DIMEC Università degli Studi di Salerno

Leaky Waves in Spatial Stability Analysis

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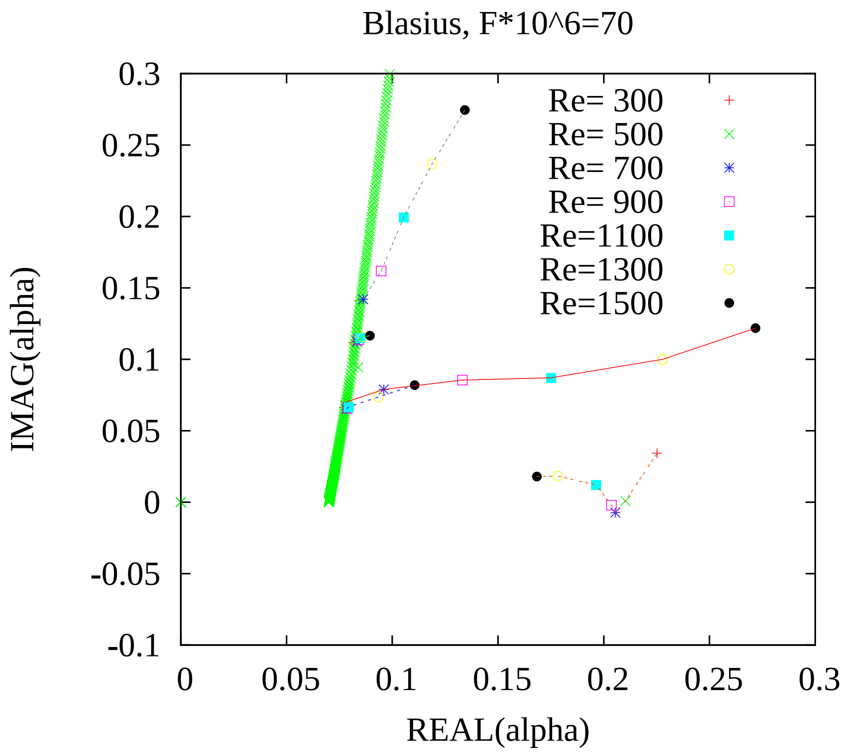
Background and motivation

- Laminar-turbulent transition in flat plate boundary layers, subject to low free strem turbulence, usually caused by infinitesimal perturbations which grow as they propagate downstream.
- Commonly analysed solving Orr-Sommerfeld equations (OSE) (eigen value problem)
- Solution of OSE composed of: a number of **discrete** modes decaying outside the boundary layer, a **continuous** spectrum behaving as $\exp(\mathrm{i}\beta y)$ in the free stream.
- The number of **discrete** modes changes with the Reynolds number, and seem to disappear behind the continuous spectrum at certain Reynolds numbers.

Investigation

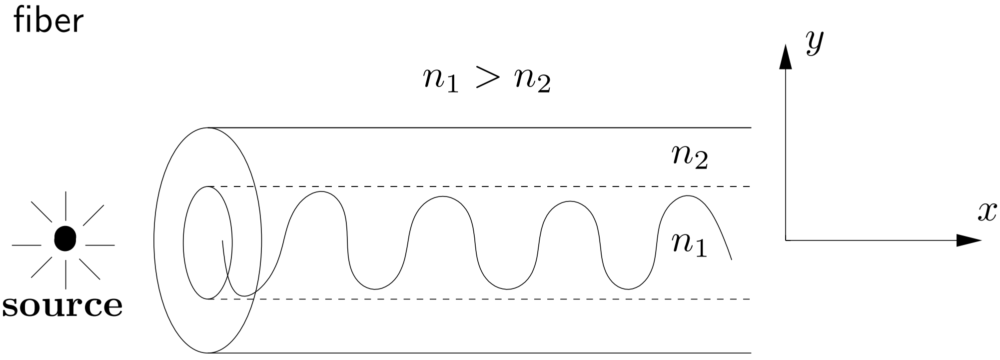
- Why does this happen ?
- Is it possible to have an all-discrete representation of the modes ?



Leaky waves

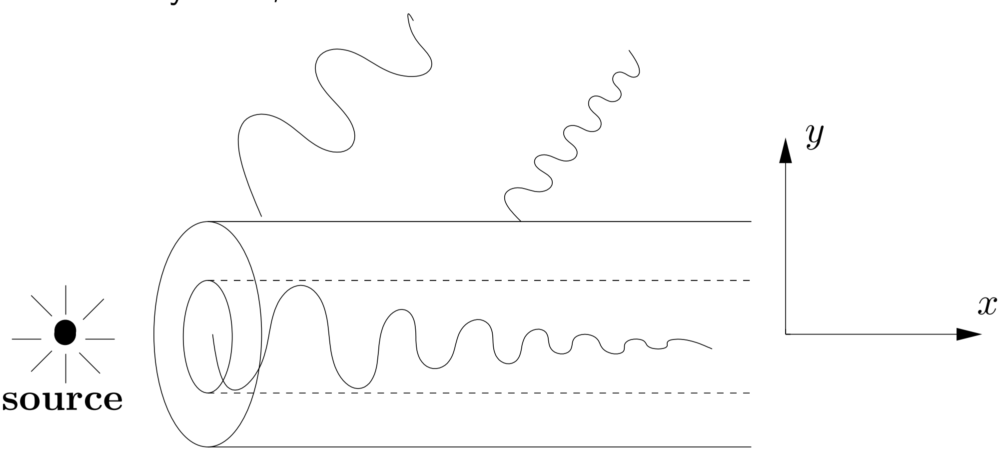
Short description of Leaky waves in fiber optics.

Ideal fiber



propagation constant α is real

Real fiber: leaky fiber, some radiation comes out



propagation constant has $Imag(\alpha) < 0$

$k^2 = \alpha^2 + \beta^2$ is real, gives $Imag(\beta) > 0$, when $Imag(\alpha) < 0$

The wave diverges as $\exp(-\mathrm{i}\beta y)$ for $y \to \infty$

when it decays for $x \to \infty$

Problem formulation

Consider the initial value problem, Orr-Sommerfeld equation

$$\left(\frac{\partial}{\partial t} + \mathrm{i}\alpha U\right)\Delta_2\, v + \mathrm{i}\alpha U'' v = \frac{1}{Re}\Delta_2\,\Delta_2 v, \quad \Delta_2 = \frac{d^2}{dy^2} - \alpha^2,$$

with initial condition

$$v(y, t = 0) = v_0(y).$$

Solution of the Laplace-transform

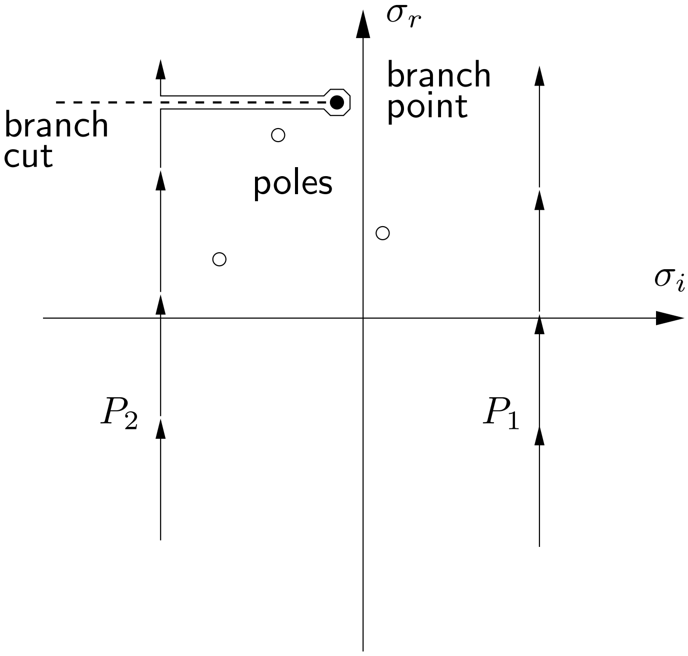
$$v = \frac{1}{2\pi\mathrm{i}} \int\int G(y, y', \sigma, \alpha)\Delta_2\, v_0(y')\, dy' e^{\sigma t} d\sigma,$$

where the Green function, G, is the solution of

$$\left((\sigma + \mathrm{i}\alpha U)\Delta_2 + \mathrm{i}\alpha U'' - \frac{1}{Re}\Delta_2\,\Delta_2\right) G = \delta_D(y - y')$$

with boundary conditions $G(0) = G'(0)$, and $G \to 0$ as $y \to \infty$

integration path's in the complex σ-plane



The singularities of G depend on the free stream boundary conditions.

The initial value problem

$$\left((\sigma + \mathrm{i}\alpha U)\Delta_2 + \mathrm{i}\alpha U'' - \frac{1}{Re}\Delta_2\,\Delta_2\right) G = \delta_D(y - y')$$

has only one solution. So does the homogeneous problem

$$\left((\sigma + \mathrm{i}\alpha U)\Delta_2 + \mathrm{i}\alpha U'' - \frac{1}{Re}\Delta_2\,\Delta_2\right) G = 0.$$

The free stream behaviour (with $U_\infty = 1$) is given by

$$\left(\sigma + \mathrm{i}\alpha - \frac{1}{Re}\Delta_2\right) \Delta_2 G = 0,$$

and the solution of $\Delta_2\, G$ can be written

$$A_1\, e^{\beta\, y} + A_2\, e^{-\beta\, y}$$

with

$$\beta^2 = \alpha^2 + Re(\sigma + \mathrm{i}\alpha).$$

A solution decaying as $y \to \infty$ can be obtained as a combination of the forced and homogeneous problems if $A_1 \neq 0$

The square-root realation between β and σ gives the branch point.

σ-formulation (original problem)

$$\left((\sigma + \mathrm{i}\alpha U)\Delta_2 + \mathrm{i}\alpha U'' - \frac{1}{Re}\Delta_2\,\Delta_2\right) G = \delta_D(y - y')$$

boundary conditions

$$G(0) = G'(0) = 0, \quad \text{and} \quad G \sim e^{-\sqrt{\alpha^2 + Re(\sigma + \mathrm{i}\alpha)}\,y}$$

This solution is multi-valued as $y \to \infty$(branch point)

β-formulation (alternative)

We write σ as a function of β

$$\sigma = -\mathrm{i}\alpha + \frac{1}{Re}(\beta^2 - \alpha^2)$$

and introduce it into the governing equation

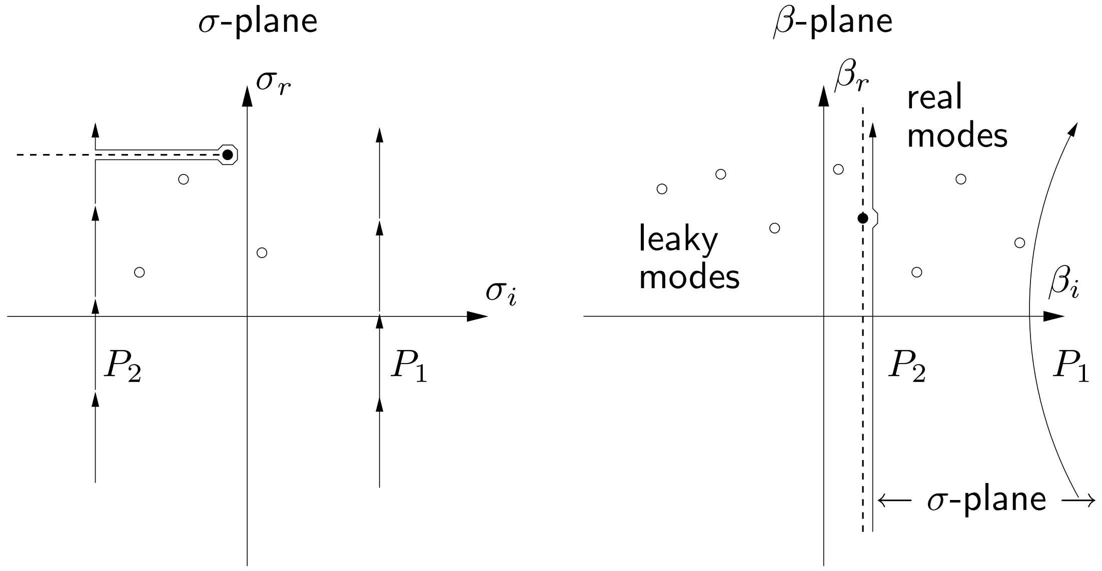
$$\left((-\mathrm{i}\alpha + \frac{1}{Re}(\beta^2 - \alpha^2) + \mathrm{i}\alpha U)\Delta_2 + \mathrm{i}\alpha U'' - \frac{1}{Re}\Delta_2\,\Delta_2\right) G = \delta_D(y - y'),$$

boundary conditions

$$G(0) = G'(0) = 0, \quad \text{and} \quad G \sim e^{-\beta\,y}$$

This solution is one-valued as $y \to \infty$

Change of variables from σ to β introduces the Leaky modes



With $G \sim e^{-\beta\, y}$, G is a one-valued function of β (as $y \to \infty$).

The integral over P_2 in the β-plane is the sum of residues of Leaky modes.

Numerical calculation

In the free stream solution, $A_1\,e^{\beta\,y} + A_2\,e^{-\beta\,y}$, want to keep growing solution, and zero decaying solution, Ill-conditioned.

Method 1: Solve in the complex y-plane

The mean flow, Blasius, converges for $\varphi \approx \pm 30°$. Here, $\tan(\varphi) = \max(y_i)/\max(y_r)$.

A direction is chosen in which the required β is dominant. Here, the analytical continuation of the real modes is found for $y_i < 0$.

Method 2: Use biorthogonality condition

$$(A - \beta\,B)\,u = 0 \qquad \text{right eigen vector}$$

$$v \cdot (A - \beta B) = 0 \qquad \text{left eigen vector}$$

From these two equations we obtain

$$(\beta_i - \beta_j)\,v_j \cdot B\,u_i = 0,$$

where

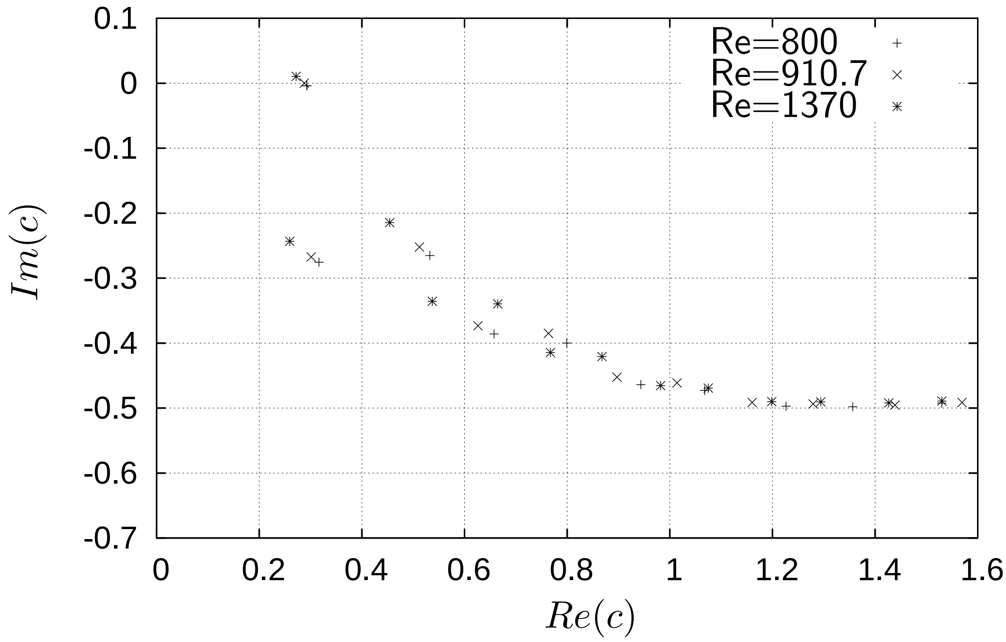
$$v_j \cdot B\,u_i = 0, \quad i \neq j.$$

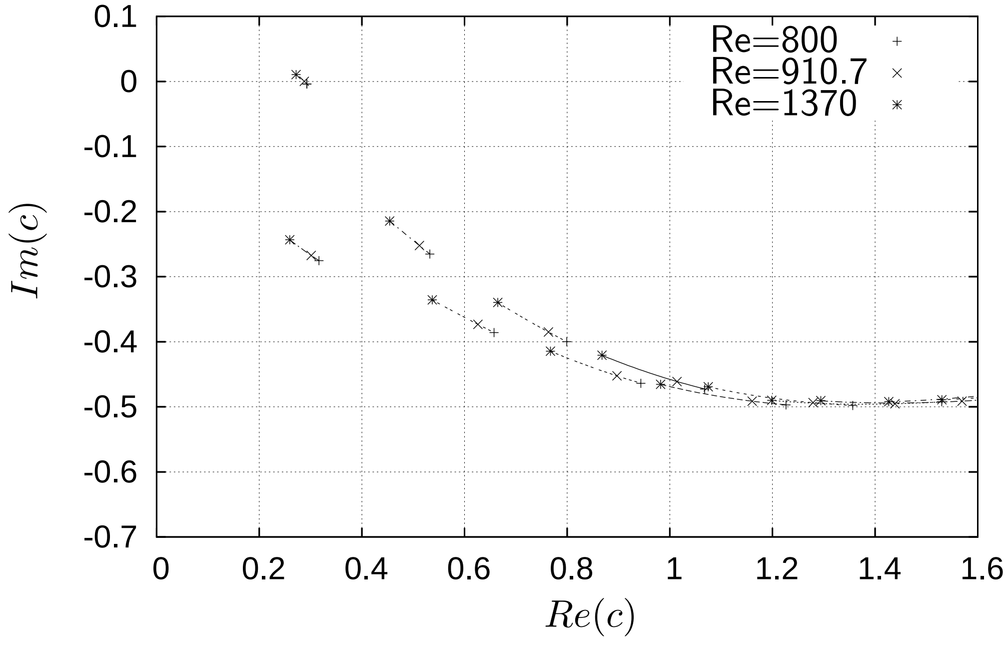
The free stream boundary condition to impose is

$$v_j \cdot B\,u = 0,$$

where subscript j denotes the solution we want to exclude.

Temporal stability analysis





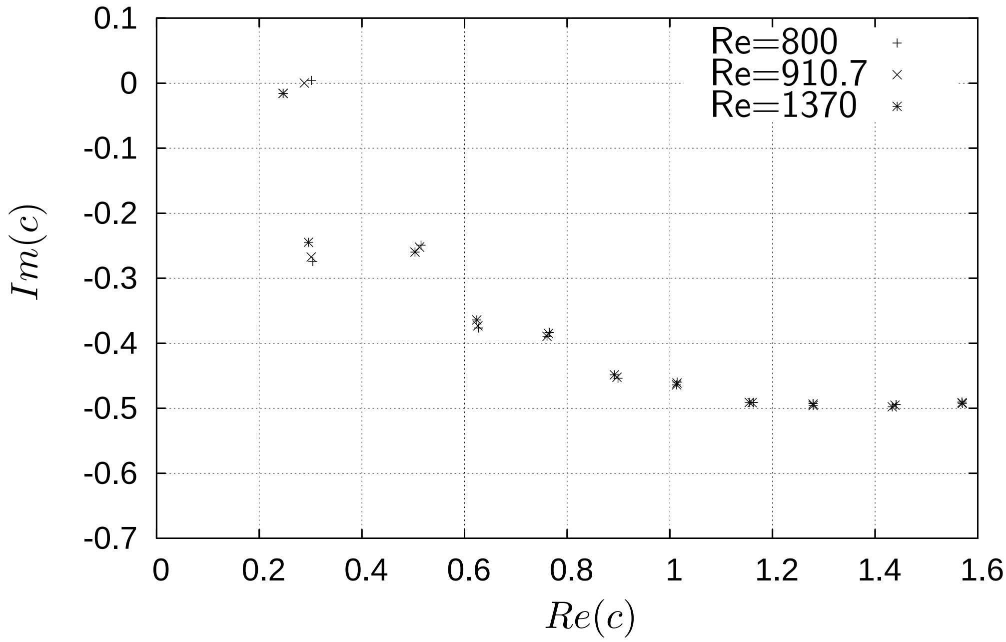
Temporal eigen value spectrum of the Blasius boundary layer for three values of Re, with $\alpha = 0.086$. At Re=910.7 $Im(c) = 0$ for the least stable eigen value. The "old" continuous spectrum: $Re(c) = 1$. Bottom: including trajectories of the discrete eigen values

Parameter dependency

Rewrite the OSE as

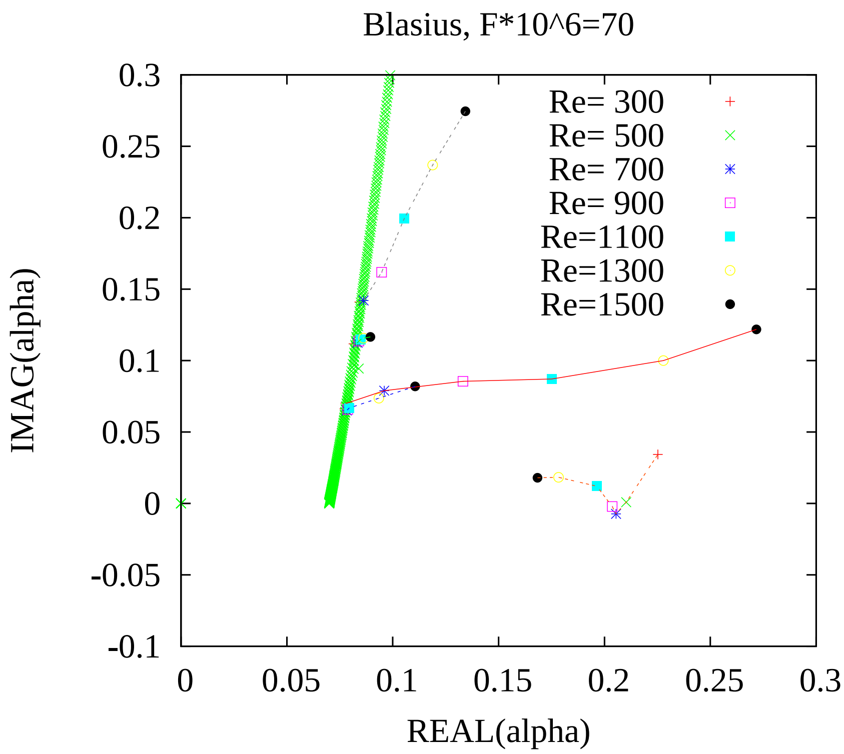
$$\left[\, \mathrm{i}\alpha Re \left\{ (U-c)(\mathcal{D}^2-\alpha^2) - U'' \right\} - (\mathcal{D}^2-\alpha^2)^2 \,\right] v = 0$$

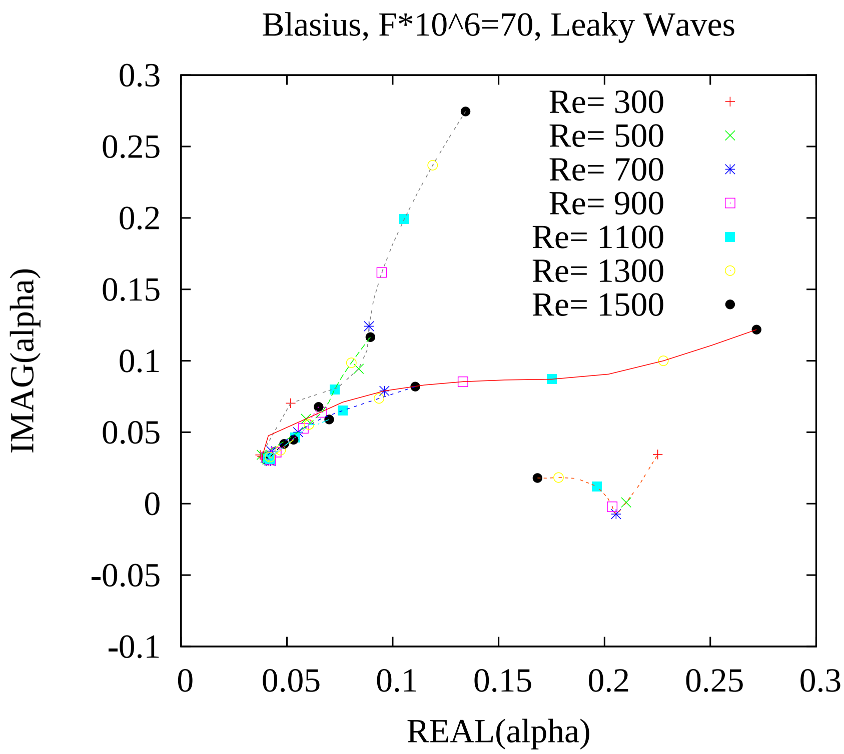
keep the product αRe constant



Temporal eigen value spectrum for three values of Re, keeping the value of αRe constant. At Re=910.7, α is chosen such that $Im(c) = 0$ for the least stable eigen value.

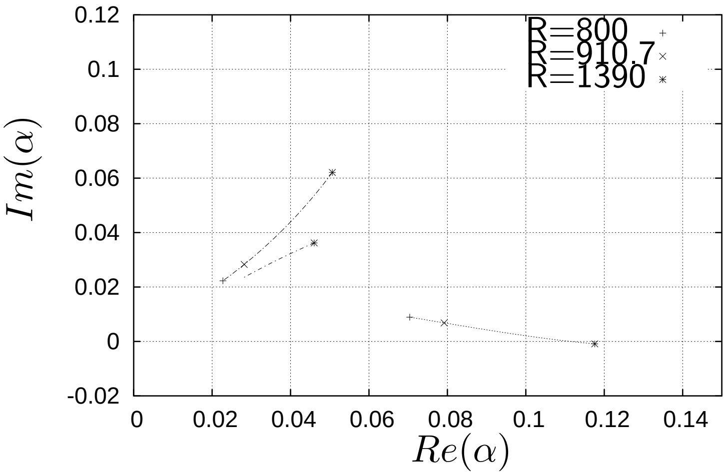
Spatial stability analysis, Blasius



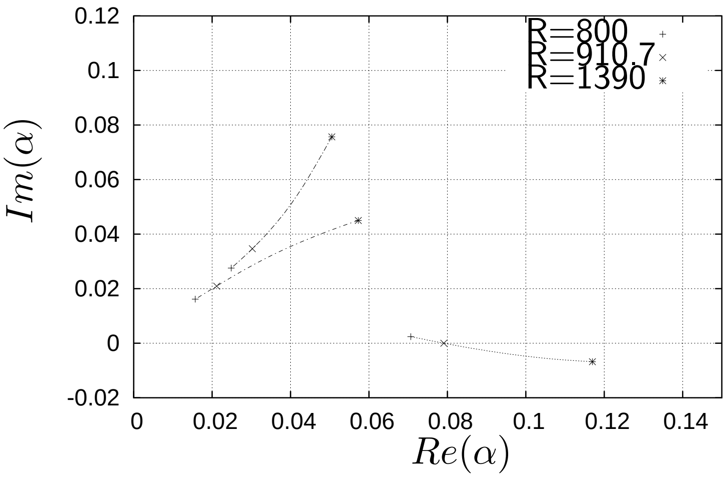


Spatial stability analysis, Falkner Skan (classic)

(a)



(b)



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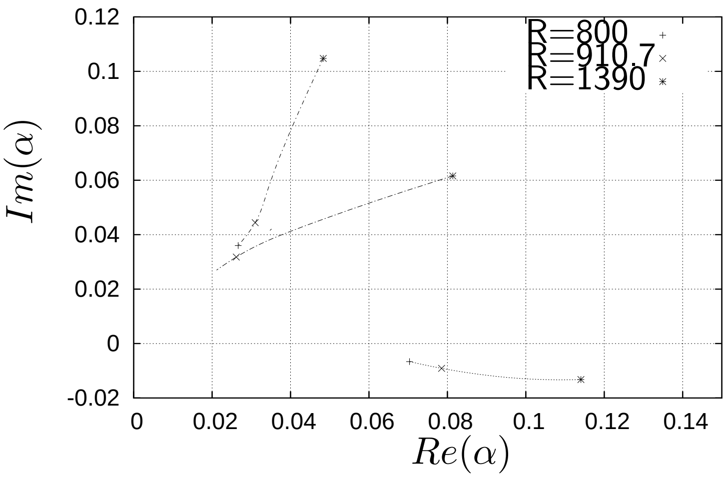
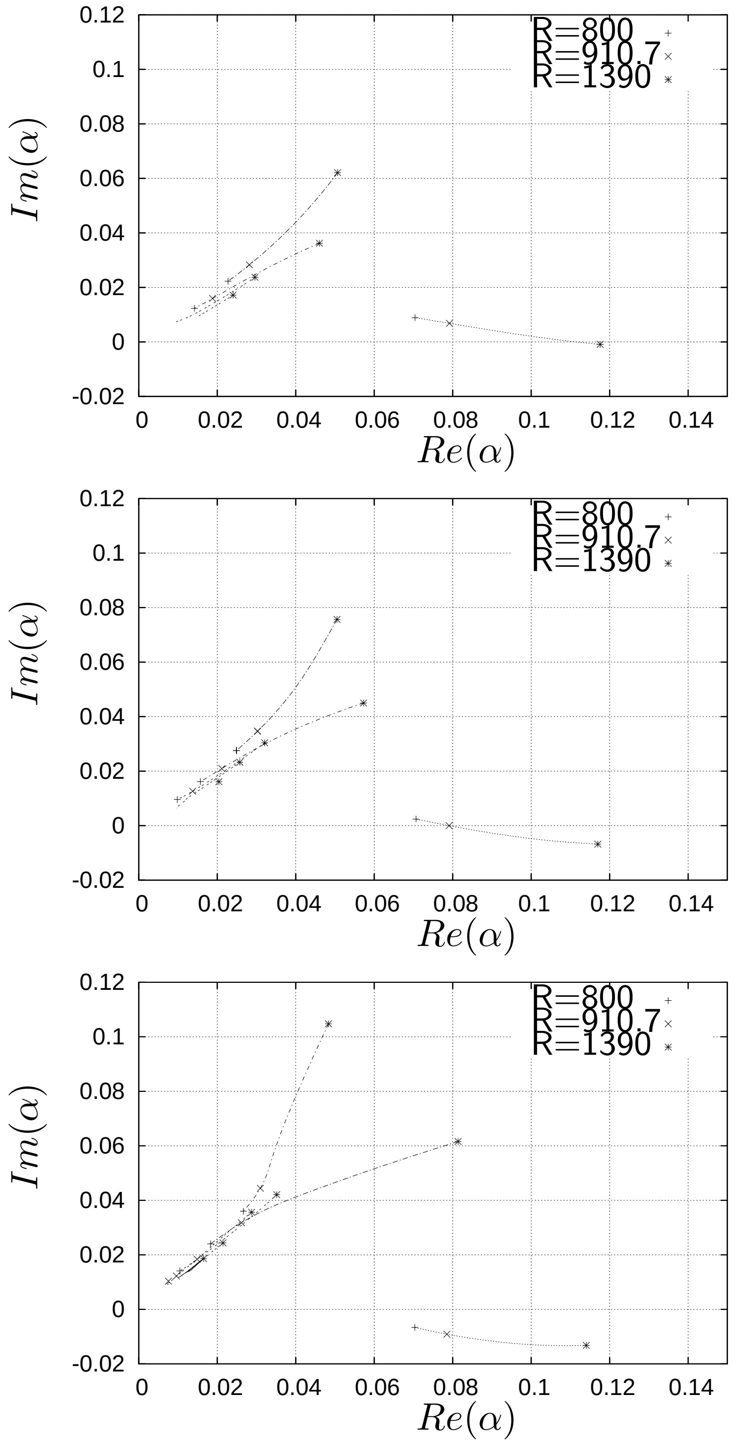


Spatial eigen value spectrum, α, three values of the Reynolds number, given $F = 25.10^{-6}$. The pressure gradient in the mean flow is given by the Hartree parameter (a) $\beta_H = 0.1$, (b) $\beta_H = 0$, (c) $\beta_H = -0.1$.

Spatial stability analysis, Falkner Skan (Leaky)

(a)
(b)
(c)



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Applications

- Increase understanding for ordering of modes
- Any application where "higher" modes are important
- Computing the first order correction of the eigen functions in Multiple-Scales analysis.

$$
\begin{aligned}
(A - \alpha_0 I)\, u_0 &= 0 \\
(A - \alpha_0 I)\, u_1 &= -\frac{du_0}{dx} \\
v_0 \cdot \frac{du_0}{dx} &= 0
\end{aligned}
$$

$$
u_1 = \sum_{j=2}^{N} \frac{1}{\alpha_j - \alpha_0} v_j \cdot \frac{du_0}{dx} u_j
$$