation will be solved in a one-dimensional infinite domain $D = (-\infty, +\infty)$ from time 0 to T, optimizing g by means in order to maximize the following quantity

$$\frac{<\phi(t=T), \phi(t=T)>}{<g, g>}$$

which represents the ratio between some measure related to the energy of the system at final and at the initial time. where $< a, b >= \int_D a \cdot b \; dx$. Initial conditions are $\phi(x, t=0) = g(x)$, whereas asymptotic boundary conditions are $\phi(x \to \pm\infty, t) \to 0$.

2 Adjoint equations

So far we have defined our *state equation* F as

$$F(\phi, g) = \frac{\partial \phi}{\partial t} + (U \frac{\partial}{\partial x} - \gamma \frac{\partial^2}{\partial x^2} - \sigma) \; \phi$$

in $0 < t < T$ with initial condition $\phi(x, t=0) = g(x)$ and boundary conditions $\phi(x \to \pm\infty, t) \to 0$, along with the following *cost function* J

$$J = \frac{< g, g >}{< \phi(t=T), \phi(t=T) >} \tag{2}$$

In order to derive the optimal condition with equality contraints with the method of Lagrangian multipliers we have to find the stationary points of the Lagrangian function $\mathcal{L}$ with respect to its variables:

$$\mathcal{L}(\phi, g, a, b, c, d) = J(\phi, g) - \int_0^T < a, F(\phi, g) > dt - < b, \phi(x, t=0) - g > +$$

$$- \int_0^T c[\phi(x \to +\infty, t) - 0]dt - \int_0^T d[\phi(x \to -\infty, t) - 0]dt$$

Whereas derivation with respect to a, b, c and d leads to the state equation, initial and boundary conditions

$$\frac{\partial \mathcal{L}}{\partial a} = 0 \Rightarrow \frac{\partial \phi}{\partial t} + (U \frac{\partial}{\partial x} - \gamma \frac{\partial^2}{\partial x^2} - \sigma) \; \phi = 0 \; ;$$

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \phi(x, t=0) = 0 \; ;$$

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \Rightarrow \phi(x \to +\infty, t) \to 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial d} = 0 \Rightarrow \phi(x \to -\infty, t) \to 0 \; .$$

derivatives of $\mathcal{L}$ with respect to ϕ and g give the *adjoint equation*

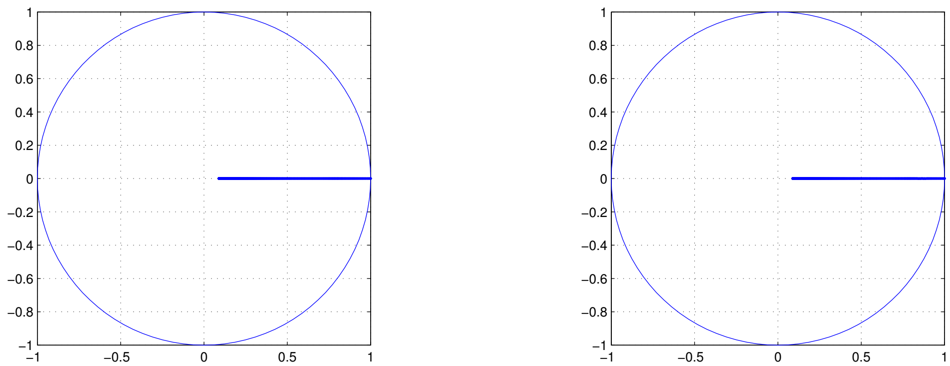
$$-\frac{\partial a}{\partial t} = (U \frac{\partial}{\partial x} + \gamma \frac{\partial^2}{\partial x^2} + \sigma) a$$

with its initial and boundary conditions

$$a(x, t=T) = \frac{2(\int_{-\infty}^{+\infty} g(\tilde{x}) g(\tilde{x}) \; d\tilde{x}) \; \phi_T}{< \phi(\tilde{x}, t=T), \phi(\tilde{x}, t=T) >^2} \quad ; \quad a(x = \pm\infty, t) = 0 \tag{3}$$

along with the following *optimality conditions* (for the full derivation see Appendix A):

$$g(x) = -\frac{a(x, t=0)}{2} \int_{-\infty}^{+\infty} \phi(\tilde{x}, t=T) \; \phi(\tilde{x}, t=T) \; d\tilde{x} \tag{4}$$



(a) (b)

Figure 1: Eigenvalues of implicit scheme used for the integration of direct (a) and adjoint (b) equations compared to the unity circle.

3 Numerical Scheme

A Matlab script has been written in order to solve numerically the optimization problem. The main steps are here briefly outlined:

- forward integration of the state equation;
- evaluation of the cost function;
- backward integration of the adjoint equation;
- assessment of a new control function via the optimality equation;

These steps have been embedded inside a loop stopping when the absolute difference between two consecutive values of J is lower than an imposed accuracy.

Both integrations of state and adjoint equations are performed using an implicit backward Euler finite difference scheme:

- State equation

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -U\frac{\phi_{i+1}^{n+1}-\phi_{i-1}^{n+1}}{2\Delta x} + \gamma\frac{\phi_{i+1}^{n+1}-2\phi_i^{n+1}+\phi_{i-1}^{n+1}}{\Delta x^2} + \sigma_i\phi_i^{n+1} \Rightarrow$$

$$\phi_{i-1}^{n+1}[-\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] + \phi_i^{n+1}[1+\frac{2\gamma\Delta t}{\Delta x^2} - \sigma_i\Delta t] + \phi_{i+1}^{n+1}[\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] = \phi_i^n$$

- Adjoint equation

$$-\frac{a_i^n - a_i^{n-1}}{\Delta t} = U\frac{a_{i+1}^{n-1}-a_{i-1}^{n-1}}{2\Delta x} + \gamma\frac{a_{i+1}^{n-1}-2a_i^{n-1}+a_{i-1}^{n-1}}{\Delta x^2} + \sigma_i a_i^{n-1} \Rightarrow$$

$$a_{i-1}^{n-1}[\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] + a_i^{n-1}[1+\frac{2\gamma\Delta t}{\Delta x^2} - \sigma_i\Delta t] + a_{i+1}^{n-1}[-\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] = a_i^n$$

Both methods have proved to be stable when investigated with the *absolute stability* condition due to the implicit method used (see Figure 4).

The accuracy of the adjoint has been checked using the *adjoint equality*

$$< a, L\phi >=< \phi, L^\dagger a > +B.T. \tag{5}$$

which in our case gives

$$\int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\ \phi]dtdx =$$

$$= \int_0^T \int_{-\infty}^{+\infty} \phi[\frac{\partial a}{\partial t} + (U\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x^2} + \sigma)a]dtdx + [a\ \phi]_0^T$$

but since both state and adjoint equation does not have any source term,

$$[a\ \phi]_0^T = 0 \Rightarrow a(0)\phi(0) = a(T)\phi(T),$$

which in all our simulation has been less than 10^{-10}, next to the machine precision.

4 Optimal perturbation

The particular choice of the control as the initial condition and the cost function as the ratio between quantities proportional to the energy of initial and final perturbation defines g as the *optimal perturbation.*

$$J = \frac{< g, g >}{< \phi(t = T), \phi(t = T) >}$$

To prove the effectiviness of the code we present different optimal perturbations g at different values of the Ginzburg-Landau parameters (U, γ and σ_0/σ_2), trying to give them an interpretation form the physical point of view (see Figure 2). In all our simulation, we obtained different values of J_{min}, which are summarized in Table 1:

different U By increasing U, the optimal perturbation tends to move slighty backward. This result is because of our approximation of the infinite domain with a finite grid, since the boundary conditions are $\phi = 0$.

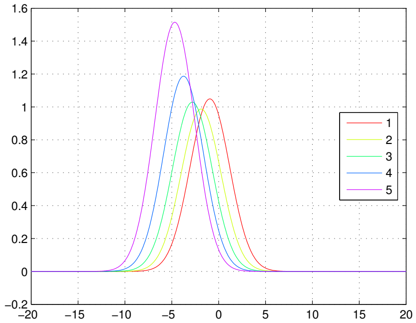
different γ As the diffusion parameter γ grows we notice that the peak of the optimal perturbation increases and the stiffness decreases, in order to minimize the diffusive effects.

different $\boldsymbol{\sigma_0/\sigma_2}$ Since σ_0 is constant positive and the bifurcation function $\sigma(x)$ is given as $\sigma_0 - (\sigma_2/2)x^2$, the increment of this ratio means a larger portion of domain in which $\sigma > 0$, i.e. where the solution exponentially grows. So, as the ratio increases the optimal perturbation does not have to be as energetic as the previous ones.

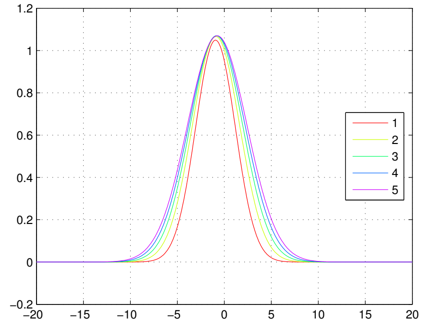
Moreover, we presents the evolution of the optimal perturbation with three different sets of parameters, whose discussion will be clarified in the section about non-modal stability analysis (Figure 3).

parameters	1	2	3	4	5
U varying, $\gamma = 1$, $\sigma_0 = 0.48$, $\sigma_0 = 0.1$	0.3817	0.4602	0.6274	0.9671	1.6850
$U = 1$, γ varying, $\sigma_0 = 0.48$, $\sigma_2 = 0.1$	0.3817	0.5504	0.7286	0.9229	1.1364
$U = 1$, $\gamma = 1$, $\sigma_0 = 0.48$, σ_2 varying	0.3817	0.5835	1.7186	20.1317	$5.3399 \cdot 10^3$

Table 1: Values of J_{min} obtained with different parameters configurations.



(a)



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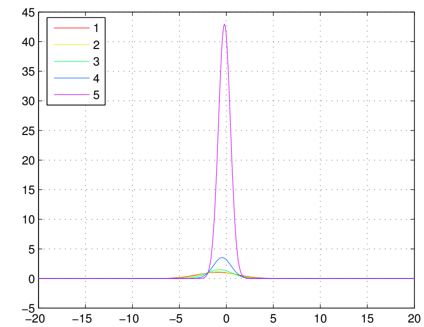
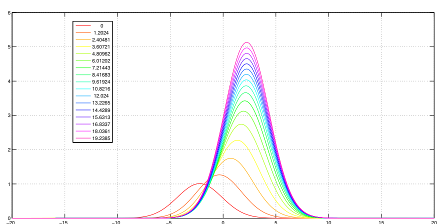
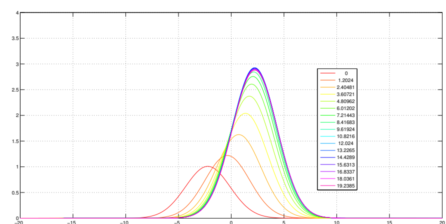


(c)

Figure 2: Optimal perturbation for systems with different values of parameters U (a), γ (b) equations compared to the unity circle and σ (c).



(a)



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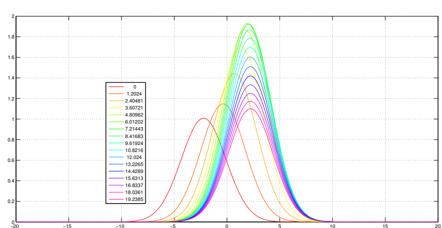


(c)

Figure 3: Different evolution of the initial perturbation in a (a) unstable ($\sigma_0 = 0.5$), (b) neutral ($\sigma_0 = 0.47$) and (c) stable ($\sigma_0 = 0.42$) sets of parameters. Note in the last figure the transient growth before the decaying of the perturbation.

5 Linear stability analysis

In order to investigate the behaviour of the solution with tools from linear stability analysis we perform the *normal mode decomposition*, substituting the solution written as

$$\phi(x,t) = \hat{\phi}(x)\ e^{\lambda t}$$

into the equation

$$\frac{\partial \phi}{\partial t} = \mathcal{A}\phi \quad (\mathcal{A} = -U\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x^2} + \sigma).$$

This transforms the linear initial-value problem into a corresponding eigenvalue problem

$$\lambda\hat{\phi}(x) = \mathcal{A}\hat{\phi}(x)$$

$$(\mathcal{A} - \lambda\mathcal{I})\hat{\phi}(x) = 0$$

where $\mathcal{I}$ is the identity operator. If $\mathcal{A}$ has at least one eigenvalue $\lambda_i > 0$ the perturbation ϕ grows exponentially with time, meanwhile it will decay exponentially if all λs are minor than 0.

In the following we evaluate the behaviour of the most unstable eigenvalue of $\mathcal{A}$ with the parameter σ_0 (see Figure 4).

6 Transient Growth

As stated in [4], linear stability theory is concerned with a quantitative description of flow behavior involving infinitesimal disturbances superimposed on a base flow. However, for our case as for most wall-bounded shear flows the spectrum is a poor proxy for the disturbance behavior as it only describes the asymptotic ($t \to \infty$) fate of the perturbation and fails to capture short-term characteristics. To accurately describe the disturbance behavior for all times, it appears necessary to introduce a finite-time horizon over which an instability is observed.

As we are investigating the temporal evolution on an initial perturbation $g(t)$, we define the gain $G(t)$ as the ratio between some measure related to the energy of the current and initial perturbation

$$G(t) = \max_{g_0} \frac{\|\phi(x,t)\ \phi(x,t)\|}{\|g_0(x)\ g_0(x)\|} \tag{6}$$

but since the evolution of the system is described by

$$\phi(x,t) = g_0(x)exp(\mathcal{A}t)$$

equation (6) becomes

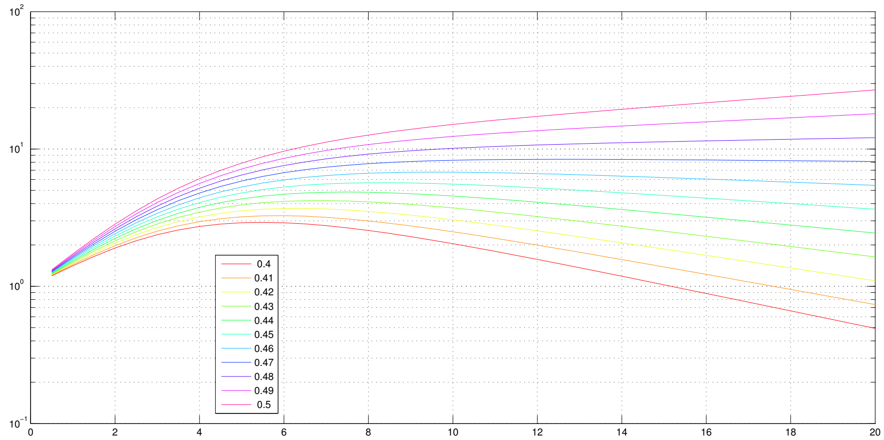
$$G(t) = \max_{g_0} \frac{\left\|g_0^2(x)\ exp(2\mathcal{A}t)\right\|}{\left\|g_0^2(x)\right\|} = \left\|\mathcal{S}exp(2\mathbf{\Lambda}t)\mathcal{S}^{-1}\right\|$$

It should become obvious that no information about the eigenvectors of $\mathcal{A}$, contained in $\mathcal{S}$, is considered when only the least stable mode is taken as a representation of the operator exponential.

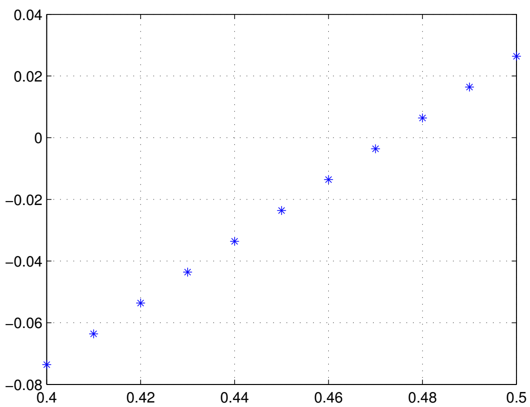
From the stability theory we know that the minimum growth-rate of the solution coincides at least with the most unstable eigenvalue, and because of the triangular disequality we can say that

$$exp(2\lambda_{max}t) \leq G(t) \leq \|\mathcal{S}\| \left\|\mathcal{S}^{-1}\right\| exp(2\lambda_{max}t)$$

The quantity $\|\mathcal{S}\| \left\|\mathcal{S}^{-1}\right\|$ represents the condition number of $\mathcal{S}$ ($k(\mathcal{S})$), a measure of the non-orthogonality of its columns. So if $k(\mathcal{S}) > 1$ (as in our case) the operator $\mathcal{A}$ is said to



(a)



(b)

Figure 4: (a) Semilogarithmic plot of the gain function $G(t)$ from which we can see the transient growth of the solution and (b) the most unstable eigenvalue of $\mathcal{A}$ for different values of σ_0 ($U = 1, \gamma = 1, \sigma_2 = 0.1$). Note that the long-term behaviour of the solution turns from stable to unstable as the most unstable eigenvalue of $\mathcal{A}$ becomes positive.

be non-normal, and systems governed by non-normal matrices can exhibit a large transient amplification of energy contained in the initial condition.

In our case we evaluated the evolution of the energy related to the perturbation for different values of σ_0, in that we can check the results from the accordance between modal and non-modal analysis. In Figure 4(a) we can see that, whereas for great times ($t \gtrsim 10$) the system undergoes a classical exponential behaviour ruled by the most unstable eigenvalue of the spatial operator $\mathcal{A}$, at lower times the system exhibits a *transient growth* explained by the non-hortogonality of the eigenvectors of $\mathcal{A}$. Results from numerical simultions agree qualitatively with the one shown in [2], given that ours refer to the energy of the perturbation and not to the perturbation itself.

7 Conclusions

In the present paper we have investigated the stability of an initial pertubation in a spatially developed flow described by the Ginzburg-Landau equation, with tools from both classic modal analysis and from recently-developed non-modal analysis.

Numerical simulations have shown that, whereas the long time behaviour is well-catched by the modal analysis, the solution exhibits a so-called transient growth on a finite-time horizon, explained by the non-normality of the spatial operator.

The results presented here are further borne out in [2], were it is stated that at the increase of σ_2 (i.e. when the flow is strongly non-parallel) the operator $\mathcal{A}$ becomes more and more non-normal.

A Derivation of adjoint equation and optimality condition

$$\mathcal{L}(\phi, g, a, b, c, d) = J(\phi, g) - \int_0^T < a, F(\phi) > dt - < b, \phi(x, t = 0) - g > +$$

$$- \int_0^T c[\phi(x \to +\infty, t) - 0]dt - \int_0^T d[\phi(x \to -\infty, t) - 0]dt =$$

$$= \frac{\int_{-\infty}^{+\infty} [g(x)\ g(x)]dx}{\int_{-\infty}^{+\infty} [\phi_T\ \phi_T]dx} - \int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\ \phi]dxdt+$$

$$- \int_{-\infty}^{+\infty} b[\phi(x, t = 0) - g]dx - \int_0^T c(\phi(x \to +\infty, t) - 0dt - \int_0^T d(\phi(x \to -\infty, t) - 0)dt$$

A.1 Derivation of $\mathcal{L}$ with respect to g

In the following we will use the notation $\phi_{\tilde{t}} = \phi(x, t = \tilde{t})$.

$$\frac{\partial \mathcal{L}}{\partial g}\delta g = \frac{\partial J}{\partial g}\delta g + \int_{-\infty}^{+\infty} b(x)\ \delta g\ dx = 0$$

but since

$$\frac{\partial J}{\partial g}\delta g = \lim_{\varepsilon \to 0} \frac{J(\phi, g + \varepsilon\delta g, a, b, c, d) - J(\phi, g, a, b, c, d)}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{\int_{-\infty}^{+\infty} (g + \varepsilon\delta g)\ (g + \varepsilon\delta g)dx - \int_{-\infty}^{+\infty} (g\ g)dx}{\varepsilon \int_{-\infty}^{+\infty} [\phi_T\ \phi_T]dx} =$$

$$= \lim_{\varepsilon \to 0} \frac{\int_{-\infty}^{+\infty} 2g\varepsilon\delta g dx}{\varepsilon \int_{-\infty}^{+\infty} [\phi_T\ \phi_T]dx} = \frac{\int_{-\infty}^{+\infty} 2g\delta g dx}{\int_{-\infty}^{+\infty} [\phi_T\ \phi_T]dx}$$

so

$$\frac{\int_{-\infty}^{+\infty} 2g\delta g dx}{\int_{-\infty}^{+\infty} [\phi_T\ \phi_T]dx} + \int_{-\infty}^{+\infty} b(x)\ \delta g\ dx = 0$$

$$\int_{-\infty}^{+\infty} 2g\delta g dx + \int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} \phi_T\ \phi_T d\tilde{x}]\ b(x)\ \delta g\ dx = 0$$

$$\int_{-\infty}^{+\infty} \delta g[2g + \int_{-\infty}^{+\infty} \phi_T\ \phi_T\ d\tilde{x}\ b(x)]dx = 0$$

Since the previous integral has to be zero $\forall g$,

$$2g + \int_{-\infty}^{+\infty} \phi_T\ \phi_T\ d\tilde{x}\ b(x) = 0$$

$$g(x) = -\frac{b(x)}{2} \int_{-\infty}^{+\infty} \phi_T\ \phi_T\ d\tilde{x}$$

A.2 Derivation of $\mathcal{L}$ with respect to ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi = \frac{\partial J}{\partial \phi}\delta\phi - \int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \delta\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\delta\phi]dxdt +$$

$$-\int_{-\infty}^{+\infty} b\delta\phi_0 dx - \int_0^T c\delta\phi_{-\infty}dt - \int_0^T d\delta\phi_{-\infty}dt = 0$$

but since

$$\frac{\partial J}{\partial \phi} = \frac{\partial}{\partial \phi}(\frac{p(\phi)}{q(\phi)}) = \frac{p'q - pq'}{q^2}$$

where $p(\phi) =< g, g >$ and $q(\phi) =< \phi_T, \phi_T >$, so

$$\frac{\partial J}{\partial \phi}\delta\phi = -\frac{2 < g, g >< \phi_T, \delta\phi_T >}{< \phi_T, \phi_T >^2}$$

Since boundary conditions impose $\phi \to 0$ as $x \to \pm\infty$, also $\delta\phi_{\pm\infty} \to 0$, so

$$\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi = -\frac{2 < g, g >< \phi_T, \delta\phi_T >}{< \phi_T, \phi_T >^2} +$$

$$-\int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \delta\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\delta\phi]dxdt - \int_{-\infty}^{+\infty} b\delta\phi(x, t = 0)dx = 0$$

Now we have to develop the second integral of this relation:

$$\int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \delta\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\delta\phi]dxdt =$$

$$= \int_0^T \int_{-\infty}^{+\infty} a\frac{\partial \delta\phi}{\partial t}dxdt + \int_0^T \int_{-\infty}^{+\infty} aU\frac{\partial \delta\phi}{\partial x}dxdt - \int_0^T \int_{-\infty}^{+\infty} a\gamma\frac{\partial^2 \delta\phi}{\partial x^2}dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

$$= \int_0^T \int_{-\infty}^{+\infty} \frac{\partial(a\delta\phi)}{\partial t}dxdt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial a}{\partial t}dxdt + \int_0^T \int_{-\infty}^{+\infty} \frac{\partial(aU\delta\phi)}{\partial x}dxdt +$$

$$-\int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial(aU)}{\partial x}dxdt - \int_0^T \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}[a\gamma\frac{\partial \delta\phi}{\partial x}]dxdt + \int_0^T \int_{-\infty}^{+\infty} \frac{\partial(a\gamma)}{\partial x}\frac{\partial \delta\phi}{\partial x}dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial a}{\partial t}dxdt + \int_0^T [aU\delta\phi]_{-\infty}^{+\infty}dt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial(aU)}{\partial x}dxdt - \int_0^T [a\gamma\frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty}dt +$$

$$+\int_0^T \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}(\frac{\partial(a\gamma)}{\partial x}\delta\phi)dxdt - \int_0^T \int_{-\infty}^{+\infty} \frac{\partial^2(a\gamma)}{\partial x^2}\delta\phi dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial a}{\partial t}dxdt + \int_0^T [aU\delta\phi]_{-\infty}^{+\infty}dt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial(aU)}{\partial x}dxdt - \int_0^T [a\gamma\frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty}dt +$$

$$+\int_0^T [\frac{\partial(a\gamma)}{\partial x}\delta\phi]_{-\infty}^{+\infty}dt - \int_0^T \int_{-\infty}^{+\infty} \frac{\partial^2(a\gamma)}{\partial x^2}\delta\phi dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

but since $\delta\phi$ goes to zero for $x \to \pm\infty$

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi \frac{\partial a}{\partial t} dxdt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi \frac{\partial (aU)}{\partial x} dxdt - \int_0^T [a\gamma \frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty} dt +$$

$$- \int_0^T \int_{-\infty}^{+\infty} \frac{\partial^2 (a\gamma)}{\partial x^2} \delta\phi dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

Rearranging members leads to

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi [\frac{\partial a}{\partial t} + (U \frac{\partial}{\partial x} + \gamma \frac{\partial^2}{\partial x^2} + \sigma)]a \ dxdt - \int_0^T [a\gamma \frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty} dt$$

Given that this holds $\forall\ \delta\phi$, the previous equation gives the following *adjoint equation*

$$-\frac{\partial a}{\partial t} = (U \frac{\partial}{\partial x} + \gamma \frac{\partial^2}{\partial x^2} + \sigma)a$$

with the corresponding *boundary conditions* $a = 0$ for $x \to \pm\infty$. Inserting the previous one in the starting equation will lead to the initial condition for the adjoint equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi = -\frac{2 < g, g >< \phi_T, \delta\phi_T >}{< \phi_T, \phi_T >^2} - \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_{-\infty}^{+\infty} b\delta\phi_0 dx = 0$$

$$-2 \int_{-\infty}^{+\infty} < g, g > \phi_T \delta\phi_T \ dx - \int_{-\infty}^{+\infty} a_T < \phi_T, \phi_T >^2 \delta\phi_T \ dx +$$

$$+ \int_{-\infty}^{+\infty} a_0 < \phi_T, \phi_T >^2 \delta\phi_0 \ dx - \int_{-\infty}^{+\infty} b < \phi_T, \phi_T >^2 \delta\phi_0 dx = 0$$

Arranging in terms of $\delta\phi_T$ and $\delta\phi_0$,

$$\int_{-\infty}^{+\infty} \delta\phi_T [-2 < g, g > \phi_T - a_T < \phi_T, \phi_T >^2] \ dx +$$

$$+ \int_{-\infty}^{+\infty} \delta\phi_0 [a_0 < \phi_T, \phi_T >^2 - b < \phi_T, \phi_T >^2] \ dx = 0$$

And so, given that this holds $\forall\ \delta\phi_T$ and $\forall\ \delta\phi_0$

$$a_T = \frac{-2 < g, g > \phi_T}{< \phi_T, \phi_T >^2}, \quad b = a_0$$

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Reduction of the instability in an aeroelasticity problem using an optimization method

Paolo Bertocchi – Marco Ferrando

SUMMARY

The aim of this work is to apply an optimization process, based on the Lagrange multipliers method, to reduce the oscillations and the instabilities of a wing due to aeroelasticity effects. The problem is governed by a linear dynamical system; an *objective* (or *cost*) function is defined, taking in account both the state of the system and the control term. The goal is to find the optimal control law in order to minimize the objective function in a specified time range, where the final time is a parameter of the problem.

Keywords: Aeroelasticity; Optimization; Adjoint method.

INTRODUCTION

The aeroelasticity problem of this work consist of a wing, with a fixed wing root and a free wing tip, provided with an aileron. The latter represents the control term of the problem. The aeroelastic model of the wing is not here reported, detailed aspects of the physics of this problem can be found in [1] and [2], here only a mention is done. In Fig. 5 (see pag. 8) is represented the model of the wing section; one can see the wing has two d.o.f., a vertical displacement $w(y)$ and a rotation $\theta(y)$ around the c.g. of the airfoil, where *y* is the coordinate in the spanwise direction. There are also stiffness and damping factors. The equation governing this system can be written as

$$\frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}\,\underline{x}(t) + \underline{B}\delta(t)$$

where $\underline{x}(t) = \left[w(t); \theta(t); \dot{w}(t); \dot{\theta}(t)\right]$ and A, B depend on the parameters of the problem. The time domain is $t \in [0, T]$ and the i.c. of the equation is $\underline{x}(0) = \underline{x}_0$. The objective function is defined as

$$J = \frac{1}{2}\gamma_1 \int_0^T \underline{x}^T(t)\underline{x}(t)\,dt + \frac{1}{2}\gamma_2 \int_0^T [\delta(t)]^2\,dt + \frac{1}{2}\gamma_3 \underline{x}^T(T)\underline{x}(T)$$

The first term of the function is associated to the evolution of the system, the second is associated to the energy of the control and the last term is associated to the state of the system at the time instant *T*. Each term is properly weighted with three coefficients γ_1, γ_2 and γ_3. Varying properly one of these weights more or less importance can be associated to each term.

DERIVATION OF THE OPTIMALITY SYSTEM

The Lagrangian function is defined as:

$$\mathcal{L}(\underline{x}(t), \delta(t), \underline{a}(t), \underline{b}) = J\left(\underline{x}(t), \delta(t)\right) - \int_0^T \underline{a}^T(t)\, \underline{F}\left(\underline{x}(t), \delta(t)\right) dt - \underline{b}^T[\underline{x}(0) - \underline{x}_0]$$

where the vectors *a* and *b* are the Lagrange multipliers, *J* is the cost function and

$$\underline{F}\left(\underline{x}(t), \delta(t)\right) = \frac{d\underline{x}(t)}{dt} - \underline{\underline{A}}\, \underline{x}(t) - \underline{B}\delta(t) = \underline{0}\,, \quad t \in [0, T]$$

that is another way to write the state equation.

Now the optimality conditions are obtained by setting the first variation of the Lagrangian function equal to zero, that means:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial \delta}\delta\delta + \frac{\partial\mathcal{L}}{\partial a}\delta a + \frac{\partial\mathcal{L}}{\partial b}\delta b = 0$$

hence $\nabla\mathcal{L} = \underline{0}$.

Performing all the derivatives one obtains:

- $\dfrac{\partial\mathcal{L}}{\partial b} = 0$ $\qquad \delta\underline{b}^T[\underline{x}(0) - \underline{x}_0] = \underline{0} \Longrightarrow \boxed{\underline{x}(0) = \underline{x}_0} \quad \forall\, \delta\underline{b}$

- $\dfrac{\partial\mathcal{L}}{\partial \delta} = 0$

$$\gamma_2 \int_0^T \delta(t)\delta\delta(t)\, dt + \int_0^T \underline{a}^T(t)\underline{B}\delta\delta(t)\, dt = 0$$

$$\int_0^T [\gamma_2\delta(t) + \underline{a}^T(t)\underline{B}]\delta\delta(t)\, dt = 0 \Longrightarrow \boxed{\delta(t) = -\frac{\underline{a}^T(t)\underline{B}}{\gamma_2}} \quad \forall\, \delta\delta(t), \;\; t \in [0, T]$$

- $\dfrac{\partial\mathcal{L}}{\partial a} = 0$

$$\int_0^T \delta\underline{a}^T(t)\underline{F}\left(\underline{x}(t), \delta(t)\right) dt = 0 \Longrightarrow$$

$$\Longrightarrow \boxed{\underline{F}\left(\underline{x}(t), \delta(t)\right) = \frac{d\underline{x}(t)}{dt} - \underline{\underline{A}}\, \underline{x}(t) - \underline{B}\delta(t) = \underline{0}} \quad \forall\, \delta\underline{a}(t), \;\; t \in [0, T]$$

- $\dfrac{\partial\mathcal{L}}{\partial x} = 0$

Using the integration by parts leads to the following expression:

$$\int_0^T \left[\frac{d\underline{a}(t)}{dt} + \underline{\underline{A}}^T \underline{a}(t) + \gamma_1\underline{x}(t)\right]^T \delta\underline{x}(t)\, dt + [\gamma_3\underline{x}(T) - \underline{a}(T)]^T \delta\underline{x}(T) + [\underline{a}(0) - \underline{b}]^T \delta\underline{x}(0) = \underline{0}$$

Hence the adjoint system is:

$$\boxed{\begin{array}{ll} \dfrac{d\underline{a}(t)}{dt} + \underline{\underline{A}}^T \underline{a}(t) + \gamma_1 \underline{x}(t) = 0 & \forall\ \delta\underline{x}(t),\ \ t \in [0, T] \\ \underline{a}(T) = \gamma_3 \underline{x}(T) & \forall\ \delta\underline{x}(T) \\ \underline{b} = \underline{a}(0) & \forall\ \delta\underline{x}(0) \end{array}}$$

DISCRETE OPTIMALITY SYSTEM

In order to implement an algorithm to solve the problem numerically the equations have to be discretized in the time domain. Regarding the state equation a *Backward Euler* method is used.

$$\frac{\underline{x}^{i+1} - \underline{x}^i}{\Delta t} = \underline{\underline{A}}\,\underline{x}^{i+1} + \underline{B}\delta^i \Longrightarrow \underline{x}^{i+1} = \underline{\underline{L}}\left(\underline{x}^i + \underline{B}\delta^i \Delta t\right) \quad \text{for } \ i = 0,1, \dots, N-1$$

where $\underline{\underline{L}} = \left(\underline{\underline{I}} - \underline{\underline{A}}\Delta t\right)^{-1}$ and $N = T/\Delta t$. The initial condition is $\underline{x}^0 = \underline{x}_0$.

Using the discrete adjoint approach and starting from the adjoint identity one gets

$$\left(\underline{a}^{i+1}\right)^T \left(\underline{\underline{L}}\,\underline{x}^i\right) = \left(\underline{\underline{L}}^T \underline{a}^{i+1}\right)^T \underline{x}^i \quad \text{for } \ i = 0,1, \dots, N-1$$

By imposing

$$\underline{\underline{L}}^T \underline{a}^{i+1} = \left(\underline{a}^i - \tfrac{1}{2}\gamma_1 \underline{x}^i \Delta t\right) \quad \text{for } \ i = 0,1, \dots, N-1$$

and substituting in the previous identity one gets

$$\left(\underline{a}^{i+1}\right)^T \left(\underline{x}^{i+1} - \underline{\underline{L}}\,\underline{B}\delta^i \Delta t\right) = \left(\underline{a}^i - \frac{1}{2}\gamma_1 \underline{x}^i \Delta t\right)^T \underline{x}^i$$

$$\left(\underline{a}^{i+1}\right)^T \underline{x}^{i+1} - \left(\underline{a}^i\right)^T \underline{x}^i - \left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\,\underline{B}\delta^i \Delta t + \frac{1}{2}\gamma_1 \left(\underline{x}^i\right)^T \underline{x}^i \Delta t = 0$$

Since the last relation has to be valid for any $i = 0,1, \dots, N-1$ one obtains an identity for evaluate the accuracy of the adjoint. Using the discrete approach the error has to be equal to the machine precision.

$$error = \left| \left(\underline{a}^N\right)^T \underline{x}^N - \left(\underline{a}^0\right)^T \underline{x}^0 - \sum_{i=0}^{N-1} \left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\,\underline{B}\delta^i \Delta t + \sum_{i=0}^{N-1} \frac{1}{2}\gamma_1 \left(\underline{x}^i\right)^T \underline{x}^i \Delta t \right|$$

The optimality condition is found using both the adjoint identity and the definition of the objective function; the latter has to be linearized before proceeding since is not linear with respect to x and δ.

$$\delta J = \sum_{i=0}^{N-1} \gamma_1 \left(\underline{x}^i\right)^T \delta\underline{x}^i \Delta t + \gamma_2 \sum_{i=0}^{N-1} \delta^i \cdot \delta\delta^i \Delta t + \gamma_3 \left(\underline{x}^N\right)^T \delta\underline{x}^N = 0$$

$$\left(\underline{a}^N\right)^T\delta\underline{x}^N-\left(\underline{a}^0\right)^T\delta\underline{x}^0-\sum_{i=0}^{N-1}\left(\underline{a}^{i+1}\right)^T\underline{\underline{L}}\,\underline{B}\delta\delta^i\Delta t+\sum_{i=0}^{N-1}\gamma_1\left(\underline{x}^i\right)^T\Delta t\delta\underline{x}^i=0$$

Since the initial condition is fixed then $\delta\underline{x}^0=0$.

$$\sum_{i=0}^{N-1}\left[\gamma_2\delta^i+\left(\underline{a}^{i+1}\right)^T\underline{\underline{L}}\,\underline{B}\right]\delta\delta^i\Delta t+\left[\gamma_3\underline{x}^N-\underline{a}^N\right]^T\delta\underline{x}^N=0$$

The optimality condition on the control is

$$\delta^i=-\frac{\left(\underline{a}^{i+1}\right)^T\underline{\underline{L}}\,\underline{B}}{\gamma_2}\quad\text{for }\ i=0,1,\dots,N-1$$

Furthermore the terminal condition on the adjoint vector is

$$\underline{a}^N=\gamma_3\underline{x}^N$$

Finally the discrete optimality system is

$$\begin{cases}\underline{x}^{i+1}=\underline{\underline{L}}\left(\underline{x}^i+\underline{B}\delta^i\Delta t\right)\\ \underline{x}^0=\underline{x}_0\\ \underline{a}^i=\underline{\underline{L}}^T\underline{a}^{i+1}+\dfrac{1}{2}\gamma_1\underline{x}^i\Delta t\\ \underline{a}^N=\gamma_3\underline{x}^N\\ \delta^i=-\dfrac{\left(\underline{a}^{i+1}\right)^T\underline{\underline{L}}\,\underline{B}}{\gamma_2}\end{cases}$$

The resolution scheme is the following:

- Resolution of the state equation from *t=0* to *t=T*, given the i.c. $\underline{x}^0=\underline{x}_0$
- Resolution of the adjoint equation proceeding backward in time, from t=T to t=0, given the terminal condition $\underline{a}^N=\gamma_3\underline{x}^N$
- Check of the accuracy and evaluation of the control δ^i using the optimality condition.

This scheme is repeated until convergence is reached, starting with an arbitrary control δ^i, e.g. constant in time or null.

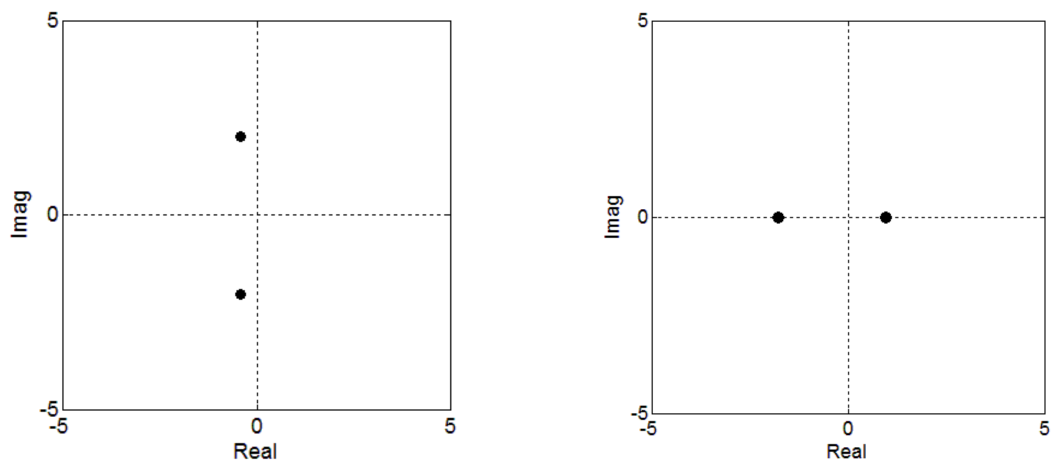
RESULTS

The previous scheme has been implemented in an algorithm using the Matlab environment. Before proceeding it is useful to recall the state equation, i.e. the dynamical system governing the problem, to understand the mathematical aspects and define the problems we want to optimize.

$$\frac{d\underline{x}(t)}{dt}=\underline{\underline{A}}\,\underline{x}(t)+\underline{B}\delta(t)\,,\quad t\in[0,T]$$

Let us consider the related homogeneous equation, i.e. without the forcing term. As one can see it is a linear, autonomous dynamical system, because the matrix A is constant in time. The point $\underline{x} = \underline{0}$ clearly is a stationary solution of the system. The stability of this point, as known, can be evaluated by inspecting the eigenvalues of the coefficients matrix A; if for all eigenvalues $\lambda_i \in \mathbb{C}$, $Re(\lambda_i) < 0$ $\left(1 \leq i \leq dim(A)\right)$ the point is said to be *stable*, otherwise, if at least one eigenvalue has positive real part, the point is said to be *unstable*.

The goal now is to find the control law in order to minimize the objective function *J*, starting from two different initial conditions, both in the neighborhood of the stationary point $\underline{x} = \underline{0}$. The first is to study and control the evolution of the system starting with the wing slightly bended, without twist, the second with the wing slightly twisted, without bending. For each case two further conditions are considered; since the matrix A (and so the eigenvalues) depends on the problem properties, varying wind speed the *stable* and the *unstable* conditions can be made. For the first case a freestream velocity $U = 12{,}5$ is used, in Fig.1 (a) is shown that all the eigenvalues of the coefficients matrix are in the negative real side of the complex plane. In Fig. 1 (b), with a freestream velocity $U = 12{,}7$, an eigenvalue has positive real part, that causes exponential growth and therefore the instability of the stationary solution.



(a) (b)

Fig. 1 *– Eigenvalues of the matrix A on the complex plane for the (a) stable case and (b) unstable case.*

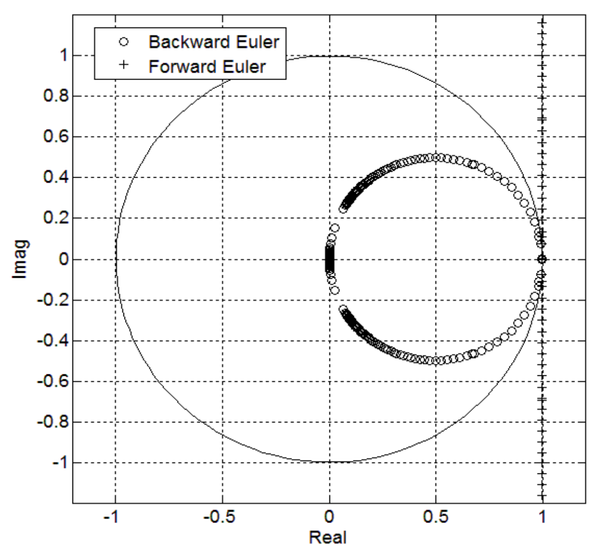
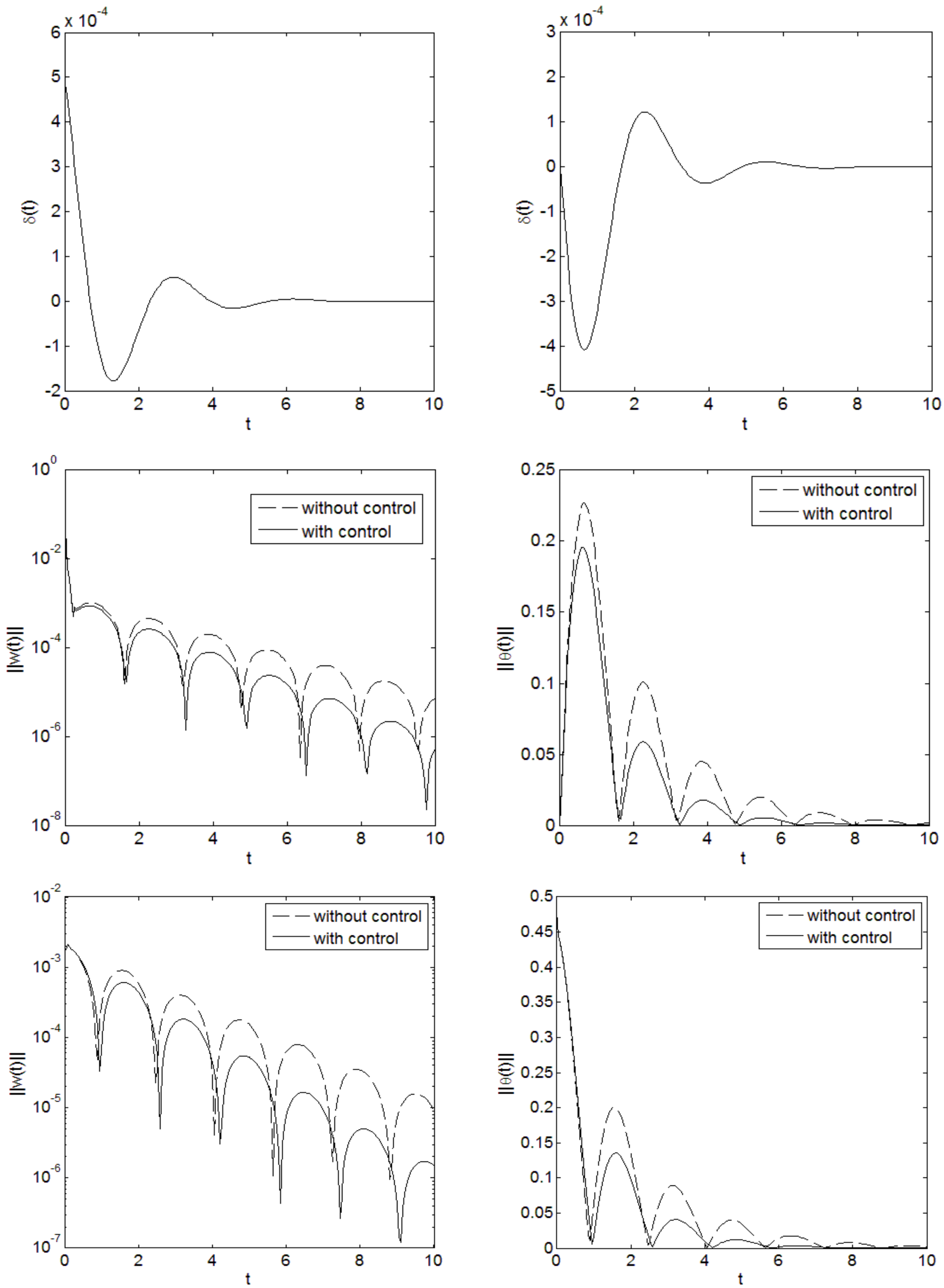


Fig. 2 *– Numerical stability of the discretized problem. For the backward Euler scheme all the eigenvalues modulus is less than one, therefore the numerical scheme is stable.*



(a) (b)

Fig. 3 *– Control law, $\|w(t)\|$ and $\|\theta(t)\|$ in the stable case for two different initial conditions:*

(a) $w_0(y) = 0{,}01\cos\left(\frac{\pi}{2}\frac{y}{L}\right) - 0{,}01;\ \theta_0(y) = 0;\ T = 10, \gamma_1 = 1, \gamma_2 = 2 \cdot 10^5, \gamma_3 = 0$

(b) $w_0(y) = 0;\ \theta_0(y) = 0{,}1\cos\left(\frac{\pi}{2}\frac{y}{L}\right) - 0{,}1;\ T = 10, \gamma_1 = 1, \gamma_2 = 2 \cdot 10^5, \gamma_3 = 0$

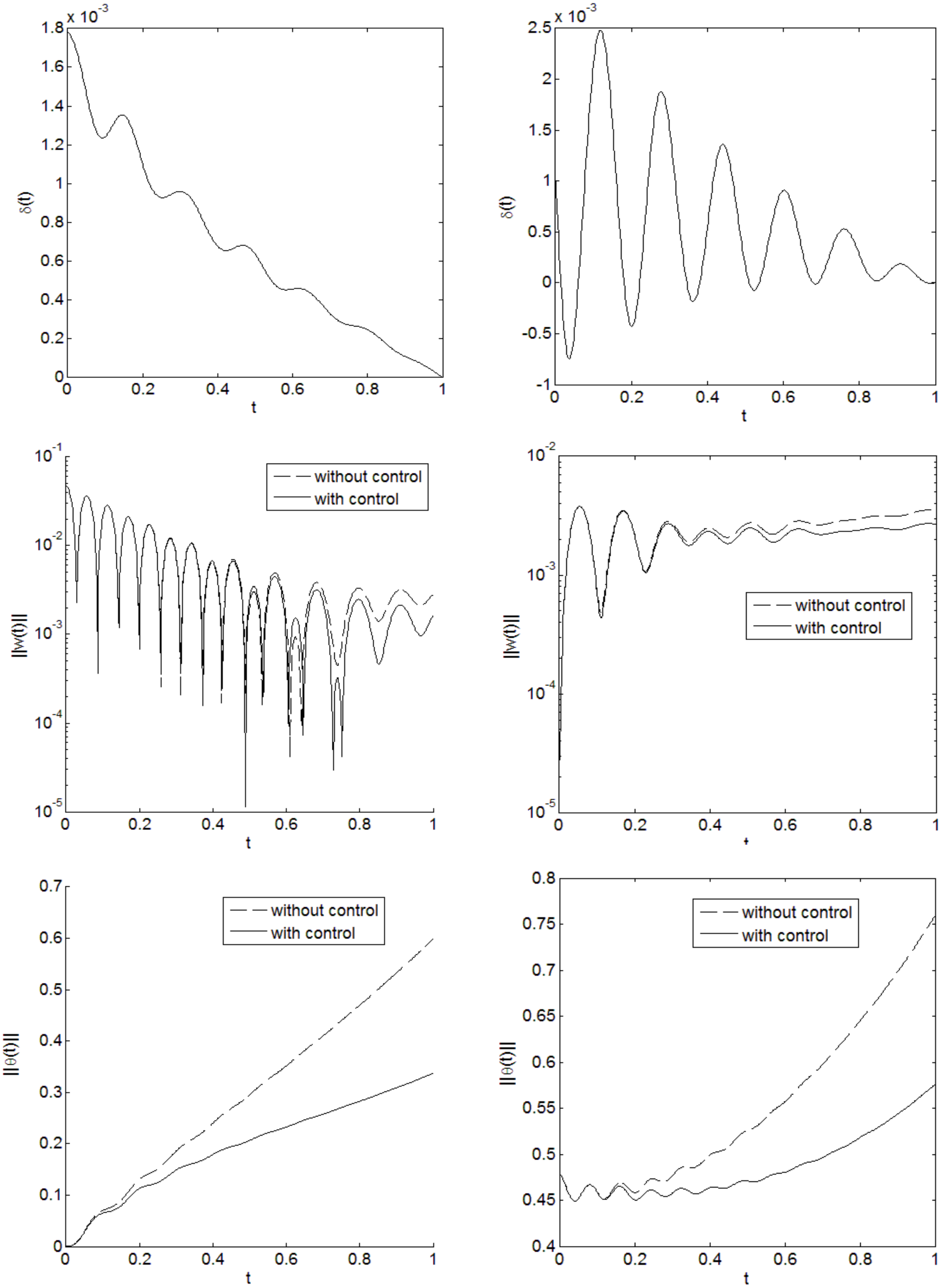


Fig. 4 *– Control law, $\|w(t)\|$ and $\|\theta(t)\|$ for the unstable case with the (a) and (b) initial conditions.*

$$(T = 1, \gamma_1 = 1, \gamma_2 = 0{,}8 \cdot 10^5, \gamma_3 = 0)$$

CONCLUSIONS

In this paper it has been developed a control model able to reduce the oscillations of a wing due to aerodynamic effects. The mathematical model used for the description of the problem is a linear dynamical system and the cost function chosen takes in account the evolution of the system, the energy of the control and the final state of the system itself; the algorithm, implemented in the Matlab environment, has been tested in a neighborhood of the stationary point in the stable and unstable cases. As one can see from the results the control is able to minimize the cost function for all the cases. In particular, in the stable case the control acts in order to reach in less time the stationary condition. In the unstable case the control is not able to stabilize the system, nevertheless this unstable behavior is delayed. This fact is probably due to the definition of the cost function, defined over a finite time range and containing both a system evolution term and an energy control term, and then the optimal condition of the control acts in order to have a compromise between this two terms.

Each term of the cost function is properly weighted through three coefficients; it is seen that, in order to have convergence, the weight of the energy control term has to be several order of magnitude greater than the system evolution term. Simulations not here reported, with the cost function taking in account only the final state of the system and the energy control term, has been performed and no appreciable differences with the previous cases were noticed.

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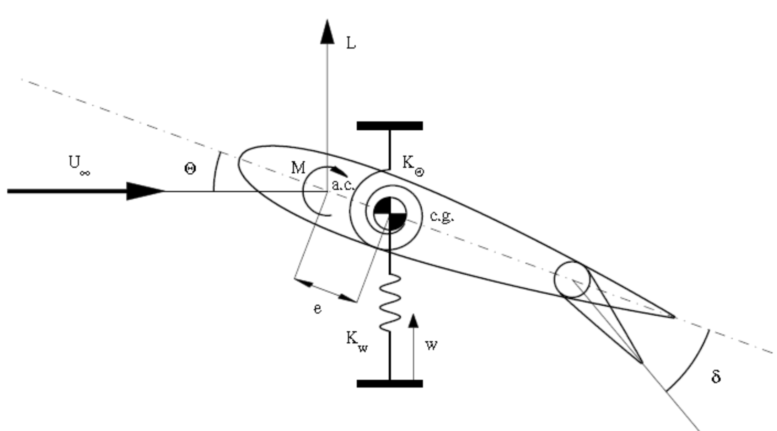

Fig. 5 *–Schematic model of the wing section.*

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OPTIMIZATION OF FISHING ACTIVITY AND REPOPULATION IN A SIMULATED MODEL

Alessandro Cavuoto

Abstract

Study of a model representing the growth of a coastal population under an external forcing.

The work first concerned the determination of the state equation governing the problem and the definition of the variables and parameters required to deal the problem.

Then has been done the analysis of the problem using the Lagrange operators method in order to obtain the fundamental equations to write down the model's code.

Once discretized the equations and defined the fundamentals matrices has been possible to implement the code and use it to simulate different dynamic situations of a coastal population growth, with and without an external forcing.

It resulted that the optimization code enables to find the optimal fishing/repopulation vector which guarantee the survival of the species.

Introduction

Today resource management is fundamental in every economic sector; energy, money, food, in a industry as much as in a natural environment.

Recently the worldwide demand for goods resulted in a critical unbalance in different sectors, leading sometimes to an environment and resources abuse which would results in a resources exhaustion.

This is the case of fishing, where the global demand of fish, crustaceans and other edible species has almost become a serious threat to the survival of several species in different places around the world.

A reckless fishing will also destroy natural habitats and will get a strong interference in the natural processes that affect the life cycle of the species, resulting thus in a destructive external forcing that must be adjusted to avoid the irreversible erasing of one or several species.

In order to establish a correct fishing way, it is of primary importance define a maximum fishing rate for each moment in a time period typical for a life cycle of a determined species.

This evaluation must be done after to have determined the natural life cycle of the species without any disturbance, in order to compare the natural evolution with the fishing forcing that can be tolerated by the marine populations.

In this work we tried to do something like this, determining a analytical model which could represent the dynamic of the population growth for some species distributed along a coastline interested by a considerable along-shore current.

The target was to find the optimal forcing vector to maintain a sufficient abundance level in the populations in order to allow the survival of the species.

Methods

First of all we started with the analysis of the growth dynamic for a generic population of species.

The situation requires the division in number j sectors of the analyzed coastline, corresponding to number j sub-populations.

Each of those populations is connected to the other with different weights, so we can represent this connection net by a connectivity matrix which reports the relationship between each site.

This matrix has a Gaussian distribution along its rows but due to the nature of the intense along-shore current the matrix must be asymmetric [1].Figure 1 shows an example.

The population growth is also affected by the natural mortality which can be represented with a diagonal matrix, by a density dependent settlement rate expressed as a matrix and at last by the fishing rate.

The relations between those factors can be written according to the following state equation which governs the problem:

$$\frac{d\underline{n}}{dt} = \left(\underline{\underline{K}}(t)\underline{\underline{S}}(n) - \underline{\underline{M}} - \underline{\underline{F}} \right)\underline{n} \qquad [1]$$

where

$\underline{\boldsymbol{n}}$ is the state vector containing the number of individuals in each population,

$\underline{\underline{\boldsymbol{K}}}(\boldsymbol{t})$ is the time dependent dispersal matrix, which defines the probability of competent larval delivery to each of j local populations per unit time per local adult at time t. Its diagonal therefore represents the level of self-recruitment of the populations. Furthermore it is composed by a constant part, $\underline{\underline{\boldsymbol{K}}}\boldsymbol{o}$, and by a time variable part, $\gamma\underline{\underline{\boldsymbol{K}}}_{\boldsymbol{t}}(\boldsymbol{t})$ normally distributed, such that $< \underline{\underline{\boldsymbol{K}}}_{\boldsymbol{t}}(\boldsymbol{x},\boldsymbol{y}) > = \boldsymbol{0}$ and

$< \underline{\underline{\boldsymbol{K}}}_{\boldsymbol{t}}(\boldsymbol{x},\boldsymbol{y}) >^2 = \boldsymbol{1}$.

$\underline{\underline{\boldsymbol{S}}}(\boldsymbol{n})$ is the density dependent settlement rate, which is expressed as

$$\underline{\underline{S}}(n) = \underline{\underline{I}} - \underline{\underline{\Sigma}}(n) \qquad [2]$$

where $\underline{\underline{\boldsymbol{\Sigma}}}(\boldsymbol{n})$ = diag(n1/N1 n2/N2 … nj/Nj) , diag(…) denotes a matrix with elements along the diagonal and zeros elsewhere and Nj is the maximum abundance in population j.

$\underline{\underline{\boldsymbol{M}}}$ is the mortality rate matrix expressed as diag(m1 m2 … mj) where mj are the local mortality rates per unit time and

$\underline{\underline{\boldsymbol{F}}}$ is the fishing rate matrix, diag(f1 f2 … fj).

Since the equation [1] shown before is not linear we must linearize it to be able to deal with it.

After linearization we have:

$$\frac{d\underline{\hat{n}}}{dt} = \left(\underline{\underline{K}} - 2\underline{\underline{K}}\,\underline{\underline{\Sigma}}(\underline{n}^*) - \underline{\underline{M}} - \underline{\underline{F}}\right)\underline{\hat{n}} \qquad [3]$$

where $\underline{\hat{\boldsymbol{n}}}$ is the perturbation defined as

$$\underline{\hat{n}} = \underline{n} - \underline{n}^* \qquad [4]$$

with $\underline{\boldsymbol{n}}^*$ solution of equation [1].

The purpose of this work is to obtain an optimal fishing rate which enables the survival of the marine population allowing at the same time the fishing in that place.

So the new state equation [3] has been decomposed as follows in order to define a forcing vector usable in the next analysis steps. What we have now is thus:

$$\frac{d\underline{\hat{n}}}{dt} = \left(\underline{\underline{K}} - 2\underline{\underline{K}}\,\underline{\underline{\Sigma}}(\underline{n}^*) - \underline{\underline{M}}\right)\underline{\hat{n}} - \underline{\underline{F}}\,\underline{\hat{n}} \qquad [5]$$

which can be written so:

$$\frac{d\underline{\hat{n}}}{dt} = \left(\underline{\underline{K}} - 2\underline{\underline{K}}\,\underline{\underline{\Sigma}}(\underline{n}^*) - \underline{\underline{M}}\right)\underline{\hat{n}} - \underline{f} \qquad [6]$$

$$\frac{d\underline{\hat{n}}}{dt} = \underline{\underline{A}}\,\underline{\hat{n}} - \underline{f} \qquad [7]$$

where $\underline{\boldsymbol{f}} = \underline{\underline{F}}\,\underline{\hat{n}}$ is the forcing of the problem.

Now we can start the determination of the required equations for the code writing; using the La Grange operators method we will find the already known state equation, the adjoint equation, the initial conditions and the optimized forcing vector.

We define our output as

$$J = \frac{\gamma_1}{2}\left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right)^T \left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right) + \frac{\gamma_2}{2}\int_0^T \underline{f}^T\,\underline{f}\,dt \qquad [8]$$

where $\underline{\hat{n}}_T$ is the target, i.e. the value desired as final state vector of the populations.

Rewriting the equation states in that way

$$F = \frac{d\underline{\hat{n}}}{dx} - \underline{\underline{A}}\,\underline{\hat{n}} + \underline{f} = 0 \qquad [9]$$

our LaGrangian is

$$L\left(\underline{\hat{n}}, \underline{f}, \underline{a}, \underline{b}\right) = J - \int_0^T \underline{a}F\,dt - \underline{b}\left(\underline{\hat{n}}(0) - \underline{\hat{n}}_0\right) =$$

$$= \frac{\gamma_1}{2}\left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right)^T\left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right) + \frac{\gamma_2}{2}\int_0^T \underline{f}^T\,\underline{f}\,dt - \int_0^T \underline{a}\left(\frac{d\underline{\hat{n}}}{dx} - \underline{\underline{A}}\,\underline{\hat{n}} + \underline{f}\right)dt - \underline{b}\left(\underline{\hat{n}}(0) - \underline{\hat{n}}_0\right) \quad [10]$$

Proceeding, placing the derivatives of L respect to $\underline{\hat{n}}, \underline{f}, \underline{a}$ *and* $\underline{b}$ equal to zero we obtain the following results in the order:

$$\frac{dL}{d\underline{a}} = 0 \Leftrightarrow -\int_0^T \delta\underline{a}\left(\frac{d\underline{\hat{n}}}{dx} - \underline{\underline{A}}\,\underline{\hat{n}} + \underline{f}\right)dt = 0 \Leftrightarrow \frac{d\underline{\hat{n}}}{dt} - \underline{\underline{A}}\,\underline{\hat{n}} + \underline{f} = 0 \qquad \textit{state equation} \qquad [11]$$

$$\frac{dL}{d\underline{b}} = 0 \Leftrightarrow -\delta\underline{b}\left(\underline{\hat{n}}(0) - \underline{\hat{n}}_0\right) = 0 \Leftrightarrow \underline{\hat{n}}(0) = \underline{\hat{n}}_0 \qquad \textit{initial condition} \qquad [12]$$

$$\frac{dL}{d\underline{\hat{n}}} = 0 \Leftrightarrow \gamma_1\left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right)\delta\underline{\hat{n}} - \left[\underline{a}(t)\delta\underline{\hat{n}}(t)\right]_0^T - \int_0^T \left(-\frac{d\underline{a}}{dt} - \underline{\underline{A}}^T\underline{a}\right)\delta\underline{\hat{n}}\,dt - \underline{b}\delta\underline{\hat{n}}_0 = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left[\gamma_1\left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right) - \underline{a}(T)\right]\delta\underline{\hat{n}} - \int_0^T \left(-\frac{d\underline{a}}{dt} - \underline{\underline{A}}^T\underline{a}\right)\delta\underline{\hat{n}}\,dt - \left[\underline{a}(0) - \underline{b}\right]\delta\underline{\hat{n}}_0 = 0 \qquad [13]$$

which gives

$$\underline{a}(T) = \gamma_1\left(\underline{\hat{n}}(T) - \underline{\hat{n}}_T\right) \qquad [14]$$

$$-\frac{d\underline{a}}{dt} = \underline{\underline{A}}^T\underline{a} \qquad \textit{adjoint equation} \qquad [15]$$

$$\underline{b} = \underline{a}(0) \qquad [16]$$

$$\frac{dL}{d\underline{f}} = 0 \Leftrightarrow \gamma_2\int_0^T \underline{f}^T\,\delta\underline{f}\,dt - \int_0^T \underline{a}^T\,\delta\underline{f}\,dt = 0 \Leftrightarrow \int_0^T \left(\gamma_2\underline{f} - \underline{a}\right)\delta\underline{f}\,dt = 0 \Leftrightarrow$$

$$\Leftrightarrow \gamma_2\underline{f} = \underline{a} \Leftrightarrow \underline{f} = \frac{\underline{a}}{\gamma_2} \qquad \textit{optimized forcing} \qquad [17]$$

Next step is the discretization of all equations found above. Since in the code implementation have been used many time steps, resulting in very small time intervals, we used the explicit method sure to keep the system stable. However, the stability check shown in fig 2 shows the perfect stability characterizing the system.

Discretizing the state equation we have

$$\frac{\underline{\hat{n}}^{i+1} - \underline{\hat{n}}^{i}}{\Delta t} = \underline{\underline{A}}\,\underline{\hat{n}}^{i} - \underline{f}^{i} \qquad [18]$$

which treated becomes

$$\underline{\hat{n}}^{i+1} = \underline{\underline{B}}\,\underline{\hat{n}}^{i} - \underline{f}^{i}dt \qquad [19]$$

$$with \qquad \underline{\underline{B}} = \left(\underline{\underline{I}} + dt\underline{\underline{A}}\right) \qquad [20]$$

Equation [19] must be put in the code loop for the calculation of the integral from 0 to T with

$\underline{\hat{n}}^{i=0} = \underline{\hat{n}}_0 = \underline{\hat{n}}(0)$ as initial condition.

For the adjoint instead we have:

$$\underline{a}^{i} = \underline{\underline{B}}^{T}\,\underline{a}^{i+1} \qquad [21]$$

which, inserted in the loop for the calculation of the integral from T to 0 with eq.[14] (which discretized is $\underline{\hat{a}}^{N} = \gamma_1\left(\underline{\hat{n}}^{N} - \underline{\hat{n}}_T\right)$ [22]) as initial condition leads to the $\underline{\hat{a}}(0)$ evaluation, essential to determine the optimized forcing vector, according to equation [17].

This method includes adjoint, whose error can be appreciated, already discretized, as

$$error = \left|\left(\underline{a}^{N}\right)^{T}\underline{\hat{n}}^{N} - \left(\underline{a}^{0}\right)^{T}\underline{\hat{n}}^{0} + \left(\frac{1}{\gamma_2}\right)\sum_{i=0}^{N-1}\left(\left(\underline{a}^{i+1}\right)^{T}\underline{a}^{1}dt\right)\right| \qquad [22]$$

Now is possible to write down the code for the simulation of the growth dynamic of the marine species living in a generic coastline. We try to achieve the species survival acting on the output reported in equation [8] choosing an appropriate value of $\underline{\hat{n}}_T$.

Results and discussions

What it results from the many simulations performed is the survival of all the populations after an established time T, although the growth trend is often decreased by the acting forcing, interpreted as a fishing action.

The particular structure of the output allows us to interpret the external forcing vector $\underline{f}$ not only as a fishing rate, but also as a repopulation forcing when, during the simulation running, a population reaches a critical value or takes a descendent growth trend.

The magnitude and the sign of the forcing vector components also depend on the initial condition $\underline{\hat{n}}_0$ and the target value $\underline{\hat{n}}_T$.

For example if we start from a low value of $\underline{\hat{n}}_0$ and we have relatively high value for $\underline{\hat{n}}_T$, i.e. higher than the final value $\underline{\hat{n}}\,(T)$ of the system without forcing, the resulting forcing will be a repopulation vector whose components will have negative sign and magnitude proportional to the difference between initial condition and target value. Figure y show it.

Instead, for all cases with $\underline{\hat{n}}_T$ sufficiently low compared to the unforced value of $\underline{\hat{n}}\,(T)$, the external forcing will simply result as a fishing vector with the most of its components with a positive sign.

In figures 4.1-4.2 we can observe the trend of a forcing vector and of its components for a simulation run with a output target set at 25% of the maximum abundance.

Therefore the code performed managed to find the optimized fishing condition for a determined environment guarantying the survival of the species which live in it. It also verifies the validity of the optimization method in such a kind of analysis.

Asymmetric Connectivity Matrix									
0,606531	1	0,606531	0,135335	0,011109	0,000335	3,73E-06	1,52E-08	2,29E-11	1,27E-14
0,135335	0,606531	1	0,606531	0,135335	0,011109	0,000335	3,73E-06	1,52E-08	2,29E-11
0,011109	0,135335	0,606531	1	0,606531	0,135335	0,011109	0,000335	3,73E-06	1,52E-08
0,000335	0,011109	0,135335	0,606531	1	0,606531	0,135335	0,011109	0,000335	3,73E-06
3,73E-06	0,000335	0,011109	0,135335	0,606531	1	0,606531	0,135335	0,011109	0,000335
1,52E-08	3,73E-06	0,000335	0,011109	0,135335	0,606531	1	0,606531	0,135335	0,011109
2,29E-11	1,52E-08	3,73E-06	0,000335	0,011109	0,135335	0,606531	1	0,606531	0,135335
1,27E-14	2,29E-11	1,52E-08	3,73E-06	0,000335	0,011109	0,135335	0,606531	1	0,606531
2,58E-18	1,27E-14	2,29E-11	1,52E-08	3,73E-06	0,000335	0,011109	0,135335	0,606531	1
1,93E-22	2,58E-18	1,27E-14	2,29E-11	1,52E-08	3,73E-06	0,000335	0,011109	0,135335	0,606531

Figure 1: Asymmetric connectivity matrix. the values inserted in each cell report the connection level between the two corresponding sectors, for example in the cell identified by row 2 and column 5 is reported the connection between population 2 and 5.

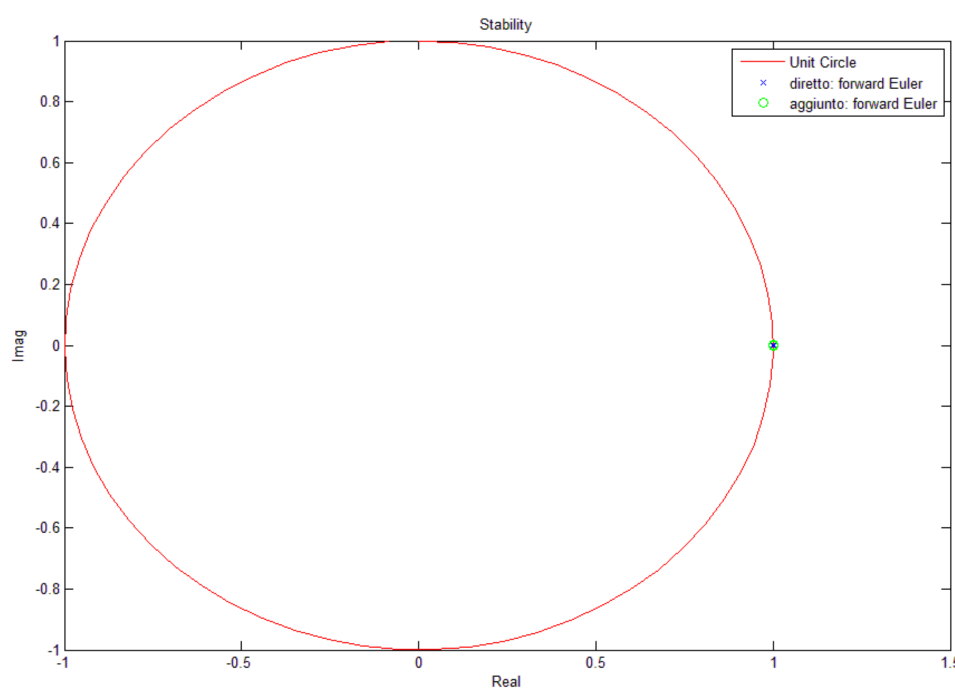

Figure 2: stability plot. It is shown a perfect system stability.

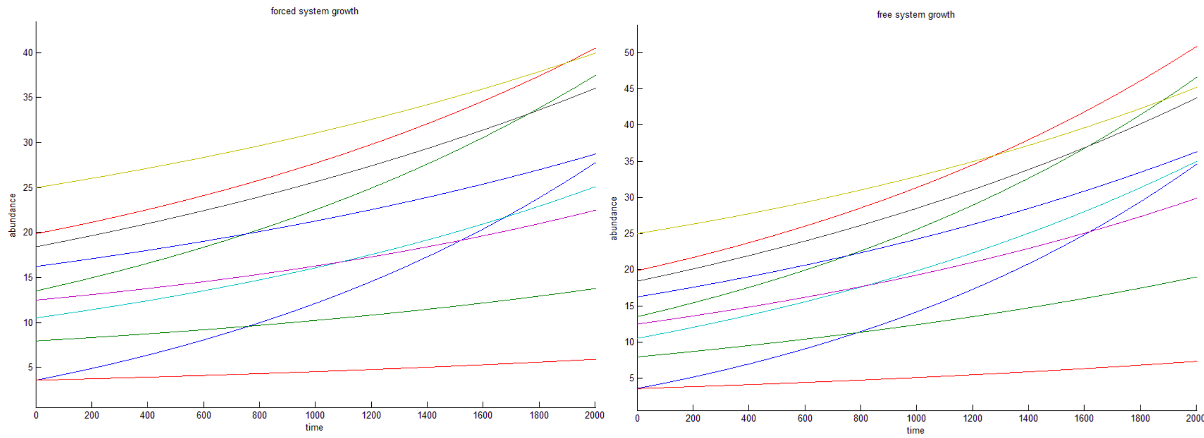

Figure 3.1: forced system growth compared with free system growth. Are plotted the different components of the state vector in function with time. It's visible an attenuation of the growth in the forced system.

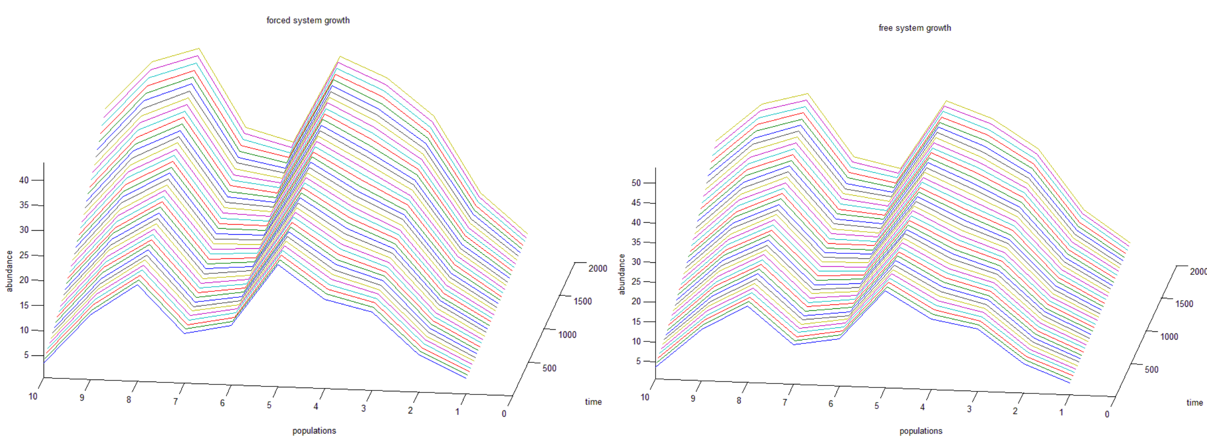

Figure 3.2: vector evolution in time for the forced system and for the free system.

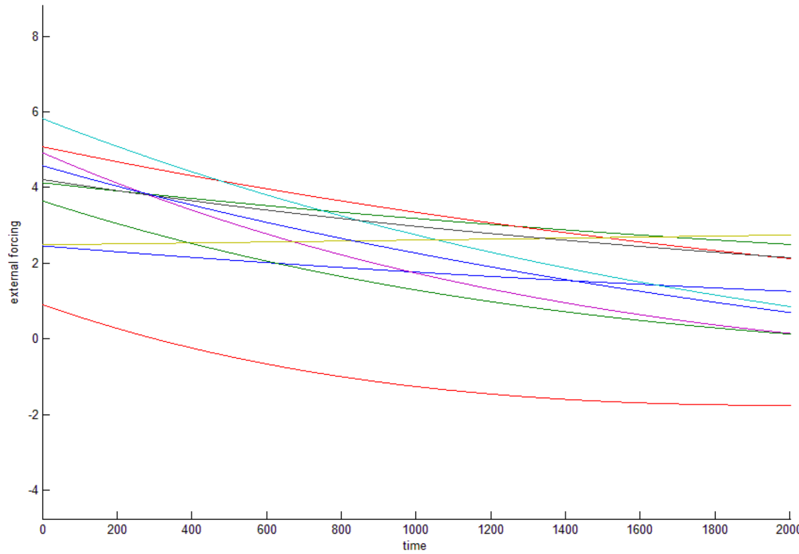

Figure 4.1: external forcing in function with time. It's evident a component which becomes negative, therefore interpretable as a repopulation component. All the other positive components are thus fishing rates.

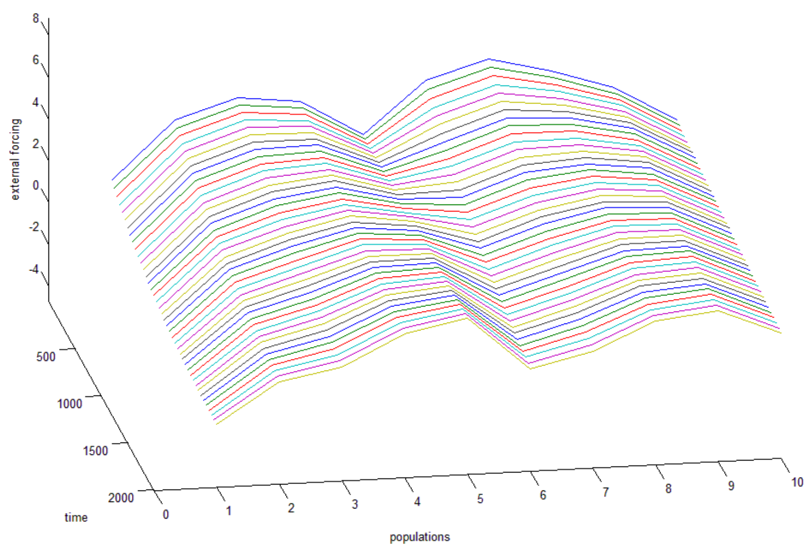

Figure 4.2: external forcing vector evolution in time.

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VALIDATION OF A POROELASTIC MODEL

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21/06/2012

Abstract

This studies aim, is to verify from a mathematical point of view the validity of a model for the infusion of a drug inside a cancer tissue. Avery important parameter has been optimised in the description of the phiscal fenomenon the hydraulic conductivity K.

The optimal value has been determined with Lagrange's approach. The function K was considered as the optimal value that better represent sperimental data. The results show how the optimised function has a reasonable tendency from a phisycal point of view and furthermore has a singular tendency to the one obtained in the model.

The model

In this reasearch we hypothize the solid tumor to have a spherical form.The changes (deformations) that the tumor undergoes are to be considered infinitesimal, for that reason the deformations are governed by Hooke's law (elastic field).

We study the transfer of the therapeudic agent inside the tumor considering a porelastic medium carachterized by hydraulic conductivity k of the tissue (given the relationship between dinamic viscosity and permeability of the medium) and by Lame's coefficient G and λ.

The relationship between the therapeutical agent and blood vessels is governed by starling's law:

$$\Omega = L_p \frac{S}{V}(p_e - p) \tag{1}$$

where:

L_p = conduttività vascolare $[\frac{cm}{mmHg \cdot s}]$
$\frac{S}{V}$ = sup erci e vasc olare p er unità di volume[$\frac{cm^2}{cm^3}$]
p_e = pressione vascolare eettiva[mmHg].
p= pressione interstiz iale[mmHg].
Ωcan act both as a well or as a source based on the difference of pressure inside or outside the blood vessel. the tumor is by nature strongly heterogeneous, we only consider it in its radial direction, which is expressed by the hydraulic conductivity of the tumor K.

another study has observed that conductivity k is strongly influenced by the deformation of the tumor.

The farmaceutical agent is introduced in the center of the tumor, creating a small radius (a) cavity, which its dimensions can be compared to the tip of the needle.

The general strength acting on the tumor would be:

$$\underline{\underline{T}} = \underline{\underline{\sigma}} - p\underline{\underline{I}} \tag{2}$$

$\underline{\underline{T}}$ =tensor of effective stress
$\underline{\underline{\sigma}}$= tensor of contact stress
p= interstizial fluid pressure (IFP)
considering that the tension deformation of the tumor is governed by hooke's law, throught the equation of the bond we get:

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \lambda(\nabla \cdot \overline{u})\underline{\underline{I}} + 2G[\frac{1}{2}((\nabla \cdot \overline{u}) + (\nabla \cdot \overline{u})^T)] \tag{3}$$

where
u=deformation of solid
$\lambda = \frac{2\nu}{1-2\nu}G$

dove

$\nu=$ is the Poisson's coefficient

assuming we found ourselves in stationary conditions, transforming the equation in cilindrical coordinates, and and remembering that all variables are hypotetically only expressed in accordance of the radial r coordinate, we obtain:

$$(2G+\lambda)\frac{d}{dr}(\frac{du}{dr}+\frac{2u}{r})=\frac{dp}{dr} \tag{4}$$

To be able to find the distribution of the pressure and deformation we need another equation. We have to consider the conservation of the mass:

$$\bar{\nabla}\cdot\bar{q}=\Omega \tag{5}$$

Please note that q has the direction and dimention of the velocity, But truthfully is not the actual velocity inside the pores, but it refers to the volumetric carrying for unit area.

The second meaning of55 represent, as already said, the relationship between the vascular net and the pharmaceutical agent during the evolution of this last one.

A fluid that evolves inside a pore is described by Darcy's law:

$$\bar{q}=-K\cdot\bar{\nabla}p \tag{6}$$

Taking into consideration the 1 and 5 becomes:

$$\bar{\nabla}\cdot(-K\cdot\bar{\nabla}p)=L_pS(p_e-p) \tag{7}$$

Taking into consideration the

$$-\frac{1}{r^2}\frac{d}{dr}(r^2K\frac{dp}{dr})=L_p\frac{S}{V}(p_e-p) \tag{8}$$

Regarding the hydraulic conductivity of the tumor it has been chosen to consider it as depending from the deformation with a semi-empiric exponential law:

$$K=K_0e^{M[\alpha\frac{du}{dr}+(1-\alpha)\frac{u}{r}]} \tag{9}$$

The equations are closed by the boundary conditions.

$$\begin{cases} p=0 & r=R' \\ p=p_{inf} & r=a' \\ \frac{du}{dr}+\frac{2\nu}{1-\nu}\frac{u}{r}=0 & r=R' \\ \frac{du}{dr}+\frac{2\nu}{1-\nu}\frac{u}{r}=0 & r=a' \end{cases}$$

where a' and R' are respectivly the inside and outside radiuous after the deformation.

Linearization of equations

The linearization of the equation is made through a perturbative analysis of the model which functions in accordance to a characteristic parameter of the problem.

$p^*=\frac{p}{p_{inf}-pe}$; $\underline{\underline{T}}^*=\frac{\underline{\underline{T}}}{2G+\lambda}$; $r^*=\frac{r}{R}$; $u^*=\frac{u}{r}$;
$K^*=\frac{K}{K_0}$;
and the equations become:

$$\frac{d}{dr^*}(\frac{du^*}{dr^*}+\frac{2u^*}{r^*})=\delta\frac{dp^*}{dr^*} \tag{10}$$

$$\frac{d}{dr^*}(r^{*2}K^*\frac{dp^*}{dr^*})=r^{*2}\gamma^2(p^*-p_e^*) \tag{11}$$

$$K^*=e^{M[\alpha\frac{du^*}{dr^*}+(1-\alpha)\frac{u^*}{r^*}]} \tag{12}$$

having $\gamma^2=\frac{L_p}{K_0}\frac{S}{V}R^2$ e $\delta=\frac{p_{inf}-p_e}{2G+\lambda}$

Its around the adimensional value of δ that the linearization is executed and stopping at the first order we have:

$$u^*=u_0^*+u_1^*\delta$$

$$K^*=K_0^*+K_1^*\delta$$

$$p^*=p_0^*+p_1^*\delta$$

Please note that we are taking into account the hypothesis of small movements and therefore we can express the exponential as the development in series of mclaurin stopping at the first order

$$K^* = 1 + M[\alpha\frac{du^*}{dr^*} + (1-\alpha)\frac{u^*}{r^*}]$$

Now lets move on to express the conditions on the boundaries of points $\frac{a}{R}$ and $\frac{R}{R}$assuming we are moving linearly from point $\frac{a}{R}$ until $\frac{a'}{R}$ and from 1 to $\frac{R'}{R}$.

For example $p(a') = p_{inf} \backsimeq p(a) + \frac{dp(a)}{dr}(a'-a)$ with $a' - a = u(a)$.

We can now write our equations in orderδ^0:

$$\begin{cases} \frac{d}{dr^*}(\frac{du_0^*}{dr^*} + \frac{2u_0^*}{r^*}) = 0 \\ \frac{d}{dr^*}(r^{*2}K_0^*\frac{dp_0^*}{dr^*}) = r^{*2}\gamma^2(p_0^* - p_e^*) \\ K_0^* = 1 + M[\alpha\frac{du_0^*}{dr^*} + (1-\alpha)\frac{u_0^*}{r^*}] \end{cases}$$

c.c.

$$\begin{cases} p_0^* = 0 & r^* = 1 \\ p_0^* = 0 & r^* = \frac{a}{R} \\ \frac{du_0^*}{dr^*} + \frac{2\nu}{1-\nu}\frac{u_0^*}{r^*} = 0 & r^* = 1 \\ \frac{du_0^*}{dr^*} + \frac{2\nu}{1-\nu}\frac{u_0^*}{r^*} = 0 & r^* = \frac{a}{R} \end{cases}$$

It can also be demonstrated that$u_0^* = 0 \Longrightarrow K_0^* = 1$. Even $p_0$has an analytical answer such as:

$$p_0^* = p_e^* + \frac{A}{r^*}e^{\gamma r^*} + \frac{B}{r^*}e^{-\gamma r^*}$$

with A and B known from boundary conditions

At the order δ^1 the equations are:

$$\begin{cases} \frac{d}{dr^*}(\frac{du_1^*}{dr^*} + \frac{2u_1^*}{r^*}) = \frac{dp_0^*}{dr^*} \\ \frac{d}{dr^*}(r^{*2}K_0^*\frac{dp_0^*}{dr^*}) = r^{*2}\gamma^2 p_1^* \\ K_1^* = 1 + M[\alpha\frac{du_1^*}{dr^*} + (1-\alpha)\frac{u_1^*}{r^*}] \end{cases}$$

c.c.

$$\begin{cases} p_1^* = -u_1^*\frac{dp_0^*}{dr^*} & r^* = 1 \\ p_1^* = -u_1^*\frac{dp_0^*}{dr^*} & r^* = \frac{a}{R} \\ \frac{du_1^*}{dr^*} + \frac{2\nu}{1-\nu}\frac{u_1^*}{r^*} = 0 & r^* = 1 \\ \frac{du_1^*}{dr^*} + \frac{2\nu}{1-\nu}\frac{u_1^*}{r^*} = 0 & r^* = \frac{a}{R} \end{cases}$$

The carrying that goes through the tumor in the non linear model has value (adimensional):

$$Q^* = 4\pi r^{*2}q^*$$

In the linear case it would be

$$Q^* = 4\pi r^{*2}(q_0^* + \delta q_1^*) \quad (13)$$

where

$$q_0^* = -\frac{dp_0^*}{dr^*}$$

$$q_1^* = -K_1^*\frac{dp_0^*}{dr^*} - \frac{dp_1^*}{dr^*}$$

Optimal K_1 determination

why K_1?

It has been decided to optimize the conductivity of the hydraulic mean K for essentially two reasons:

- - The relationship between K and the deformation u is crucial for it's use in the poroelastic theory. It is as a matter of fact the mainly responsible for the interation between fluid and pharmaceutical

- - Both in literature and in our model it is still not clear how to combine K with the deformation u.

This study does not doubt the exponential relationship that goes on between deformation and conductivity, but wants to verify if the exponential curve that better approximates the experimental values (strictly determined by a mathematical method) is qualitatively similar to the one chosen in our model(determined with a physical-empiric method).

The experimental data reported in the 1were taken from the work of McGuire et al. and for each value of p_{inf} (37,52,69 $mmHg$) ... is calculated Q_{inf} media ($Q^I = 0.15$,$Q^{II} = 1.4$,$Q^{III} = 0.35$).

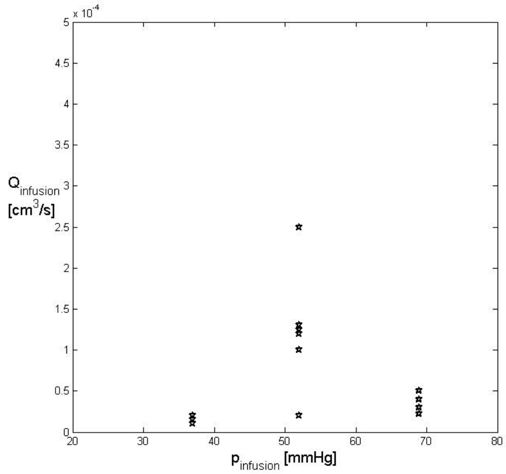

Figure 1: experimental values

Lagrangian approach

To make it easier from now on we will omit $*$remembering that all measurements are adimensional

From the optimization we have K_{ott} for every value of Q_{inf}.

For this reason the size that has to be minimized has been expressed as follows:

$$J = (Q(a') - Q^I)^2 + \frac{\xi}{2}\int_a^1 K_1^2 dr \qquad (14)$$

The optimization has been done for the linear equations stopping at the 1 order, furthermore, for reasons tied to the method of functionality ... is expressed as follows

$$Q(a') = \int_a^1 Q(r)\delta_D dr \qquad (15)$$

where δ_D is a known function *delta di Dirac.*

Taking into account that 13 13 and remembering that 14 linearization has stopped on order 1 the 14 becomes :

$$J = \chi_0 + \int_a^1 (\chi_1 \frac{p_1}{dr} + \chi_2 K_1)\delta_D dr + \frac{\xi}{2}\int_a^1 K_1 dr \quad (16)$$

where:

- $\chi_0 = (Q^I)^2 + 16\pi^2 a'^4 (\frac{dp_0}{dr})^2 + 8Q\pi a'^2 \frac{dp_0}{dr}$
- $\chi_1 = 32\pi^2 \frac{dp_0}{dr}\epsilon r^4 + 8Q^I \pi r^2 \epsilon$
- $\chi_2 = 32\pi^2 \epsilon r^4 (\frac{dp_0}{dr})^2 + 8Q^I \pi r^2 \epsilon \frac{dp_0}{dr}$

We can now define the lagrangiana definition as follows:

$\mathcal{L}(p_1, K_1, \alpha, b, c) = J - \int_a^1 \alpha \cdot (\frac{d^2 p_1}{dr} + \frac{2}{r}\frac{dp_1}{dr} - \gamma^2 p_1 + (\frac{2}{r}\frac{dp_0}{dr} + \frac{d^2 p_0}{dr^2})K_1 + (\frac{dp_0}{dr})(\frac{dK_1}{dr}))dr - b \cdot (p_1(a) + u_1(a)\frac{dp_0(a)}{dr}) - c \cdot (p_1(1) + u_1(1)\frac{dp_0(1)}{dr})$

where α, b, c are moltiples of Lagrange.

imposing:

$$\nabla\mathcal{L} = 0 \qquad (17)$$

We obtain the conditions necessary for the resolution of the problem. Therefore we have:

- $\frac{\partial\mathcal{L}}{\partial p_1} = 0$
  $\frac{d^2\alpha}{dr} - \frac{2}{r}\frac{d\alpha}{dr} - \gamma^2\alpha = -8\pi\epsilon\frac{d}{dr}((4\pi\frac{dp_0}{dr}r^4 + Q^I r^2)\delta_D)$
  $\alpha(a) = 0$
  $\alpha(1) = 0$
  $b = \frac{d\alpha}{dr}(a))$
  $c = -\frac{d\alpha}{dr}(1)$
- $\frac{\partial\mathcal{L}}{\partial K_1} = 0$
  $K = \frac{1}{\xi}[-8\pi\epsilon\frac{d}{dr}((4\pi\frac{d^2 p_0}{dr^2}r^4 + Q^I\frac{dp_0}{dr}r^2)\delta_D) + \alpha(\frac{2}{r}\frac{dp_0}{dr} + \frac{d^2 p_0}{dr^2}) - \frac{d}{dr}(\alpha\frac{dp_0}{dr})]$
- $\frac{\partial\mathcal{L}}{\partial\alpha} = 0$
  $\frac{d^2 p_1}{dr} + \frac{2}{r}\frac{dp_1}{dr} - \gamma^2 p_1 + (\frac{2}{r}\frac{dp_0}{dr} + \frac{d^2 p_0}{dr^2})K_1 + (\frac{dp_0}{dr})(\frac{dK_1}{dr}) = 0$
- $\frac{\partial\mathcal{L}}{\partial b} = 0$
  $p_1(a) = -u(a)\frac{dp_0(a)}{dr}$
- $\frac{\partial\mathcal{L}}{\partial c} = 0$
  $p_1(1) = -u_1(1)\frac{dp_0(1)}{dr}$

please note that:

- Deriving from p_1 we are able to determine the three multiplicators of lagrange α, b, c
- - Deriving in respect to K_1 we obtain $K_{1,ott}$

- deriving in repect to α we obtain the direct

- deriving in respect to$b$ e c we obtain the initial conditions of the direct

Discretization of the method

The equation system generated from the 15 becomes discretized by an explicit method at the second order

Results

As stated before, we face the problem separately for each of the three values of the infusion range.

Given the obvious impossibility to impose a punctual condition on the function it has become necessary the introduction of a gaussiana function such as $f = C(\theta)e^{-\theta r^2}$(delta di Dirac approssimato) ....which would have the effect of blocking the information on all of r given different weights for every point. In this way, acting onθ We can choose the strength of our control. Obviously according to the choices made on δ_D the calculus grid must be modified. To make the calculations easier, the study will be executed with a $\delta_D$the most ample possible t.c. $\int_a^1 \delta_D dr$ which doesn't have to move from the unit for more than a variation of 1.5%.

Finally, for a particular case we must demonstrate the convergence of the results at the varying of δ_D As long as the grid was chosen accurately. In this case study will be shown 3 different cases with a δ_D ever more forced.

$\int_a^1 \delta_{D_1} dr = 0.9831$; $\int_a^1 \delta_{D_2} dr = 0.9999$; $\int_a^1 \delta_{D_3} dr = 1$.

δ_{D_1}:

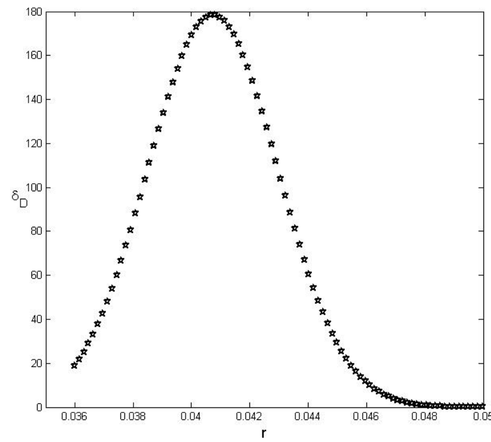

Figure 2: first case and 6000 points

If we know decided to keep on using a grid of 6000 points even forδ_{D_3}we would obtain something like this:

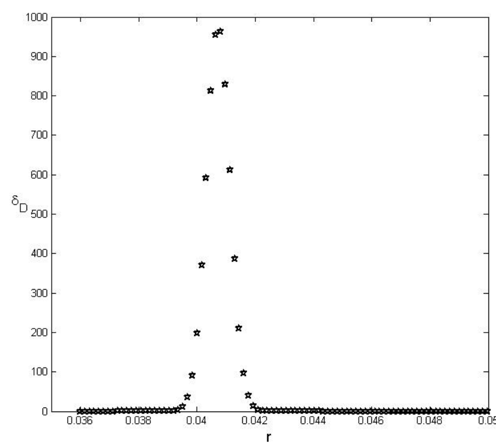

Figure 3: third case and 6000 points

It is clear that if we wish to keep on using this grid we would have to raise the number of points on the grid. The following will show a grid of 10000 points:

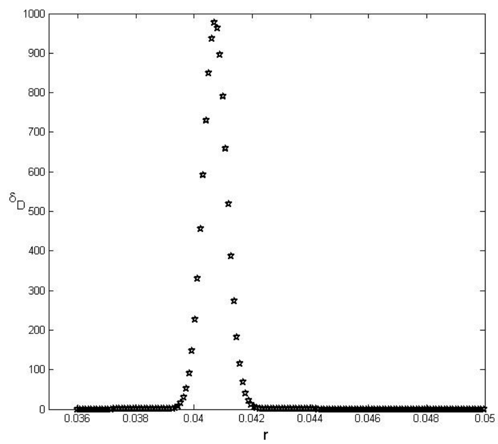

Figure 4: fourth case and 10000 points

Whileδ_{D_2}has the following form (grid of 8000)

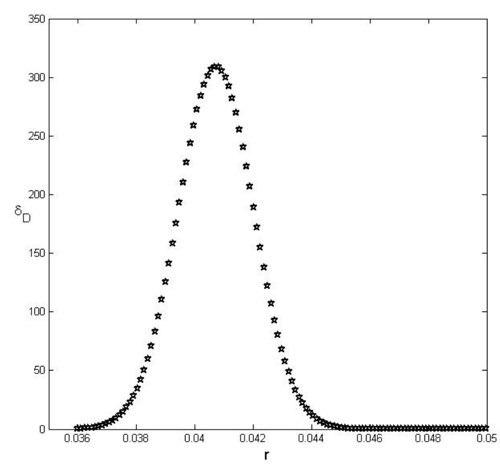

Figure 5: second case and 8000 points

The optimization is obviously strongly influenced by the freedom that is given to the control. The parameter responsible for such choice is ξ,the more it assumes small values the more the control is free to go its own way despite the energy spent to make all this possible.

In the figure below are reported the values of the solutions not optimized and not linearized for each of the three experiments (Q^I, Q^{II}, Q^{III}) confronted with the values of the optimization at the change of the parameter ξ:

Case 1 (Q^I)

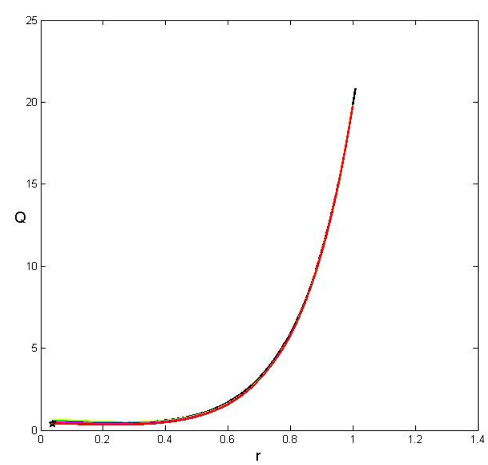

Figure 6: Q for different values of ξ

Which enlarged in the point of interest shows the validity of the optimization:

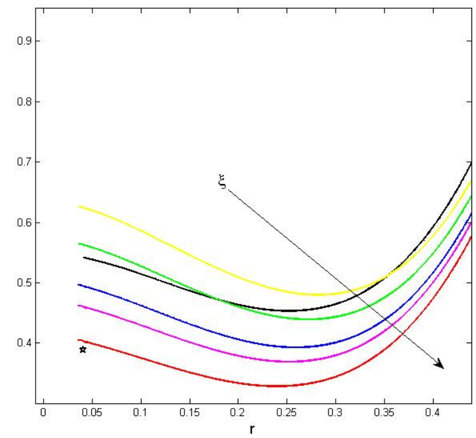

Figure 7: zoom

The black curve represents the non linear solution the others have a value of ξ respectively of $1, 0.1, 0.05, 0.04, 0.03$.

The figure beneath, is in reference to the different hydraulic conductivity of the mean K:

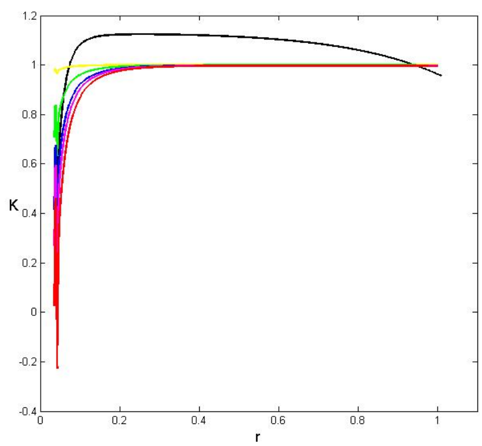

Figure 8: K for different values of ξ

Which enlarged in the point of interest:

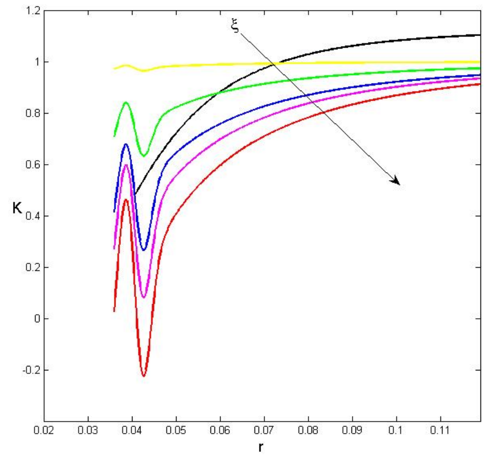

Figure 9: zoom

Beneath are reported the values taken by J at the varying of the control parameter ξ:

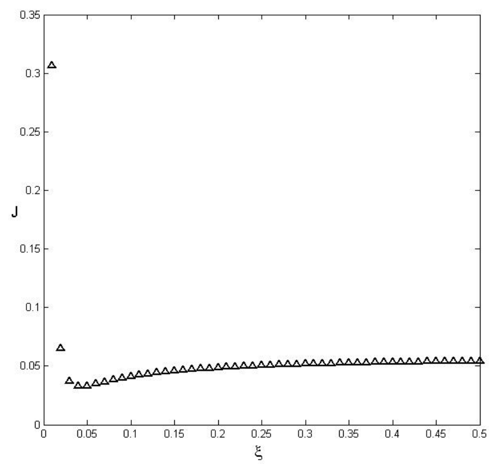

Figure 10: J for different values of ξ

In the end, to verify if the chosen grid is sufficient the values are reported for one of the cases calculated above before, with the 6000 point grid and then with a 1000 point grid

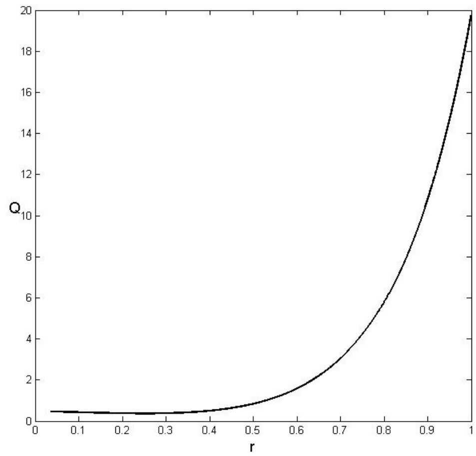

Figure 11: conversetion

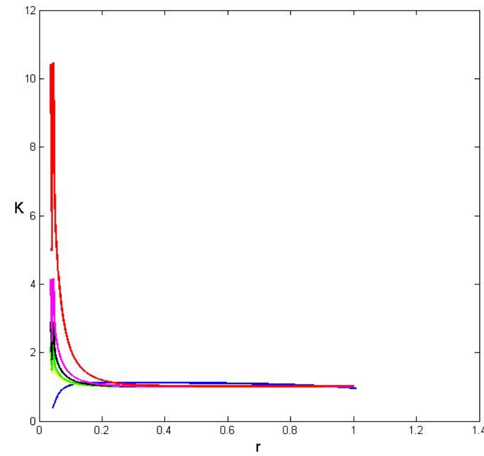

Figure 13:

The problem has now reached perfect conversion.

Case 2 Q^{II}

We report in the same order of the first case the results obtained.

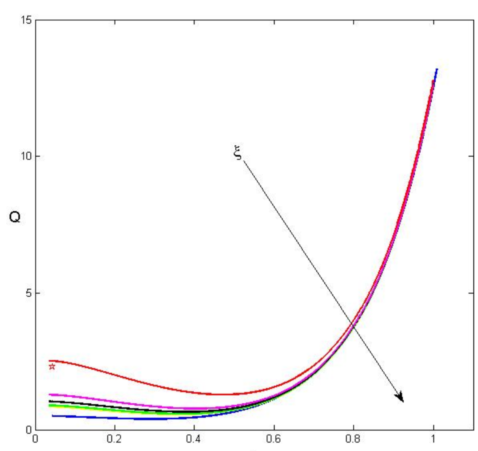

Figure 12:

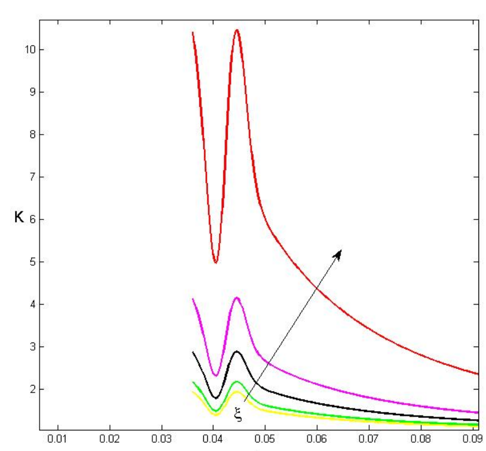

Figure 14:

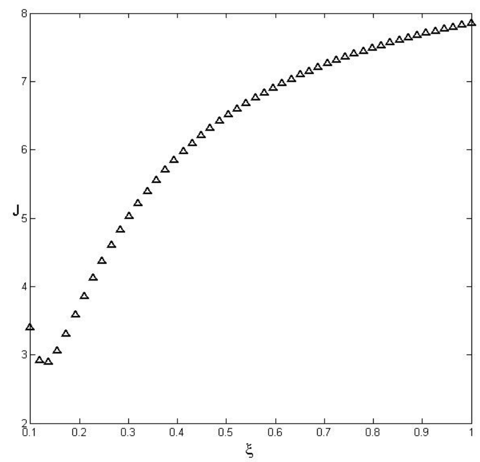

Figure 15:

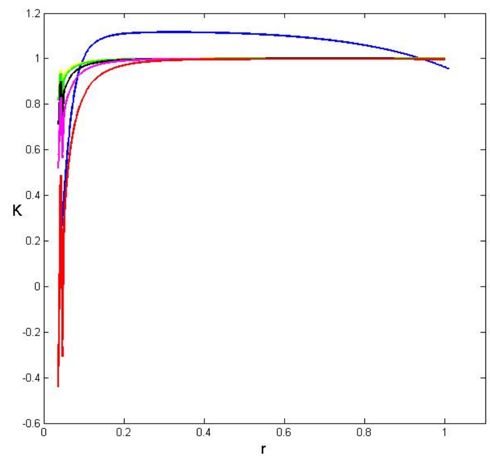

Figure 17:

case 3 (Q^{III})

The optimization in the third case has taken the following values:

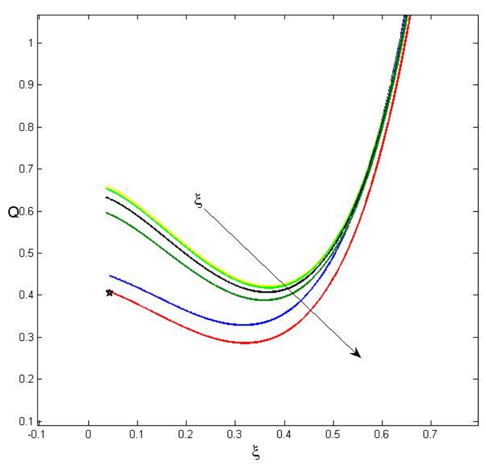

Figure 16:

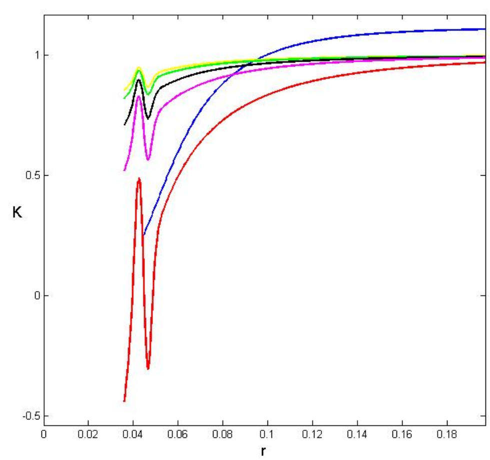

Figure 18:

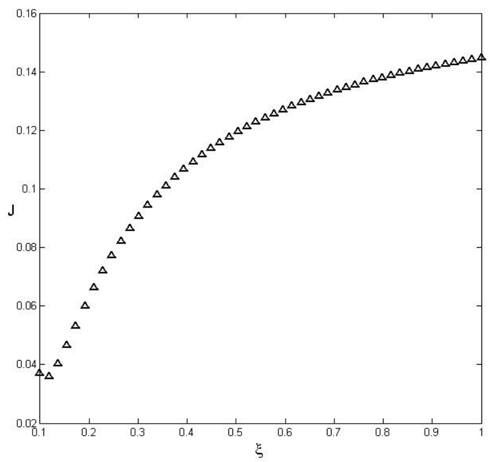

Figure 19:

Convergence

The graphic shows the convergence of the compute when you increase the dots on the grid:

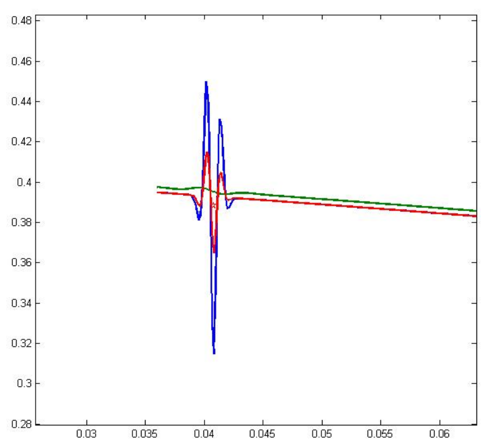

Figure 20:

Optimal control of a non-homogeneous convective wave equation in a mono-dimensional resonator: a variational approach.

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September 24, 2012

Abstract

Low level of pollutants can be achieved by a lean and premixed burning. Unfortunately, these are the conditions causing the undesirable phenomenon of self-excited thermo-acoustic oscillations, responsible for inefficient burning and structural stresses so intense that they can lead to engine and combustor failure. The phenomena is well described by the non-homogeneous convective wave equation that, in its simplest application, could be written in a one dimensional space domain. The article wants to let the reader gain sensitivity on the effect of the heat released from a source located in bounded flow. A variational analysis will be performed to show the optimal time-dependence of the heat source in order to minimize the oscillations inside the resonator.

1 Introduction

Thermo-acoustic instabilities may occur whenever combustion takes place inside a resonator. The phase difference between heat release oscillations and pressure waves at the injection holds responsibility for the phenomenon, as described by Lord Rayleigh [1]. Strong vibrations at low frequencies may establish inside the resonator causing the humming phenomenon that irremediably affects the functioning and the efficiency of the system. A simple analysis on a mono-dimensional problem based on a variational approach is performed to find out the optimal shape of the heat release. Step by step derivation of the math is explicitly given as well as main set up of the MATLAB code.

2 The physical model and the equations

The following problem can be easily inferred from a combination of linearised conservation principles of mass, momentum and energy. It differs from the well known D'Alembert Wave Equation due to the presence of the material derivative in place of the ordinary time derivative in order to take into account a non zero superimposed mean flow. The source term is the material derivative of the heat release $Q(x,t)$. Since such a kind of energy transfer is usually represented by a flame or, in experimental set up, by an heated grid, its space dependency could not be represented by continuous functions and piece-wise functions are needed (heaviside $H[(x-a)(b-x)]$ or Dirac Delta $\delta(x-f)$). For the sake of clarity, and for easier derivation, we choose the step-function H. Nonetheless, thank to this choice it is possible to modify the thickness of the flame shrinking it to a flat sheet when $a=b$. Boundary conditions are chosen in order to model an open-ended duct in both inlet and outlet. Initial conditions are chosen between the easiest harmonic function.

$$\begin{cases} \dfrac{D^2p(x,t)}{Dt^2} - c^2\dfrac{\partial^2 p(x,t)}{\partial x^2} = \dfrac{DQ(x,t)}{Dt} & t>0, \quad 0<x<L, \quad c>0 \\ \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} & \\ Q(x,t) = \hat{Q}q(t)H\left[(x-a)\cdot(b-x)\right] & 0<a\leq b<L \\ p(0,t) = p(L,t) = 0 & \\ p(x,0) = \tilde{p}(x) = sin\left(\dfrac{n\pi x}{L}\right) & \\ \dfrac{Dp(x,0)}{Dt} = \dot{\tilde{p}}(x) = 0 & \end{cases} \tag{1}$$

3 The direct system

3.1 Continuous form

In order to cast the above described problem as follows:

$$[\mathbb{C}]_c\frac{\partial \mathbf{\Phi}(x,t)}{\partial t} + [\mathbb{A}]_c\mathbf{\Phi}(x,t) = [\mathbb{B}]_c\mathbf{q}(t),$$

the hereinafter proposed definition of $\mathbf{\Phi}(x,t)$ is introduced:

$$\mathbf{\Phi}(x,t) = \begin{Bmatrix} p \\ \dot{p} \end{Bmatrix}$$

where $\dot{p} = \frac{Dp}{Dt}$. The expression of the source term is:

$$\frac{DQ}{Dt} = \dot{q}(t)H\left[(x-a)\cdot(b-x)\right] + u(b+a-2x)q(t)\delta\left[(x-a)\cdot(b-x)\right]$$

The resulting system of equations is:

$$\frac{\partial}{\partial t}\begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} u\frac{\partial}{\partial x} & -1 \\ -c^2\frac{\partial^2}{\partial x^2} & u\frac{\partial}{\partial x} \end{bmatrix}\begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{DQ}{\partial t} \end{Bmatrix} \tag{2}$$

The Cost Function J to be minimized is defined as follows:

$$\begin{cases} J(\mathbf{\Phi}(x,t)) = \dfrac{\gamma_1}{2}\displaystyle\int_0^T \mathbf{\Phi}^T(x,t)[\mathbb{K}]_c\mathbf{\Phi}(x,t)dt + \dfrac{\gamma_2}{2}\displaystyle\int_0^T \mathbf{q}^2(t)dt \\ [\mathbb{K}]_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$$

3.2 Discrete form

The space-time domain $T-L$ is dived into N time steps indexed as n giving $dt = T/N$ as a time resolution and M space steps indexed as m giving $dx = L/M$ as a space resolution. To gain awareness on the stability of the scheme the space discretization is superimposed and the time step is given by the condition on the CFL number $c\cdot dt/dx$. Therefore the following discretization scheme (implicit-2^{nd}-order Crank-Nicolson) is given for each point contained in the discrete domain, boundary excluded.

$$\frac{\partial \mathbf{\Phi}}{\partial t} = \frac{\mathbf{\Phi}_j^{n+1} - \mathbf{\Phi}_j^n}{\Delta t} \qquad \frac{\partial \mathbf{\Phi}}{\partial x} = \frac{\mathbf{\Phi}_{j+1}^n - \mathbf{\Phi}_{j-1}^n}{2\Delta x} \qquad \frac{\partial^2 \mathbf{\Phi}}{\partial x^2} = \frac{\mathbf{\Phi}_{j+1}^n - 2\mathbf{\Phi}_j^n + \mathbf{\Phi}_{j-1}^n}{\Delta x^2}$$

A different graphical notation for the involved variables will be used to remind their new discrete nature. The system (2) is written in discrete terms as follows:

$$[\mathbb{C}]\frac{\mathtt{phi(:,n+1)} - \mathtt{phi(:,n)}}{\Delta t} + [\mathbb{A}]\frac{\mathtt{phi(:,n+1)} + \mathtt{phi(:,n)}}{2} = [\mathbb{B}]\mathtt{q(n,1)} \tag{3}$$

Where the matrices are just inferred with the proper boundary conditions in the first and last lines. For the sake of clarity, despite the size of Φ is $[2,1]$ there is only one boundary condition in $x = 0$ and $x = L$ where $p = \Phi(1,1) = 0$ and no condition on $Dp/Dt = \Phi(2,1)$ is provided. The answer is to be found in the definition of the state $\Phi(x,t)$ where $\Phi(2,1)$ is a function of $\Phi(1,1)$ and its value on the boundary is directly calculated from the neighbourhood with a different discretization of the spatial derivatives (2^{nd}-order as well as ones in the body of the matrix $[\mathbb{A}]$). The Cost Function J is inferred in discrete terms as follows and the adjoint system can be directly derived in discrete terms granting an exact adjoint solution for any chosen resolution.

$$\begin{cases} \mathtt{J} = \dfrac{\gamma_1}{2}\displaystyle\int_0^T \mathtt{q(n,1)}^T\mathtt{q(n,1)} + \dfrac{\gamma_2}{2}\displaystyle\int_0^T \mathtt{phi(:,n)}^T\,[\mathbb{K}]\,\mathtt{phi(:,n)}dt \\ \mathtt{k(i,j)} = 1 \quad \textit{when} \quad \mathtt{i} = \mathtt{j} = \mathtt{2m-1} \\ \mathtt{k(i,j)} = 0 \quad \textit{when} \quad \mathtt{i}\,,\,\mathtt{j} \neq \mathtt{2m-1} \end{cases}$$

Before going on with the set up of the optimality system let us recap the definition and dimension of each matrix involved in the discrete formulation:

$$\mathbb{A} = \begin{bmatrix} -3u/2\Delta x & -1 & 2u/\Delta x & 0 & -u/2\Delta x & 0 & .. & .. \\ 1 & 0 & 0 & 0 & 0 & 0 & .. & .. \\ -u/2\Delta x & 0 & 0 & -1 & u/2\Delta x & 0 & .. & .. \\ -c^2/\Delta x^2 & -u/2\Delta x & 2c^2/\Delta x^2 & 0 & -c^2/\Delta x^2 & u/2\Delta x & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & -u/2\Delta x & 0 & 0 & -1 & u/2\Delta x & 0 \\ .. & .. & -c^2/\Delta x^2 & -u/2\Delta x & 2c^2/\Delta x^2 & 0 & -c^2/\Delta x^2 & u/2\Delta x \\ .. & .. & u/2\Delta x & 0 & -2u/\Delta x & 0 & 3u/2\Delta x & -1 \\ .. & .. & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (2M \times 2M)$$

$$\mathbb{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & -1/\Delta t & u(b+a-2x(m)) + 1/\Delta t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\Delta t & 1/\Delta t & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0t & -1/\Delta t & u(b+a-2x(m)) + 1/\Delta t & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \quad (2M \times N)$$

$$\mathbb{C} = \begin{bmatrix} 1 & .. & .. & .. & .. & .. \\ .. & 0 & .. & .. & .. & .. \\ .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & 1 & .. \\ .. & .. & .. & .. & .. & 0 \end{bmatrix} \quad (2M \times 2M) \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ .. \\ q_n \\ .. \\ q_N \end{bmatrix} \quad (N \times 1)$$

$$\mathtt{phi} = \begin{bmatrix} \Phi(1,1) & .. & \Phi(1,n) & .. & \Phi(1,N) \\ \dot{\Phi}(1,1) & .. & \dot{\Phi}(1,n) & .. & \dot{\Phi}(1,N) \\ .. & .. & .. & .. & .. \\ \Phi(m,1) & .. & \Phi(m,n) & .. & \Phi(m,N) \\ \dot{\Phi}(m,1) & .. & \dot{\Phi}(m,n) & .. & \dot{\Phi}(m,N) \\ .. & .. & .. & .. & .. \\ \Phi(M,1) & .. & \Phi(M,n) & .. & \Phi(M,N) \\ \dot{\Phi}(M,1) & .. & \dot{\Phi}(M,n) & .. & \dot{\Phi}(M,N) \end{bmatrix} \quad (2M \times N)$$

Stability analysis The eigenvalues of the matrix $[\mathbb{C}]+\frac{dt}{2}[\mathbb{A}]$ has been eveluated in order to check the stability of the system. Results are shown in figure 1.

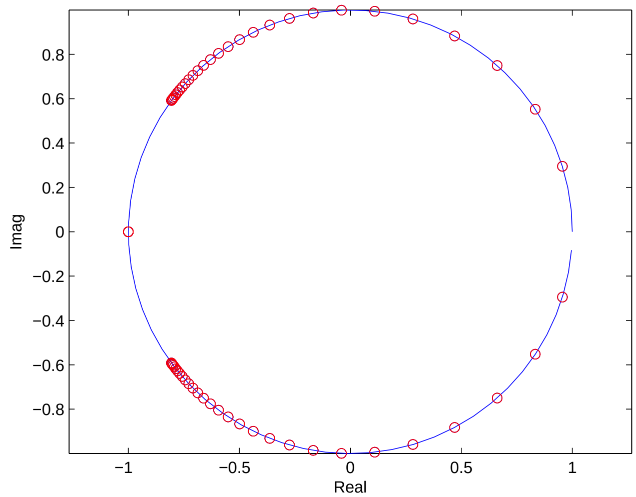

Figure 1: Eigenvalues of the matrix $[\mathbb{C}]+\frac{dt}{2}[\mathbb{A}]$ compared to the unit circle.

4 Optimization process

Let us write the Lagrange Operator to minimize the Cost Function J.

$$\mathcal{L}(\mathtt{phi},\mathtt{q},\mathtt{a},\mathtt{b}) = \frac{\gamma_1}{2}\int_0^T \mathtt{q(n,1)}^T\mathtt{q(n,1)} + \frac{\gamma_2}{2}\int_0^T \mathtt{phi(:,n)}^T\,[\mathbb{K}]\,\mathtt{phi(:,n)}dt+$$
$$-\int_0^T <\mathtt{a}\,,\left([\mathbb{C}]\frac{\partial\mathtt{phi}}{\partial t}+[\mathbb{A}]\mathtt{phi}-[\mathbb{B}]\mathtt{q(n,1)}\right)> dt - \mathtt{b}\left(\mathtt{phi(:,0)}-\Phi_0\right)$$

Differentiating the Lagrange Operator with respect to **a** and **b** trivially leads to the definition of the direct system and to the initial conditions on state $\Phi(x,0)$. On the other hand differentiating $\mathcal{L}$ with respect to $\Phi(x,t)$ and $\mathbf{q}(t)$ leads to the adjoint system and to the optimality condition respectively. The analytical details are shown hereinafter.

$$\frac{\partial\mathcal{L}(\mathtt{phi},\mathtt{q},\mathtt{a},\mathtt{b})}{\partial\mathtt{phi}} = \int_0^T \gamma_1[\mathbb{K}]\mathtt{phi(:,n)}\partial\delta\mathbf{\Phi}(x,t)dt$$
$$-\int_0^T <\mathtt{a}\,,\left([\mathbb{C}]\frac{\partial\delta\mathbf{\Phi}(x,t)}{\partial t}+[\mathbb{A}]\delta\mathbf{\Phi}(x,t)\right)> dt - \mathtt{b}\delta\mathbf{\Phi}(x,0)=0$$

Integrating by parts the expression we get:

$$\int_0^T \gamma_1[\mathbb{K}]\mathtt{phi(:,n)}\partial\delta\mathbf{\Phi}(x,t)dt - \left[[\mathbb{C}]^T\mathtt{a}\delta\Phi(x,t)\right]_0^T +$$
$$+\int_0^T\left([\mathbb{C}]^T\frac{\partial\mathtt{a}}{\partial t}-[\mathbb{A}]^T\mathtt{a}\right)\delta\mathbf{\Phi}(x,t)dt - \mathtt{b}\delta\mathbf{\Phi}(x,0)=0$$

$$\rightsquigarrow \qquad \begin{cases} [\mathbb{C}]^T\frac{\partial}{\partial t}\mathtt{a(:,n)}-[\mathbb{A}]^T\mathtt{a(:,n)}+\gamma_1[\mathbb{K}]\mathtt{phi(:,n)}=0 \\ \mathtt{a}(:,N)=0 \\ \mathtt{b(:,1)}=[\mathbb{C}]^T\mathtt{a(:,1)} \end{cases}$$

And finally the optimality condition:

$$\frac{\partial \mathcal{L}(\mathtt{phi}, \mathtt{q}, \mathtt{a}, \mathtt{b})}{\partial \mathtt{q}} = \int_0^T \gamma_2 \mathtt{q}(\mathtt{n}, \mathtt{1}) dt - \int_0^T < \mathtt{a}\,, [\mathbb{B}] > dt = 0$$
$$\rightsquigarrow \qquad \mathtt{q}(\mathtt{n}) = \frac{\mathbb{B}(\,:\,, n)^T \mathtt{a}(\,:\,, \mathtt{n})}{\gamma_2}.$$

The adjoint system shows the same behaviour of the direct one; the same schemes will be applied:

$$[\mathbb{C}]^T \frac{\mathtt{a}(\,:\,, \mathtt{n}+\mathtt{1}) - \mathtt{a}(\,:\,, \mathtt{n})}{\Delta t} - [\mathbb{A}]^T \frac{\mathtt{a}(\,:\,, \mathtt{n}+\mathtt{1}) + \mathtt{a}(\,:\,, \mathtt{n})}{2} + \gamma_1 [\mathbb{K}] \mathtt{phi}(\,:\,, \mathtt{n}) = 0$$

Here follows the evaluation of the accuracy of the adjoint:

$$\int_0^T < \mathtt{a}\,, \left([\mathbb{C}] \frac{\partial \mathtt{phi}}{\partial t} + [\mathbb{A}] \mathtt{phi} - [\mathbb{B}] \mathtt{q}(\mathtt{n}, \mathtt{1}) \right) > dt = [\mathtt{a}[\mathbb{C}]\mathtt{phi}]_0^T - \int_0^T \left([\mathbb{C}]^T \frac{\partial \mathtt{a}}{\partial t} - [\mathtt{A}]^T \mathtt{a} \right) \mathtt{phi} + [\mathbb{B}]^{\mathtt{T}} \mathtt{aq} dt$$
$$\rightsquigarrow \qquad [\mathtt{a}(\,:\,, \mathtt{N})[\mathbb{C}]\mathtt{phi}(\,:\,, \mathtt{N}) - \mathtt{a}(\,:\,, \mathtt{1})[\mathbb{C}]\mathtt{phi}(\,:\,, \mathtt{1})] = \int_0^T \left([\mathbb{B}]^T \mathtt{aq} - \gamma_1 \mathtt{phi}^T [\mathbb{K}] \mathtt{phi} \right) dt$$

In discrete terms the above expression stands for:

$$\mathtt{a}(\,:\,, \mathtt{n}+\mathtt{1}) \cdot L\mathtt{phi}(\,:\,, \mathtt{n}+\mathtt{1}) = \left(L^T \mathtt{a}(\,:\,, \mathtt{n}+\mathtt{1}) \right) \mathtt{phi}(\,:\,, \mathtt{n})$$
$$\rightsquigarrow \qquad \mathtt{a}(\,:\,, \mathtt{n}+\mathtt{1}) \left(\mathtt{phi}(\,:\,, \mathtt{n}+\mathtt{1}) - [\mathbb{B}] q^n \right) = \mathtt{a}(\,:\,, \mathtt{n}) \mathtt{phi}(\,:\,, \mathtt{n}) - \gamma_1 \mathtt{phi}(\,:\,, \mathtt{n}) [\mathbb{K}] \mathtt{phi}(\,:\,, \mathtt{n}) \Delta t$$

Integration over the whole time time domain leads to the condition:

$$\cancelto{0}{\mathtt{a}(\,:\,, \mathtt{N})\mathtt{phi}(\,:\,, \mathtt{N})} - \mathtt{a}(\,:\,, \mathtt{1})\mathtt{phi}(\,:\,, \mathtt{1}) =$$
$$\sum_{n=1}^{N} \left(\mathtt{a}(\,:\,, \mathtt{n}+\mathtt{1}) [\mathbb{B}] \mathtt{q}(\mathtt{n}, \mathtt{1}) - \gamma_1 \mathtt{phi}(\,:\,, \mathtt{n}+\mathtt{1}) [\mathbb{K}] \mathtt{phi}(\,:\,, \mathtt{n}+\mathtt{1}) \right) \Delta t.$$

The evaluation of the accuracy should lead to the machine precision thank to the discrete derivation of the adjoint system. In such a case this not occur and the accuracy never shrink beyond 10^{-3}. The reason of this unexpected behaviour is not clear and might be found in the strong gradients appearing in the adjoint solution that could produce relevant diffusion phenomena (figure 3).

5 Results

Here follows the parameters chosen for the optimization.

L	1 m	T	0.05 s
m	31	n	CFL$\cdot \Delta x / c$
c	343 m/s	CFL	3
Mach	0.2	$\hat{Q}$	1
a	0.2 L	b	0.3 L
γ_1	1	γ_2	10^-5

Table 1: Parameters of the system.

In order to understand the reason of the little value of the ratio γ_2/γ_1 a brief sensitivity analysis of the Cost Function $J(\Phi(\mathbf{q}), \mathbf{q})$ with respect to the control $\mathbf{q}$) is performed.

$$\frac{\partial J}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} \left(\frac{\gamma_1}{2} \int_0^T \Phi^T \Phi dt + \frac{\gamma_2}{2} \int_0^T \mathbf{q}^T \mathbf{q} dt \right) = \gamma_1 \int_0^T \Phi \frac{\partial \Phi}{\partial \mathbf{q}} + \gamma_2 \int_0^T \mathbf{q} dt$$

Following to the definition of the direct system, $\frac{\partial \Phi}{\partial \mathbf{q}}$ is something proportional to the matrix $\mathtt{inv}([\mathbb{C}] + \frac{dt}{2}[\mathbb{A}]) \cdot ([\mathbb{B}] dt \hat{Q})$. Given that $[\mathbb{B}]$ is a nearly empty matrix to be integrated over the whole time domain, the reason of the little value of γ_2/γ_1 is straightforwardly highlighted. Such an analysis is able to outline that the control $\mathbf{q}$ is able to have relevant effect on the solution only for large value of $\hat{Q}$. This is actually the case of gas-turbine thermo-acoustics where the dimension of $\hat{Q}$ is a power density (W/m^3) usually with the order of magnitude of 10^6.

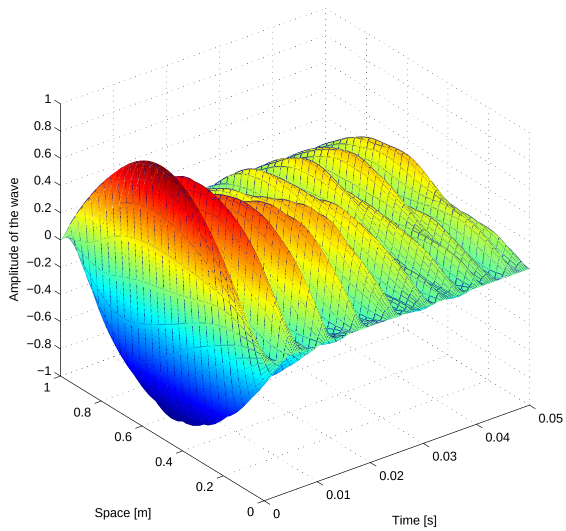

Figure 2: Screen shot after the optimization process showing the shape of the optimized state in a space-time representation.

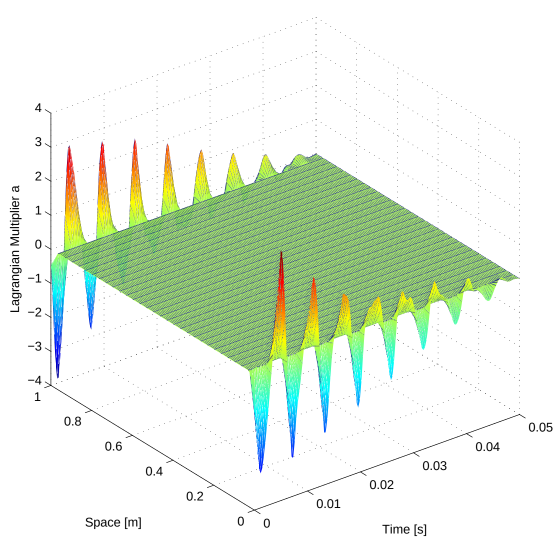

Figure 3: Screen shot after the optimization process showing the shape of an adjoint variable in a space-time representation.

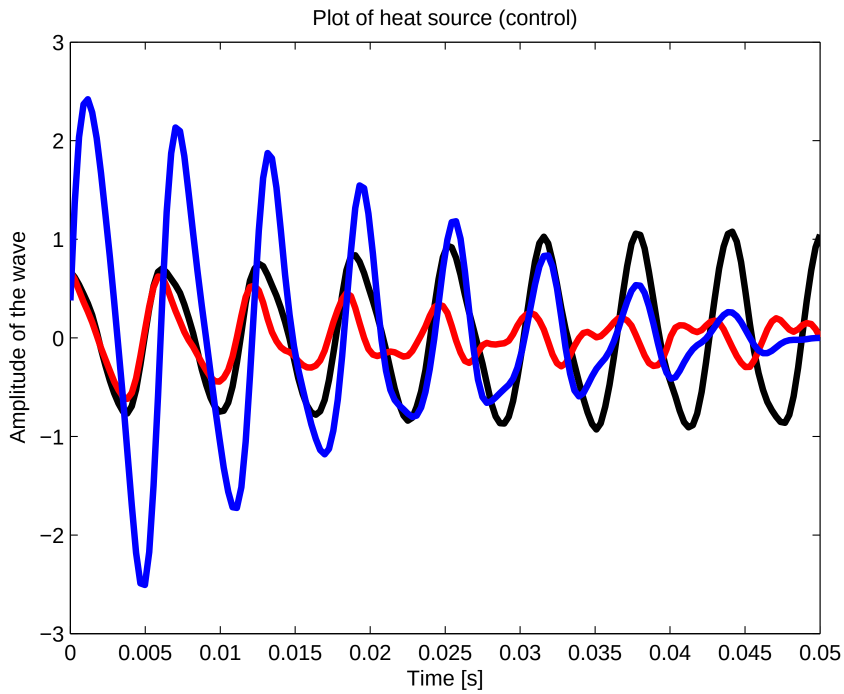

Figure 4: Plot of the state shape before (black) and after (red) the control in $x = L/4$ (center of the heat source) superimposed to the plot of the heat source (blue) in the same location (different scale).

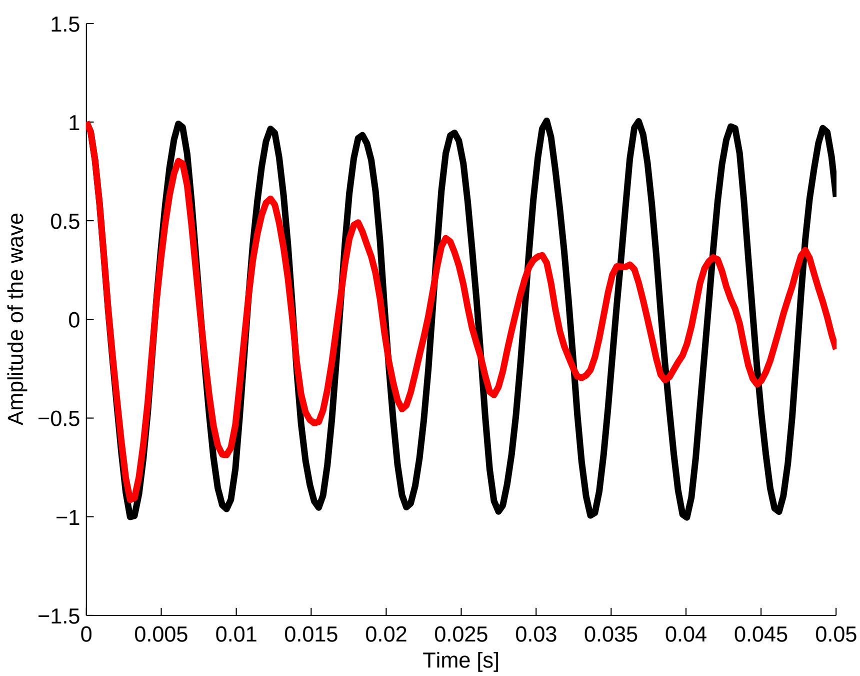

Figure 5: Plot of the state shape before (black) and after (red) the control in $x = L/2$.

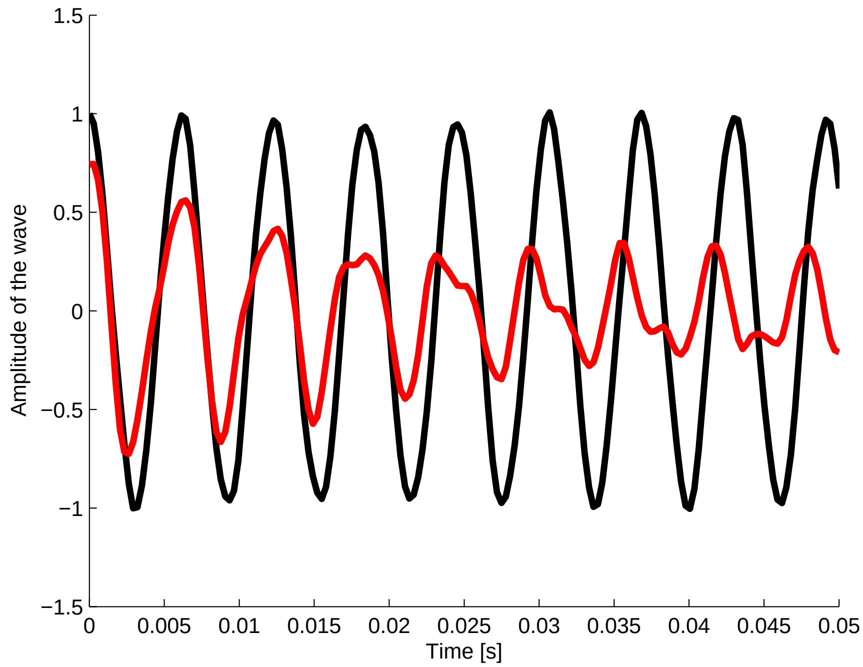

Figure 6: Plot of the state shape before (black) and after (red) the control in $x = 3L/4$.

6 Conclusion

An optimization tool based on a variational approach has been developed and tested for a simple hyperbolic equation. A sensitivity analysis of the state has been performed in order to grab the order of magnitude involved in the problem. Further development could be planned in order to get a more robust derivation of the numerical scheme. We claim this due to the fact that, at present, convergence seems to be too weak and strongly affected by lots of parameters negatively influencing the possible extents of such a tool.

A Listing of the main scripts

Main script

```
%% -----------------------------------------------------------------------
% MAIN
% -----------------------------------------------------------------------
%% -----------------------------------------------------------------------
% GENERAL PARAMETERS
% -----------------------------------------------------------------------
loadParameters;

%% -----------------------------------------------------------------------
% DEFINITION OF THE INITIAL CONDITIONS ON THE DIRECT SYSTEM
% -----------------------------------------------------------------------
defInitialValues;

%% -----------------------------------------------------------------------
% DEFINITION OF PARAMETERS OF THE HEAT SOURCE (location, width)
% -----------------------------------------------------------------------
```

```
heatSource;

%% ----------------------------------------------------------------------
% DEFINITION OF THE MATRIX A, C, and RELATED ONES....
% -----------------------------------------------------------------------
matrixA;
C=eye(2*M); C(2,2)=0; C(2*M,2*M)=0;
Aplus      = C +dt/2*A ;
Aminus     = C -dt/2*A ;
traspAplus = C'+dt/2*A';
traspAminus= C'-dt/2*A';

%% ----------------------------------------------------------------------
% DEFINITION OF OBJECTIVE FUNCTION PARAMETERS
% -----------------------------------------------------------------------
%              / T                       / T
%      gamma2 |                  gamma1 |
% J = --------| q'(t)q(t) dt + -----    | phi'(x,T)[K] phi(x,T)
%         2   |                    2    |
%            / 0                       / 0

K=zeros(2*M,2*M); for m=1:M, K(2*m-1,2*m-1)=1; end
Jactual=10^10;
gamma1=1; gamma2=10^-5*gamma1;

%% ------------------------------------------------------------------------
% MAIN LOOP
%-------------------------------------------------------------------------
iter=1;dJrel=1;

while dJrel>10^-1

    directSystem; adjointSystem;

    Jold=Jactual; Jactual=Jiter; dJrel=abs((Jold-Jactual)/Jactual);
    iter=iter+1;

    plotResults;

    %% --------------------------------------------------------------------
    % ACCURACY OF THE ADJOINT
    % ---------------------------------------------------------------------
    adjointAccuracy(iter)=abs(a_in'*C*phi_out-a_out'*C*phi_in-errorSum)

end

disp(['Number of iterations: ',num2str(iter),'.'])
```

Definition of the main matrix $[\mathbb{A}]$**.**

```
% ---------------------------------------------------------------------
% DEFINITION OF THE MATRIX A
% ---------------------------------------------------------------------
% p1       dp1      p2       dp2     p3       dp3      p4       p4
% p j-1    dp j-1   p j      dp j    p j+1    dp j+1   p j+1    dp j+1
% p M-3    dp M-3   p M-2    dp M-2  p M-1    dp M-1   p M      dp M
```

```
% initialize the matrix
%A=zeros(2*M,2*M);

% ----------------------------------------------------------------------
% boundary condition in j=1 (second order accurate)
% ----------------------------------------------------------------------
A(1:2,1:6)=[
% def of material derivative
-1.5*u/dx  -1 2*u/dx  0  -0.5*u/dx  0;
% BOUNDARY CONDITION p(1)=0
1          0  0       0   0         0;
];

% ----------------------------------------------------------------------
% body of the matrix (second order accurate)
% ----------------------------------------------------------------------
subA= [
% def of material derivative
-u/2/dx    0        0           -1  u/2/dx    0     ;
% WAVE EQUATION
-c^2/dx^2  -u/2/dx  2*c^2/dx^2  0  -c^2/dx^2  u/2/dx;
];

for j=1:M-2
A(1+2*j:2+2*j,2*j-1:2*j+4)=subA;
end

% ----------------------------------------------------------------------
% boundary condition in j=M (second order accurate)
% ----------------------------------------------------------------------
A(2*M-1:2*M,2*M-5:2*M)=[
% def of material derivative
0.5*u/dx  0   -2*u/dx   0  1.5*u/dx   -1;
% BOUNDARY CONDITION p(M)=0
0         0   0         0  1            0;
```

Definition of the source term matrix $[\mathbb{B}]$**.**

```
%% -------------------------------------------------------------------
% DEFINITION OF THE MATRIX B
% -------------------------------------------------------------------
B=zeros(2*M,N);
for n=2:N
   for m=1:M
      if (m-aGrid)*(bGrid-m)>=0
          if (m-aGrid)*(bGrid-m)==0
              B(2*m,n-1)=-1/dt;
              B(2*m,n)=u*(bFlame+aFlame-2*x(m))+1/dt;

          else
              B(2*m,n-1)=-1/dt;
              B(2*m,n)=1/dt;
          end

      end
```

```
    end
end
```

References

[1] Rayleigh, L., 1896. *The Theory of Sound*. McMillan.