Structural Sensitivity of the Finite-Amplitude Vortex Shedding Behind a Circular Cylinder

P. Luchini, F. Giannetti, and J. Pralits

Abstract In this paper we study the structural sensitivity of the nonlinear periodic oscillation arising in the wake of a circular cylinder for $Re47$. The sensibility of the periodic state to a spatially localised feedback from velocity to force is analysed by performing a structural stability analysis of the problem. The sensitivity of the vortex shedding frequency is analysed by evaluating the adjoint eigenvectors of the Floquet transition operator. The product of the resulting neutral mode with the nonlinear periodic state is then used to localise the instability core. The results obtained with this new approach are then compared with those derived by Giannetti & Luchini [8]. An excellent agreement is found comparing the present results with the experimental data of Strykowski & Sreenivasan [7].

Keywords Fluid mechanics · Nonlinear global modes · Structural sensitivity · Adjoint

1 Introduction

Spatially developing flows such as mixing layers, wakes and jets, may sustain in specific parameter ranges synchronised periodic oscillations over extended regions of the flow field, displaying there an intrinsic dynamics characterised by a sharp frequency selection. Under these conditions the whole flow field behaves like a global oscillator and the structure underlying the spatial distribution of the fluctuations is usually termed "global mode". The spatio-temporal evolution of such flows has been clarified considerably only in recent years: progress was made through model equations, experiments, stability analysis and direct numerical simulations. A theoretical approach to this class of problems was formulated by Chomaz et al. [1], Monkewitz et al. [6] and Le Dizés et al. [4] in the context of flows with properties slowly varying in space. Relying only on a local analysis, they were able to show that such flows may exhibit internal resonance when a region of absolute instability

P. Luchini (✉), F. Giannetti, and J. Pralits
DIMEC, Università di Salerno, 84084 Fisciano (SA), Italy
e-mail: luchini@unisa.it

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of sufficient size develops. The resonance is self-excited and is characterised by a well defined frequency. The important link between the global and local instability properties, both in the linear and fully nonlinear regime, is obtained via a WKBJ approach: the theory identifies a specific spatial position in the absolutely unstable region which acts as a *wavemaker*, providing a precise frequency selection criterion and revealing some important insights pertaining to the forcing of these modes. In particular, in a linear setting, the complex global frequency ω_g is obtained by the saddle-point condition

$$\omega_g = \omega_0(X_s) \quad \text{with} \quad \frac{\partial \omega_0}{\partial X}(X_s) = 0 \tag{1}$$

based on the analytic continuation of the local absolute frequency curve $\omega_0(X)$ in the complex X-plane, with X denoting here the slow streamwise variable. Although this asymptotic theory yields accurate predictions for slowly evolving flows, in many real configurations the assumptions underlying the WKBJ approach are not met very closely. This is the case of bluff-body wakes, where strong non-parallel effects prevent us from using asymptotic theory. In such cases a numerical modal analysis must be used to determine the characteristics of the instability and to find its critical Reynolds number. One of the most common examples is given by the flow around an infinitely long circular cylinder. In order to study the global properties of such flow, Giannetti & Luchini [8] performed a 2D stability and receptivity analysis of the steady base flow using the properties of the adjoint eigenfunctions. The asymptotic theory developed by Chomaz et al. [1], Monkewitz et al. [6] and Le Dizés et al. [4] in the context of slowly evolving quasi-parallel flows endows the region around the saddle point with the fundamental role of *wavemaker* in the excitation of the global mode. In the context of a two-dimensional modal analysis a concept similar to that of *wavemaker* can be introduced by investigating where in space a modification in the structure of the problem is able to produce the greatest drift of the eigenvalue. Using this approach Giannetti and Luchini (2007) determined the regions where the feedback from velocity to force is maximum and consequently identified the regions were the instability acts. Qualitative agreement was obtained with the numerical and experimental data of Strykowski and Sreenivasan [7]. From a theoretical point of view a similar approach, being based on the properties of the steady base flow, is only valid in a neighbourhood of the neutral point. When $Re > Re_c \approx 47$ the flow becomes unsteady and a Karman vortex street develops.

In this paper we extend the approach developed by Giannetti and Luchini (2007) to study finite-amplitude vortex shedding, in order to assess how unsteadiness and saturation can modify the previous results.

2 Problem Formulation

We investigate the stability characteristics of the two-dimensional flow arising around an infinitely long circular cylinder invested by a uniform stream. A Cartesian coordinate system has its origin in the cylinder's centre, with the x axis pointing in the flow direction. For $Re < Re_{c,2} \approx 180$ the fluid motion can be described by the two-dimensional unsteady incompressible Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} \tag{2a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2b}$$

where $\mathbf{u}$ is the velocity vector and p is the reduced pressure. Equations (2a,b) are made dimensionless using the cylinder diameter D^* as the characteristic length scale, the velocity of the incoming uniform stream u^*_∞ as the reference velocity and $\rho^* u^{*2}_\infty$ as the reference pressure. Thus

$$Re = \frac{u^*_\infty D^*}{\nu^*} \tag{3}$$

is the Reynolds number based on the cylinder diameter.

Equations (2a,b) must be supplemented by appropriate boundary conditions. In particular, on the surface of the cylinder Γ_c the no-slip and no-penetration conditions require both velocity components to vanish, while in the far field the flow approaches asymptotically the incoming uniform stream. For $Re > Re_c$ the steady symmetric flows becomes unstable and a Karman vortex street develops. In such conditions, after an initial transient, the flow becomes periodic:

$$\mathbf{u}(t + T) = \mathbf{u}(t)\ , \ \ p(t + T) = p(t) \tag{4}$$

with period T, Strouhal number $St = 1/T$ and angular pulsation $\omega = 2\pi/T$.

In order to locate the *wavemaker* of the instability, Giannetti and Luchini (2007) determined the space distribution of the sensitivity of the eigenvalue to a structural perturbation of the problem. The analogous quantity for the nonlinear periodic oscillation is the space distribution of the sensitivity of its frequency to a structural perturbation of the problem. This is the objective of the present paper.

Suppose now we give a structural perturbation to this problem, in the form of a body force depending on the local velocity $\mathbf{h}(\mathbf{u})$. If the perturbation is small the new solution will remain periodic but with a different period (in contrast with the corresponding linear problem whose frequency will in general become complex and bring about either amplification or damping). In order to be able to treat the problem perturbatively and avoid secular effects, it is convenient to scale the time variable on the period of the solution itself. Thus introducing the scaled time

$$\tau = \frac{t}{T} \tag{5}$$

the equations can be rewritten as

$$\frac{1}{T}\frac{\partial \mathbf{u}}{\partial \tau} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} \tag{6a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6b}$$

where T is an additional unknown and the period in the variable τ is constant and equal to 1.

Writing the perturbed solution as

$$\{\mathbf{u_0}(\tau) + \mathbf{u}(\tau), p_0(\tau) + p(\tau)\},$$

where $\{\mathbf{u_0}, p_0\}$ denote the unperturbed periodic flow and $\{\mathbf{u}, p\}$ have become the small perturbations induced by the added forcing, and inserting it into the equations together with the small external forcing $\mathbf{h} = \mathbf{C}\mathbf{u_0}$ we obtain

$$\frac{1}{T+\delta T}\frac{\partial(\mathbf{u_0}+\mathbf{u})}{\partial\tau} + (\mathbf{u_0}+\mathbf{u})\cdot\nabla(\mathbf{u_0}+\mathbf{u}) + \nabla(p_0+p) = \frac{1}{Re}\Delta(\mathbf{u_0}+\mathbf{u}) + \mathbf{h} \tag{7a}$$

$$\nabla\cdot(\mathbf{u_0}+\mathbf{u}) = 0 \tag{7b}$$

If the effect of the structural perturbation is small, we can linearize these equations and obtain an equation for the perturbation

$$\frac{1}{T}\frac{\partial\mathbf{u}}{\partial\tau} + \mathbf{u_0}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u_0} + \nabla p - \frac{1}{Re}\Delta\mathbf{u} = \frac{\delta T}{T^2}\frac{\partial\mathbf{u_0}}{\partial\tau} + \mathbf{h} \tag{8a}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{8b}$$

This linear problem can be studied through Floquet analysis, and as is well known the resulting perturbation will in general *not* be periodic, but modified by the Floquet exponent. The condition, implicit in the definition of τ, that a constant period equal to 1 be maintained, constitutes a compatibility condition determining δT, which is exactly the variation of period induced by the structural perturbation $\mathbf{h} = \mathbf{C}\mathbf{u_0}$.

2.1 Adjoint Equations

Just as in the corresponding linear stability problem, if we just wanted to determine the variation of period for a specific form of structural perturbation we could solve the problem as stated above; but we can obtain a much more powerful result, i.e. the sensitivity of the period to an *arbitrary* structural perturbation with the aid of adjoint equations. Key to this approach is the observation that the unperturbed equations (6a,b) have a non-unique solution, insofar as if $\mathbf{u_0}(\tau)$ is a periodic solution, $\mathbf{u_0}(\tau + \delta\tau)$ is as well. Linearizing with respect to $\delta\tau$ we find that $\partial\mathbf{u_0}/\partial\tau$ is a solution of the linearized equations (8) in homogeneous form (e.g., with $\mathbf{h} = 0$). Since Equations (8a,b) with periodic boundary conditions have a nontrivial solution with zero forcing and zero δT, the original inhomogeneous linear problem only has a solution if a compatibility condition is satisfied. This compatibility condition can be derived through adjoint equations.

The adjoint of the linearized Navier–Stokes operator (8) is defined, as usual, by multiplying both sides of the equation by suitably differentiable fields $\{\mathbf{f}^+, m^+\}$, integrating over all space and a period in time, and then using integration by parts to shift differentiation operators from the direct to the adjoint fields. We thus obtain:

$$\begin{aligned}
&\int \mathbf{f}^+ \cdot \left(\frac{\delta T}{T^2} \frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \\
&= \int \left[\mathbf{f}^+ \cdot \left(\frac{1}{T} \frac{\partial \mathbf{u}}{\partial \tau} + \mathbf{u_0} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u_0} + \nabla p - \frac{1}{Re} \Delta \mathbf{u} \right) + m^+ \nabla \cdot \mathbf{u} \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \\
&= \int \left[\mathbf{u} \cdot \left(-\frac{1}{T} \frac{\partial \mathbf{f}^+}{\partial \tau} - \nabla \cdot (\mathbf{u_0} \mathbf{f}^+) + \nabla \mathbf{u_0} \cdot \mathbf{f}^+ - \nabla m^+ - \frac{1}{Re} \Delta \mathbf{f}^+ \right) + \right. \\
&\qquad \left. - p \nabla \cdot \mathbf{f}^+ \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t
\end{aligned} \tag{9}$$

where periodicity eliminates finite terms in time, and spatial boundary conditions for the adjoint are assumed to eliminate finite terms in space that are not already eliminated by boundary conditions for the direct problem. Equation (9) constitutes a generalized Green's identity (Morse and Feshbach [9]) for the LNSE. It is self-evident that if $\{\mathbf{f}^+, m^+\}$ are chosen to nullify the *r.h.s.* of Equation (9) independently of $\mathbf{u}$ and p, i.e. to satisfy the adjoint equations, the *l.h.s.* must be zero as well. Recalling that $\mathbf{h} = \mathbf{C}\mathbf{u_0}$, we thus obtain the compatibility condition

$$N \frac{\delta T}{T} = - \int \mathbf{f}^+ \cdot \mathbf{C}\mathbf{u_0}\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \quad \text{where} \quad N = \int \mathbf{f}^+ \cdot \frac{1}{T} \frac{\partial \mathbf{u_0}}{\partial \tau}\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \tag{10}$$

Since $\delta\omega/\omega = -\delta T/T$, we have obtained the structural sensitivity $\mathbf{S}$ of the oscillation frequency ω to the structural perturbation $\mathbf{C}(\mathbf{x})$:

$$\mathbf{S} = \frac{\delta \omega}{\delta \mathbf{C}} = \frac{\omega}{N} \int \mathbf{u_0} \mathbf{f}^+\, \mathrm{d}t \tag{11}$$

Notice that $\mathbf{C}$ is a tensor quantity, relating a force to a velocity, and so is $\mathbf{S}$. The notation $\mathbf{u_0}\mathbf{f}^+$ must be read as a dyadic product.

3 Numerical Approach

The time dependent flow around the cylinder is solved by discretizing the equations with finite differences on a staggered Cartesian grid. The advancement in time is obtained by the classical Runge-Kutta Crank-Nicholson scheme of Rai and Moin.

The presence of the cylinder is represented by an immersed-boundary technique similar to that used by Fadlun et al. [2]. Thus, the entire domain is covered by computational cells and there is no need for body-fitted coordinates. The boundary conditions on the surface of the cylinder Γ_c are imposed through a linear interpolation. Several interpolation procedures have been proposed in the past: in Fadlun et al. [2] the velocity at the first grid point external to the body is obtained by linearly interpolating the velocity at the second grid point (which is instead obtained by directly solving the Navier–Stokes equations) and the velocity at the body surface: in their numerical algorithm this condition is approximately enforced by applying momentum forcing inside the flow field. The interpolation direction is either the streamwise or the transverse direction, but the choice between them is not specified. Mohd-Yusof [5] used a more complex interpolation scheme which involved forcing the Navier–Stokes equations both inside and on the surface of the body. In particular the no-slip conditions were imposed at the point of the boundary touched by the wall-normal line passing through the closest internal point, using bilinear interpolations for this purpose. Finally, Kim et al. [3] introduced a mass injection forcing to satisfy the continuity equation for the cells containing the immersed boundary. A slightly different and easier approach has been used by Giannetti and Luchini (2007) to study the structural sensibility of the first instability of the cylinder wake. In this paper we follow this last approach: the interpolation is performed using the point closest to the body surface (which can be either an internal or an external point) and the following point on the exterior of the cylinder. The interpolation is performed either in the streamwise or transverse direction according to which one is closest to the local normal.

The linear system of algebraic equations deriving form the discretization of the nonlinear equations, along with their boundary conditions is solved at each substep through a sparse LU decomposition. Both the nonlinear equations and the adjoint equations are marched in time until a time-periodic state is reached.

4 Numerical Results

Figure 1 shows our typical result: a space distribution of the structural sensitivity **S** defined by Equation (11), in this case at a Reynolds number slightly above threshold. Since **S** is a tensor, various representative quantities may be chosen to be plotted. In Fig. 1 the choice is the spectral radius, which represents the sensitivity to a force of the worst possible direction. Other choices can be the Frobenius norm (sum of the squares of all four components) or the absolute value of the trace (sensitivity to a force locally aligned with velocity, i.e. a pure resistance).

Similar data for Re = 80 and 100 are shown in Figs. 2 and 3. All three figures agree remarkably well with the experimental data of Strykowski and Sreenivasan [7], who introduced a small perturbing cylinder in the wake of a larger one and reported the variation in critical Reynolds number as a function of position of the perturbing cylinder (Fig. 4).

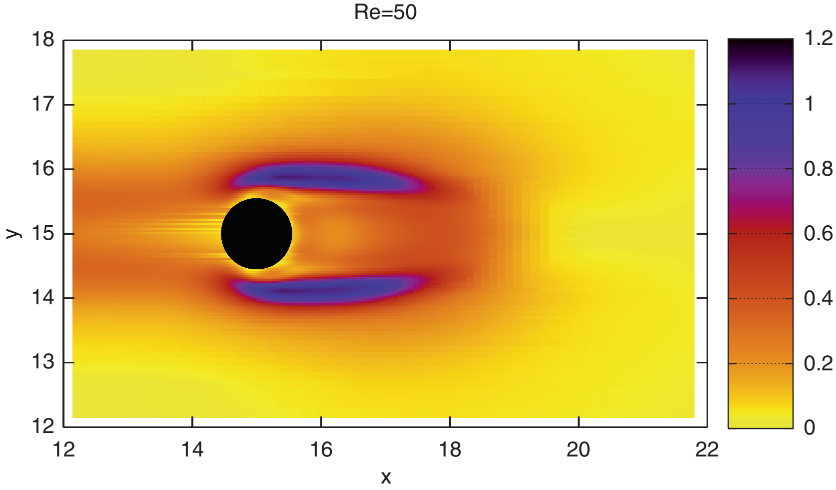

Fig. 1 Structural sensitivity of the periodic wake at Re = 50

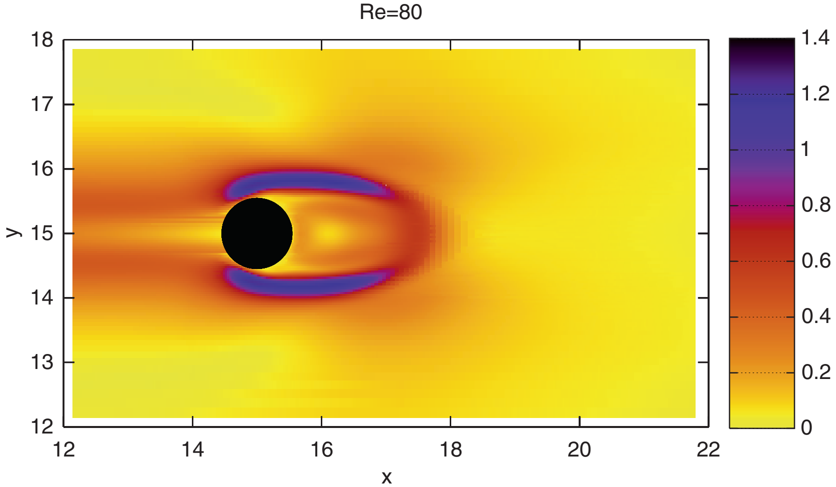

Fig. 2 Structural sensitivity of the periodic wake at Re = 80

4.1 Comparison with the Linear Results

In fact, it is a surprise that the structural sensitivity of the saturated periodic oscillation, even at the relatively low Reynolds number of 50, does not agree as satisfactorily with the structural sensitivity of the linear eigenmode as calculated by Giannetti and Luchini (2007) (Fig. 5). Actually, if attention is paid to the colour scale, it will be seen that the two are quite different in amplitude and not just in shape.

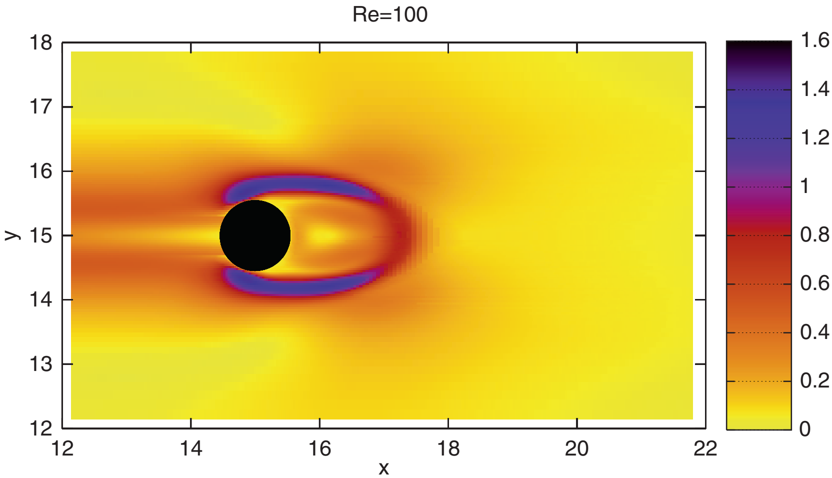

Fig. 3 Structural sensitivity of the periodic wake at Re = 100

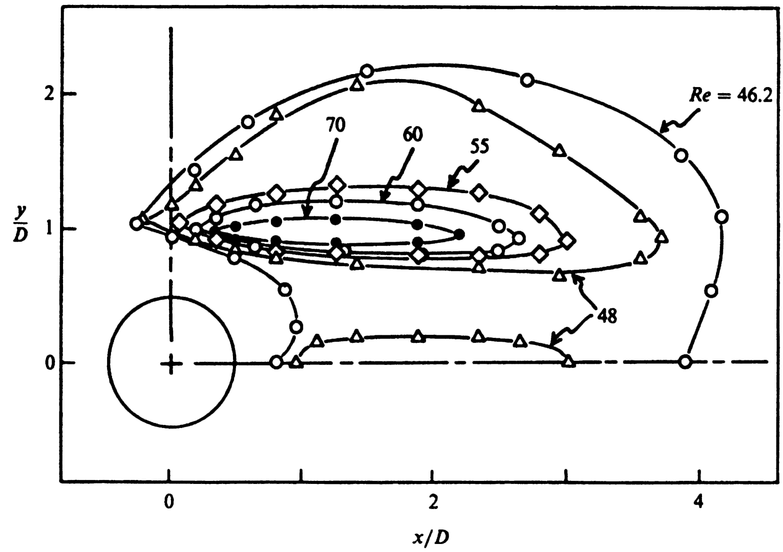

Fig. 4 Experiment of Strykowski and Sreenivasan [7]

This was a puzzle until we realized that the frequency of the nonlinear oscillation can be influenced in two different ways: by a structural perturbation force determined by the fluctuating velocity alone (as implicitly assumed in our linear results), or by a force that responds both to the mean and to the fluctuating velocity. Neither is wrong: they serve different purposes. The structural perturbation depending on the fluctuation only was the appropriate tool to study the position of the *wavemaker*,

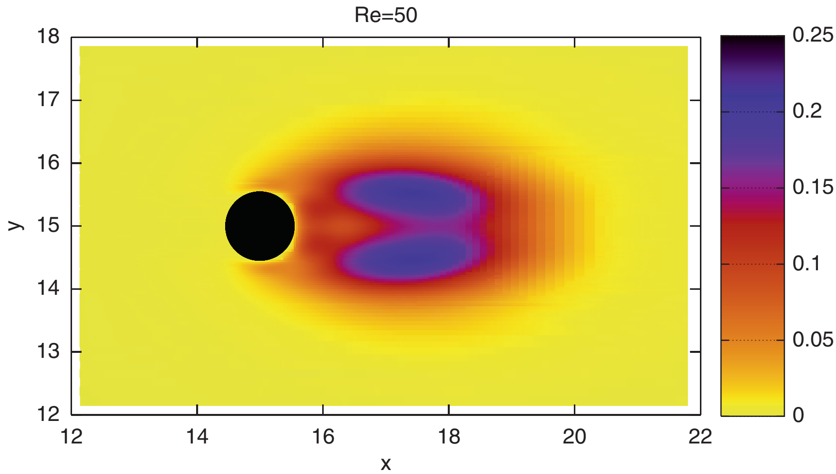

Fig. 5 Structural sensitivity of the linear instability mode at $\mathrm{Re} = 50$

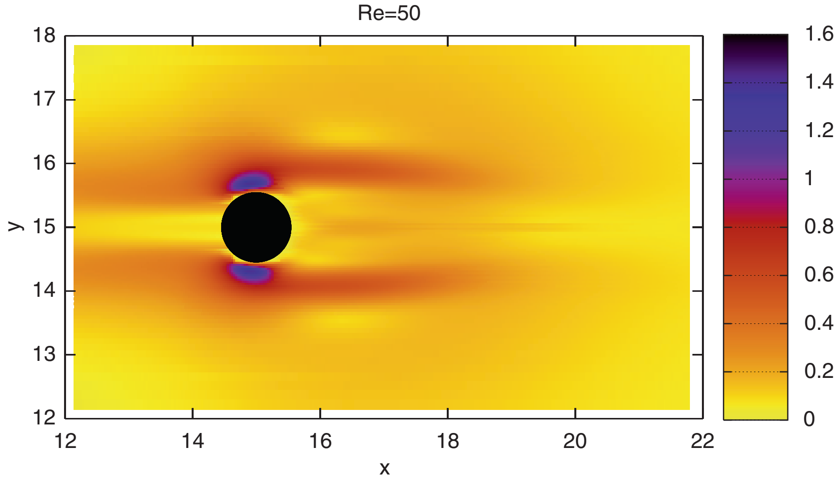

Fig. 6 Structural sensitivity of the linear instability mode at $\mathrm{Re} = 50$ with the base-flow modification included

but the perturbation depending on the full velocity field is the one that was implicitly assumed in the present nonlinear results, and of course is the one that occurs in the experiments. Once this difference is identified, it is not difficult to extend the linear eigenmode calculation to account for the frequency variation induced by a perturbation influencing the mean flow. On so doing Fig. 5 becomes Fig. 6 and, all of a sudden, a more satisfactory agreement is recovered with both experiments and nonlinear sensitivity results.

5 Conclusion

The structural sensitivity map of the frequency of a periodically oscillating wake to a perturbing force locally proportional to velocity has been determined for two-dimensional flow past a cylinder at various Reynolds numbers. The results, meant as an extension of the eigenmode structural sensitivity of Giannetti and Luchini (2007) have actually uncovered a dominant effect of the frequency variation induced by a modification of the base flow over the frequency variation induced by the direct structural perturbation of the eigenmode, thus clarifying that the former effect was also dominant in the experiments of Strykowski and Sreenivasan [7]. When the nonlinear results are epurated of the contribution of the mean flow, or vice versa this effect is included in the eigenmode calculation, agreement for near-threshold Reynolds number is actually recovered.

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