Course on Advanced Fluid Dynamics

Stability of a liquid film flowing down an inclined flat plate.

Giuseppe Zampogna

Abstract

A linear stability analysis of a liquid film flowing down an inclined flat plane is carried out. In particular, we focus on how the variations of some parameters, which represent, for example, the contact angle among the solid-liquid-air interface, modify the stability of the base flow. This kind of studies are very important to understand the formation of drops. For this reason the theoretical model utilized takes into account microscopical quantities like the van der Waals potential which rules the dynamics near the contact line. The implemented model is a simplified case of the theoretical model, but it underlines some important characteristics of the real film. As the first step, the free surface profile and the displacement velocity are determined. Starting from here some stable and unstable modes for the steady base flow are analyzed. The stability of the constant base flow is performed analytically.

1 Introduction

Consider a liquid film flowing down a flat plate of lenght L as in Figure 1. Assuming that the motion is independent of the z-axis, if $\mathbf{u}$ is the velocity field depending on the position x on the plate and the time t, the governing equations are

$$\begin{cases} u_x + v_y = 0 \\ \rho(u_t + uu_x + vu_y) = -p_x + \rho g sin(\theta) + \mu \nabla^2 u - \left(\frac{A'}{6\pi h^3}\right)_x \\ \rho(v_t + uv_x + vv_y) = -p_x + \rho g cos(\theta) + \mu \nabla^2 v \end{cases} \tag{1}$$

where A' is the Hamaker constant and $\phi = \frac{A'}{6\pi h^3}$ is the van der Waals potential that represents the inter-molecular interaction. If we want to know which region in space is occupied by the liquid we have to introduce a new variable

h that is the height of the liquid. Calling x_c the liquid-solid-air interface the domain of the equation is

$$D = \{(x, y)|0 \leq x \leq x_c,\ 0 \leq y \leq h(x, t)\} \tag{2}$$

Now we are able to introduce initial and boundary conditions:

- at $y = 0$ we have
$$u = \frac{\tilde{\alpha}}{3h} u_y, \qquad v = 0; \tag{3}$$
where the first equation represents the slip condition due to the contact point among solid-liquid-air interfaces.

- at $y = h$ we have
$$v = h_t + u h_x \tag{4}$$

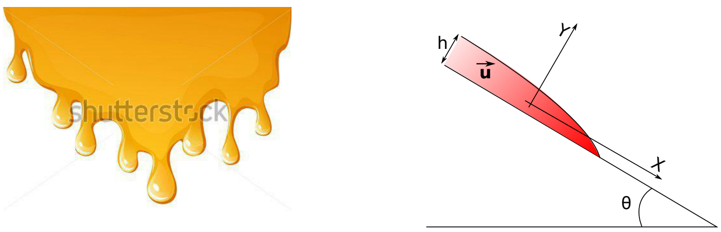


Figure 1: Schematic of a real three dimensional problem and simplified two dimensional problem after assumption of z-independence.

Our goal is to study the stability of the solution of this differential problem. A strategy to do this is to focus our attention on the shape of the film and consider the stability of this one. Hence, we have to find an equation for h. We introduce h_0 that is the mean height of an infinity long film far away from the contact point, and the mean velocity of the same film $\bar{u}$ defined by

$$\bar{u} = \frac{1}{h_0} \int_0^{h_0} u_\infty \, dy = \frac{1}{h_0} \int_0^{h_0} \frac{gy \sin\theta}{2\nu} (2h_0 - y) \, dy = \frac{g h_0^2 \sin\theta}{3\nu} \tag{5}$$

where u_∞ is the velocity of the film driven only by gravity, neglecting the surface tension. Once we have introduced these quantities we can non-dimensionalize the problem:

$$u^* = \frac{u}{\bar{u}}, \quad v^* = \frac{v}{\delta \bar{u}}, \quad y^* = \frac{y}{h_0}, \quad h^* = \frac{h}{h_0}, \quad x^* = \frac{\delta x}{h_0},$$

$$t^* = \frac{t}{\frac{h_0}{\delta \bar{u}}}, \quad p^* = \frac{p}{\frac{\mu \bar{u}}{\delta h_0}}$$

where $\delta << 1$ is the characteristic length in y; it underlines that the fact that there is a big difference between the length scale in x and y. At this point, rewriting equations (1), (3), (4) and using the non-dimensional quantities we obtain an equation for the thickness of the film:

$$h_t + \left[\left(3 + A\frac{h_x}{h^4} + Sh_{xxx} \right) \left(\frac{h^3}{3} + \tilde{A}h \right) \right]_x = 0 \tag{6}$$

with the following initial and boundary conditions

$$\text{I.C.} \qquad h(x,0) = f(x) \tag{7}$$

$$\text{B.C.} \qquad h(0,t) = 1,\ h_x(0,t) = 0,\ h(x_c,t) = 0,\ h_x(x_c,t) = -tg(\alpha) \tag{8}$$

where f is a given initial shape for the film and x_c is the contact point and S, A, $\tilde{A}$ are constant

$$S = \frac{3\delta^3 \sigma_s}{\rho g h_0^2 sin(\theta)}, \qquad A = \frac{3\delta A'}{2\pi \rho g h_0^4 sin(\theta)}, \qquad \tilde{A} = \frac{\tilde{\alpha}}{3h_0^2}. \tag{9}$$

with $\delta << 1$.

Notation 1.1. In order to clarify the meaning of the constant S, A, $\tilde{A}$ we list the notations:

- σ_s: surface tension for an air-water interface;
- ρ: density of the liquid;
- h_0: height of the film far away from the contact point;
- θ: inclination of the solid wall;
- α: contact angle;
- $\tilde{\alpha}$: slip constant;
- δ: characteristic length in y;
- A': Hamaker constant.

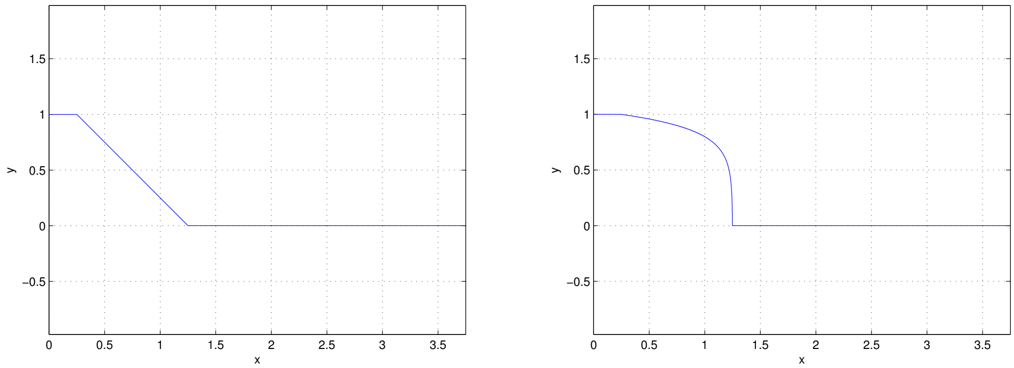


Figure 2: Two possible initial conditions $f(x)$ (linear and logarithmic).

Assumptions. Defining the problem in this way, we impose the contact angle α between solid liquid air interface. Actually α is unknown. In order to be more realistic we introduce the precursor film; it is a thin film of liquid that flows over the plate. Introducing this trick, α is able to assume a natural value. Obviously this value depends on the thickness ϵ of the film and this will be studied in the following: we need to find a value for which the solution do not change. Moreover, to adjust the contact angle as we want, we can modify the constant $\tilde{\alpha}$. Another important assumption is that, for simplicity, we will take the constant A equal to zero; this means that we are not accounting for the Van der Waals potential in equation (1). Under this assumptions, the problem becomes

$$h_t + \left[(3 + Sh_{xxx})\left(\frac{h^3}{3} + \tilde{A}h\right)\right]_x = 0, \tag{10}$$

with initial and boundary conditions

$$h(x,0) = f(x), \tag{11}$$

$$h(0,t) = 1,\ h_x(0,t) = 0,\ h(x_{max},t) = \epsilon,\ h_x(x_{max},t) = 0, \tag{12}$$

where $x_{max} >> x_c$.

2 Numerical resolution

The numerical resolution of the problem is made by two fundamental steps; in the first step we have implemented an integration in time of equation (10); in the second step we have studied the stability of the solution.

ϵ	S	$\tilde{A}$	θ
10^{-2}	0.92	$0.35 - 10.7$	$7^o - 30^o$
10^{-3}	0.0092	$0.35 - 10.7$	$7^o - 30^o$
10^{-4}	0.000092	$0.35 - 10.7$	$7^o - 30^o$

Table 1: Here is shown for each value of ϵ the corresponding values of the other utilized parameters (S, $\tilde{A}$, θ).

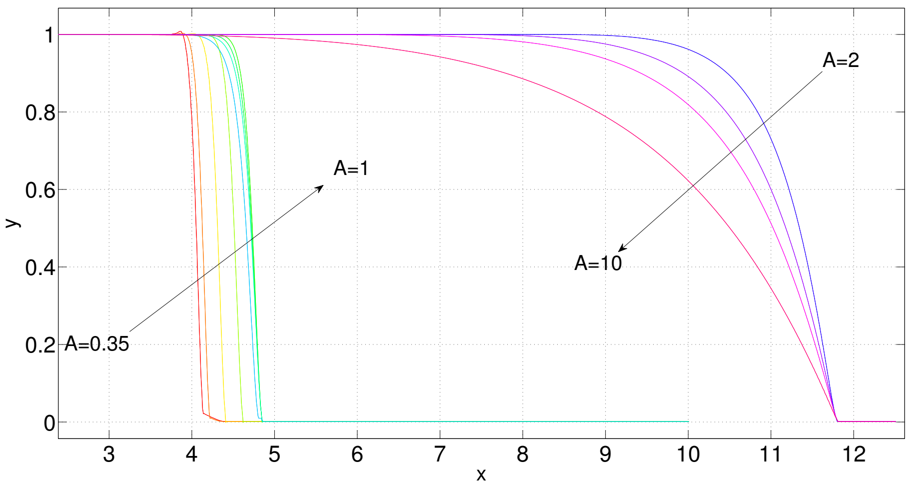


Figure 3: Profiles of the steady solution after 30000 step in time, varying $\tilde{A}$ from 0.35 to 10.

Time integration. Setting as initial conditions the profiles in Figure 2, equation (10) is integrated in time; there are two possible behaviors depending on the choice of the parameters and initial condition: the solution becomes a steady profile or evolves in a state of instability. Table 1 lists the values of the parameters employed.

In order to validate the code we have tested that the solution does not change if we vary some important parameters as:

- precursor film
- spatial and temporal step

We have tested the time convergence considering the relative error

$$\frac{|SOL_N - SOL_{N+1}|}{|SOL_N|} << 1 \tag{13}$$

where SOL_N is the solution profile after N steps in time. For $N = 30000$ this condition is always satisfied and the solution profile is assumed to be steady.

Stability for a constant base flow. Here we study the stability of some solutions; first of all we will consider the stability of a constant base flow. This is the case of a semi-infinite film analyzed very far from the contact point. If $\overline{h}(x) = \overline{H}_0$ is the constant base flow, perturbing it with

$$\overline{h}(x) + \tilde{h}(x)e^{-i\omega t}, \qquad \omega \in \mathbb{C}$$

and linearizing equation (10) we obtain

$$\left(\frac{S}{3}\overline{H}_0^3 + S\tilde{A}\overline{H}_0\right)\tilde{h}_{xxxx} + \left(3\overline{H}_0^2 + 3\tilde{A}\right)\tilde{h}_x - i\omega\tilde{h} = 0. \tag{14}$$

Supposing that $\tilde{h}$ is of the form

$$\tilde{h}(x) = e^{ikx} \qquad k \in \mathbb{R}$$

substituting into (14) the following equation arises:

$$\left(\frac{S}{3}\overline{H}_0^3 + S\tilde{A}\overline{H}_0\right)k^4 + \left(3\overline{H}_0^2 + 3\tilde{A}\right)ik - i(\omega_r + i\omega_i) = 0, \tag{15}$$

so that

$$\begin{cases} \omega_r = \left(3\overline{H}_0^2 + 3\tilde{A}\right)k \\ \omega_i = -\left(\frac{S}{3}\overline{H}_0^3 + S\tilde{A}\overline{H}_0\right)k^4. \end{cases} \tag{16}$$

In this way we have an analytical expression for ω varying the parameter $\tilde{A}$; ω_r is the frequency of oscillations of the mode, whereas ω_i represents the growth rate. A positive value of ω_i denotes that the mode is unstable.

Stability for a steady base flow. By time integration we have obtained a solution that we will use as base flow for the linear stability. We recall the equation for $\overline{h}$

$$\overline{h}_t + \left[\left(3 + S\overline{h}_{xxx}\right)\left(\frac{\overline{h}^3}{3} + \tilde{A}\overline{h}\right)\right]_x = 0. \tag{17}$$

If we consider a steady solution of equation (17) as base flow, then we can consider the following change of variable

$$X = x - ct. \tag{18}$$

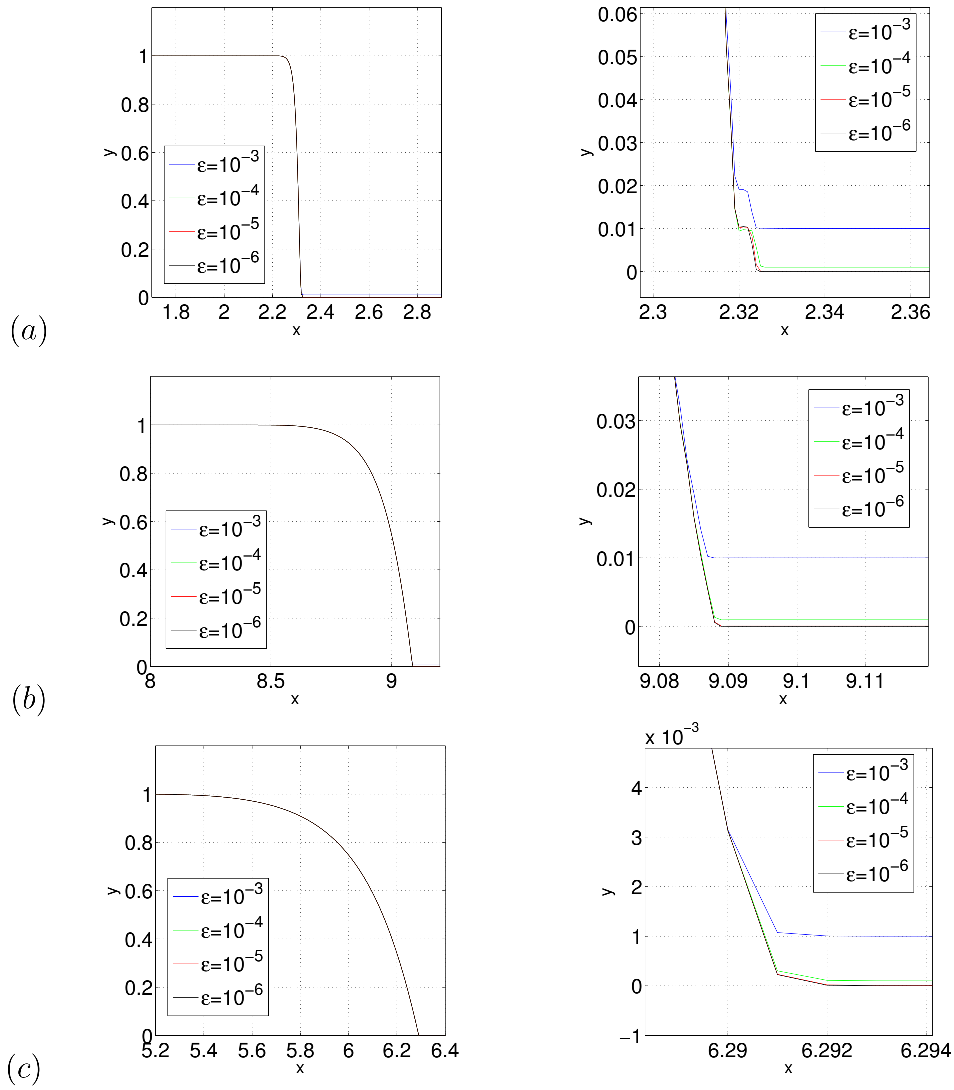
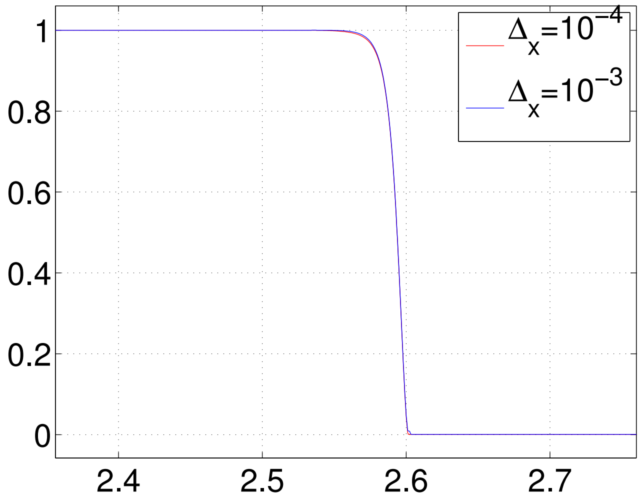
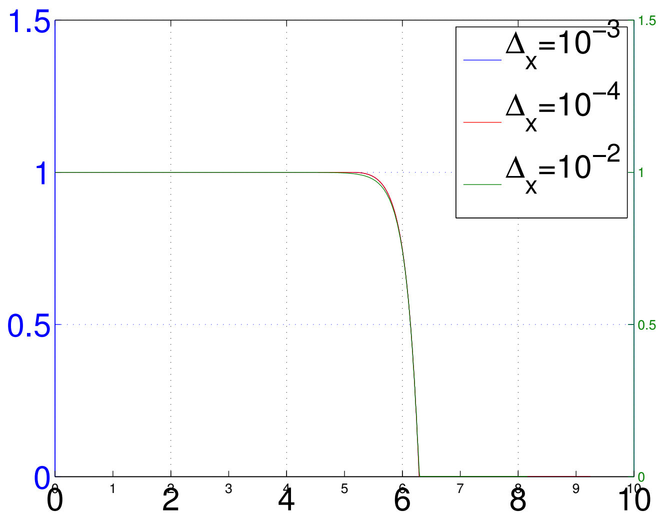


Figure 4: Steady solution for varying thickness ϵ of the precursor film, with $\Delta_x = 10^{-3}$,$\Delta_t = 10^{-2}$, $\tilde{A} = 1$ (frame (a)), $\tilde{A} = 5$ (frame (b)) and $\tilde{A} = 10$ (frame (c)), after 30000 steps in time.



(*a*)



(*b*)

Figure 5: Steady solution for varying Δ_x for $\Delta_t = 10^{-2}$ at $\tilde{A} = 1$ (frame (a)) and $\tilde{A} = 10$ (frame (b)), after 30000 steps in time.

If $\overline{h}$ is the solution, a perturbation of this one is

$$h = \overline{h}(X) + \tilde{h}(X)e^{\sigma t}, \qquad \tilde{h}_t = \left[\tilde{h}(X)e^{\sigma t}\right]_t = e^{\sigma t}\left(-c\tilde{h}_X + \sigma\tilde{h}\right) \tag{19}$$

with $c \in \mathbb{R}$ the wave velocity, $\overline{h}_x = \overline{h}_X$ and $\tilde{h}_x = \tilde{h}_X$. It will be clear in the following paragraph that $\sigma \in \mathbb{C}$ is related to an eigenvalue problem. In the approach of stability, $\mathrm{Re}(\sigma)$ represent the growth rate and $\mathrm{Im}(\sigma)$ the frequency of the solution; in particular, if $\mathrm{Re}(\sigma) > 0$, the solution is unstable. Substituting into equation (17) we obtain

$$-c\overline{h}_X - c\tilde{h}_X + \sigma\tilde{h} + \left[(3 + Sh_{XXX})\left(\frac{h^3}{3} + \tilde{A}h\right)\right]_X = 0. \tag{20}$$

Linearizing with respect to $\tilde{h}$

$$-c\overline{h}_X - c\tilde{h}_X + \sigma\tilde{h} + \left[\left(3 + S\overline{h}_{XXX}\right)\left(\frac{\overline{h}^3}{3} + \tilde{A}\overline{h}\right)\right]_X +$$
$$+\left[\left(S\tilde{h}_{XXX}\right)\left(\frac{\overline{h}^3}{3} + \tilde{A}\overline{h}\right)\right]_X + \left[\left(3 + S\overline{h}_{XXX}\right)\left(\overline{h}^2\tilde{h} + \tilde{A}\overline{h}\right)\right]_X = 0$$

and using the fact that $\overline{h}$ is a solution of (17)

$$-c\tilde{h}_X + \sigma\tilde{h} + \left[S\tilde{h}_{XXX}\left(\frac{\overline{h}^3}{3} + \tilde{A}\overline{h}\right)\right]_X + \left[\left(3 + S\overline{h}_{XXX}\right)\left(\overline{h}^2\tilde{h} + \tilde{A}\overline{h}\right)\right]_X = 0$$

$$\begin{array}{rc} \sigma\tilde{h} = & \tilde{h}\left(6\overline{h}\overline{h}_X + 2S\overline{h}\overline{h}_X\overline{h}_X XX + S\overline{h}^2\overline{h}_{XXXX} + \tilde{A}S\overline{h}_{XXXX}\right) + \\ + & \tilde{h}_X\left(c + 3\overline{h}^2 + S\overline{h}^2\overline{h}_{XXX} + 3\tilde{A} + \tilde{A}S\overline{h}_{XXX}\right) + \\ + & \tilde{h}_{XXX}\left(S\overline{h}^2\overline{h}_X + S\tilde{A}\overline{h}_X\right) + \\ + & \tilde{h}_{XXXX}\left(S\frac{\overline{h}^3}{3} + S\tilde{A}\overline{h}\right) \end{array} \tag{21}$$

that is the linearized equation for $\tilde{h}$ in which we have put in evidence the eigenvalue σ. To perform an eigenvalues problem from this equation we have to discretize it. The velocity c is computed in the time integration code observing the displacement of the steady profile; some values of c are shown in Table 2.

$\tilde{A}$	c	α
0.4	0.04	88.51^o
1	0.07	83.17^o
5	0.26	63.06^o
10	0.88	52.53^o

Table 2: Here we show how c and α change for varying $\tilde{A}$.

Eigenvalue problem. The first step in the discretization is to observe that here $\tilde{h}$ and $\overline{h}$ are vector and to compute the derivatives $\tilde{h}_X$, $\tilde{h}_{XXX}$, $\tilde{h}_{XXXX}$, $\overline{h}_X$, $\overline{h}_{XXX}$ and $\overline{h}_{XXXX}$; in this case a central scheme is used. In order to clarify the structure of the matrix in this eigenvalue problem we define the vectors ζ, ξ, ψ, χ as

$$\begin{aligned}
\zeta = &\quad 6\overline{h}\overline{h}_X + 2S\overline{h}\overline{h}_X\overline{h}_{XXX} + S\overline{h}^2\overline{h}_{XXXX} + \tilde{A}S\overline{h}_{XXXX}, \\
\xi = &\quad c + 3\overline{h}^2 + S\overline{h}^2\overline{h}_{XXX} + 3\tilde{A} + \tilde{A}S\overline{h}_{XXX}, \\
\psi = &\quad S\overline{h}^2\overline{h}_X + S\tilde{A}\overline{h}_X, \\
\chi = &\quad S\frac{\overline{h}^3}{3} + S\tilde{A}\overline{h}.
\end{aligned}$$

Now we are ready to understand the structure of our matrix.

$$\Omega = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \omega_{i\ i-2} & \omega_{i\ i-1} & \omega_{i\ i} & \omega_{i\ i+1} & \omega_{i\ i+2} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$\omega_{i\ i-2} = -\psi_i/(2\delta) + \chi_i/\delta,$$

$$\omega_{i\ i-1} = -\xi_i/(2\delta) + \psi_i/\delta - 4\chi_i/\delta,$$

$$\omega_{i\ i} = \zeta_i + 6\chi_i/\delta,$$

$$\omega_{i\ i+1} = \xi_i/(2\delta) - \psi_i/\delta - 4\chi_i/\delta,$$

$$\omega_{i\ i+2} = \psi_i/(2\delta) + \chi_i/\delta.$$

Regarding boundary conditions, we need to modify the matrix in the following way

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \omega_{i\ i-2} & \omega_{i\ i-1} & \omega_{i\ i} & \omega_{i\ i+1} & \omega_{i\ i+2} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

After doing this, equation (21) can be written as

$$\sigma I \mathbf{h} + \Omega \mathbf{h} = 0 \tag{22}$$

where $\mathbf{h}$ is the vector of the solution.

3 Results and conclusions.

In this section we will show some results, focusing our attention on the spectrum of Ω matrix, eigenfunctions and contact angles. In Figures 6 and 7 respectively we can see the linear and parabolic dependence of ω_r and ω_i from the wave number k. Regarding the stability for a constant base flow, since $\omega_i < 0\ \forall k$ (Figure 7), the solution is always stable. In the case of steady base flow, the behavior of the solution and its stability is strictly related to the value of $\tilde{A}$: for $\tilde{A} \leq 1$ the profile h increases before the wavefront, like a drop, and the contact angle tends to 90 degrees (see Figure 3 and Table 2). In Fiugure 8 we can see how the spectrum of Ω changes for varying $\tilde{A}$: for $\tilde{A} = 0.4$ the imaginary part σ_i of the eigenvalues is near the x-axis, increasing $\tilde{A}$ σ_i becomes larger for each eigenvalue. In Figure 9 one real stable eigenfunction for each $\tilde{A}$ is shown. In Figure 10 we can observe that the largest unstable eigenfunction looks like a peak in correspondence of the contact point. Moreover for $\tilde{A} \leq 1$ we have more than one unstable eigenvalue (the corresponding eigenfunctions are shown in Figure 11), for $\tilde{A} \geq 1$ there is only one unstable eigenvalue. Eventually, in Figure 12 we can see the profile of the solution for varying the inclination of the flat plate. What we need to complete this work is to decrease the lower limit for $\tilde{A}$ which now is 0.35 and to add in the equation for h the terms multiplied by A which represent the Van der Waals potential and, hence, the micro-scale interactions.

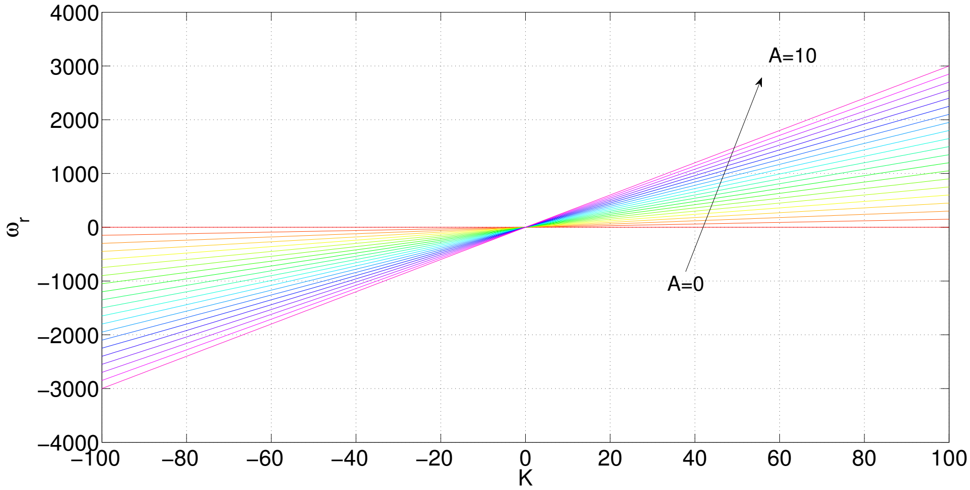


Figure 6: Analytical solution for Re(ω).

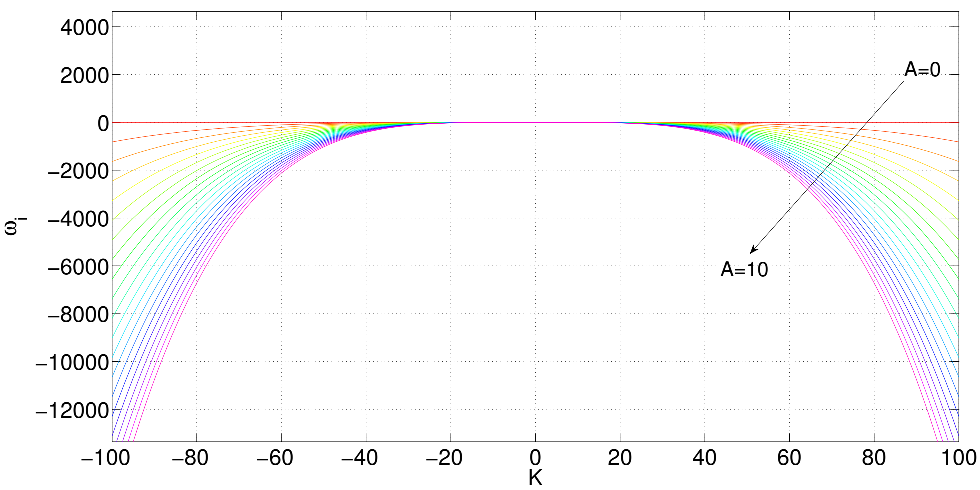


Figure 7: Analytical solution for Im(ω). No unstable modes can be seen.

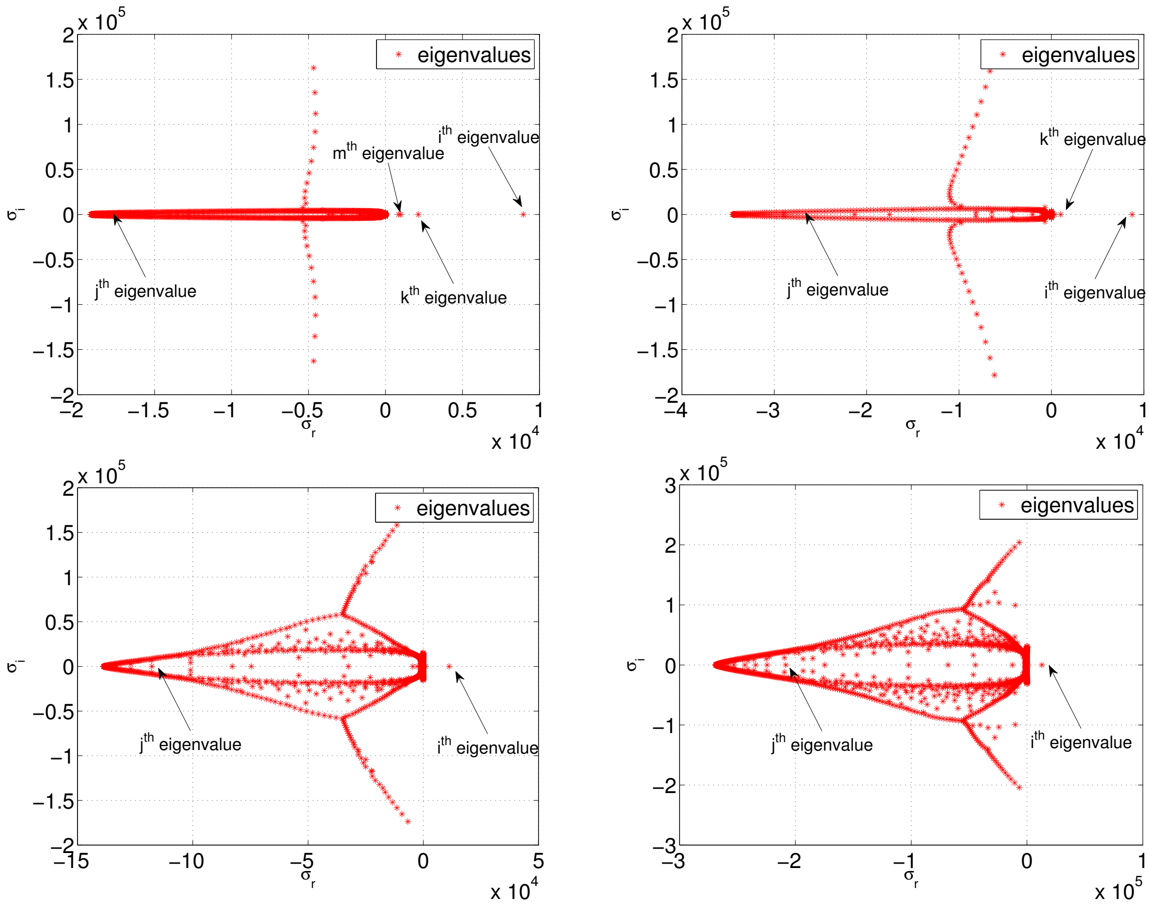


Figure 8: Spectrum of Ω for $\tilde{A} = 0.4,\ 1,\ 5,\ 10$ with $\Delta_x = 10^{-3}$ and $\theta = \pi/24$.

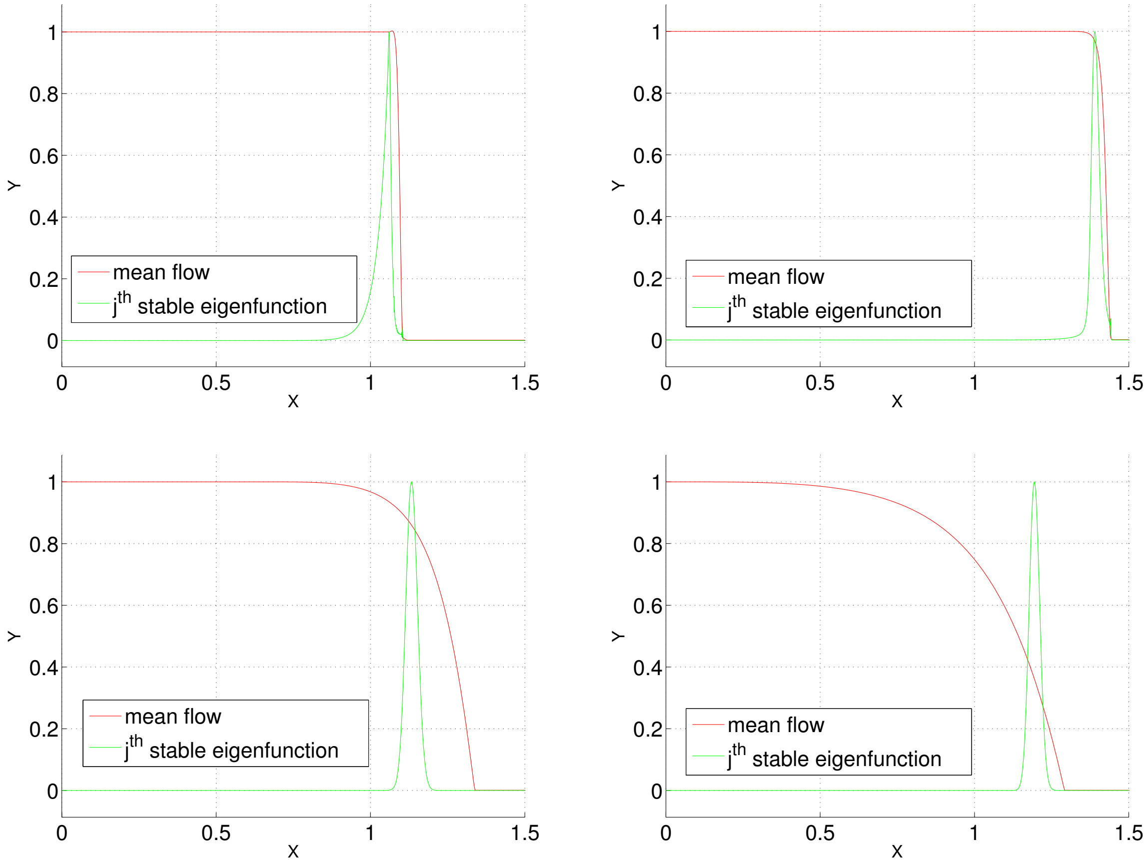


Figure 9: Some stable eigenfunctions for $\tilde{A} = 0.4,\ 1,\ 5,\ 10$ with $\Delta_x = 10^{-3}$ and $\theta = \pi/24$.

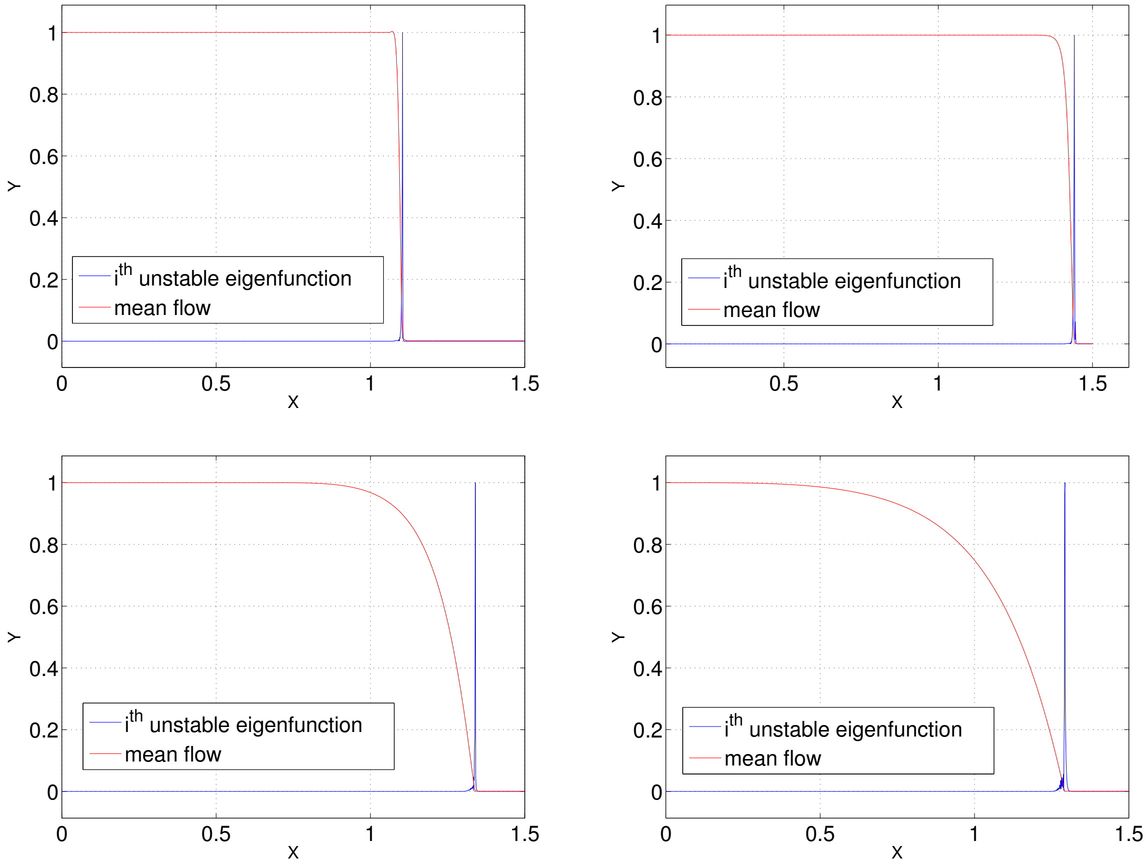


Figure 10: Largest unstable eigenfunctions for $\tilde{A} = 0.4,\ 1,\ 5,\ 10$ with $\Delta_x = 10^{-3}$ and $\theta = \pi/24$.

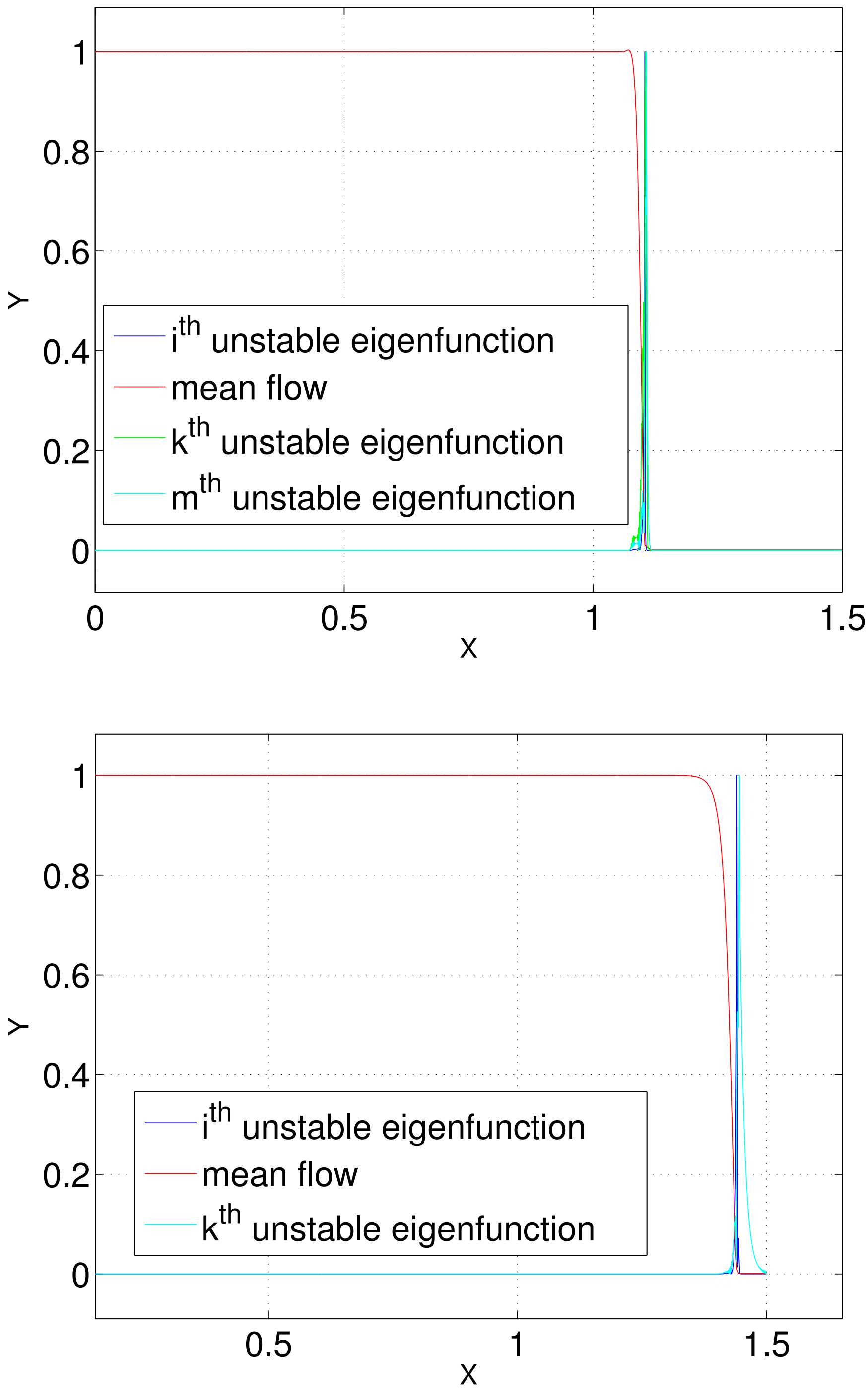


Figure 11: All unstable eigenfunctions for $\tilde{A} = 0.4$ and $\tilde{A} = 1$ with $\Delta_x = 10^{-3}$ and $\theta = \pi/24$. (For $\tilde{A} = 5$ and $\tilde{A} = 10$ there is only one unstable eigenfunction).

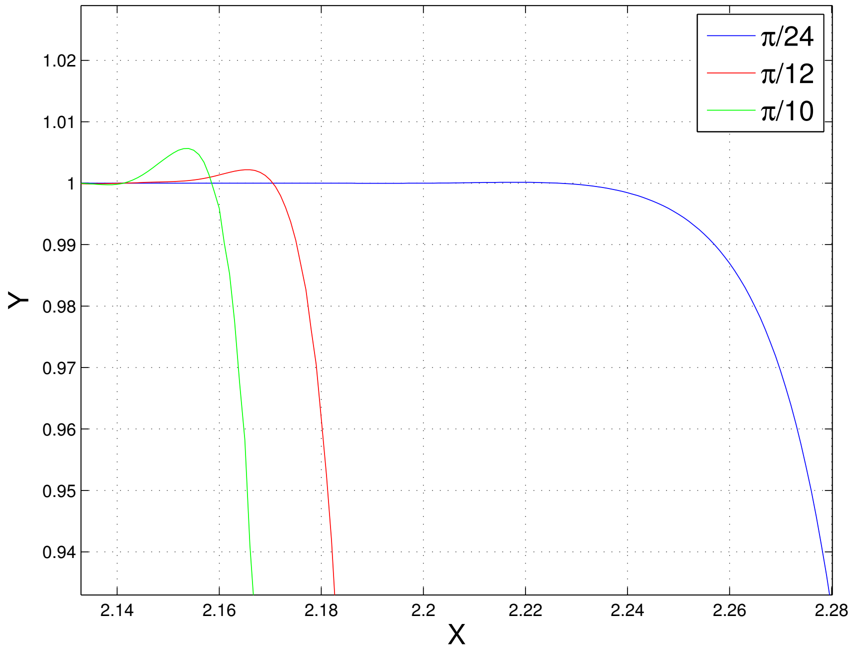


Figure 12: Zoom of the profile of the solution for varying the inclination angle θ with $\tilde{A} = 1$.

References

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