Sensitivity Analysis of the Finite-Amplitude Vortex Shedding behind a Cylinder

J. Pralits, F. Giannetti, P. Luchini

DIMEC, Università di Salerno
84084 Fisciano (SA), Italy
jpralits@unisa.it

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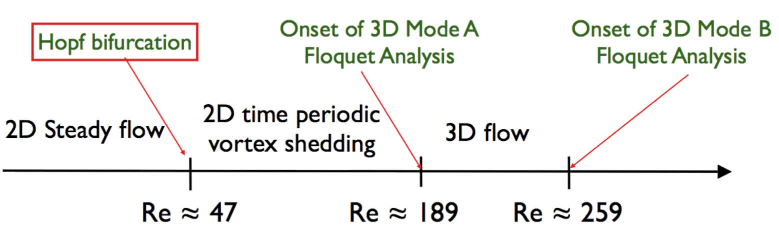
Outline

- Introduction
- Unstable stationary base flow
  - Eigenvalue sensitivity due to localized structural perturbations
  - Effects of localized base-flow modifictions
- Finite-amplitude vortex shedding
  - Sensitivity of the limit-cycle frequency
  - Sensitivity of the limit-cycle amplitude
  - Comparisons and implications
- Conclusions

The cylinder wake



- Global Stability

  Jackson (1987), Zebib (1987)

- "Wavemaker"

  Giannetti & Luchini (2007)

- Effects of base-flow variations

  Marquet et al. (2008), Luchini et al. (2008), Pralits et al. (2010)

streak-100-0-zoom.avi

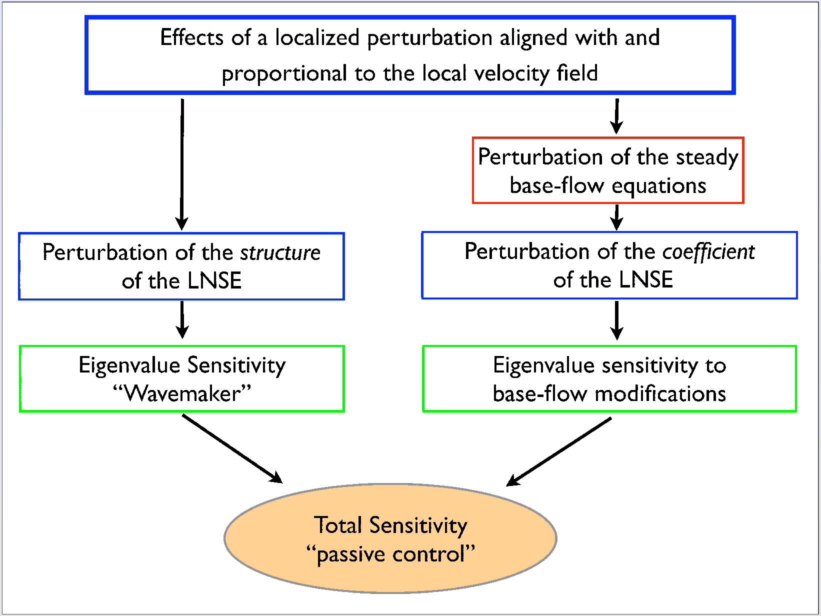
Localised structural sensitivity

- In the context of a two-dimensional modal analysis the core of the instability (wavemaker) can be found by investigating where in space a modification in the structure of the problem produces the largest drift of the eigenvalue.
- We consider structural perturbations consisting of a *localized* external force proportional to and aligned with the local velocity (*i.e.*, a small solid object[1]).

$$F(x,y) \approx \delta(x-x_0, y-y_0)\delta A I \cdot u(x,y) \qquad \delta A = \frac{4\pi}{Re \ln(7.4/Re_c)}$$

- With this choice it is possible to evaluate, separately, the effects induced on the frequency and growth rate of the instability.

[1] Pozrikidis (1996), Dyke (1975)


Effects of a localized perturbation aligned with and proportional to the local velocity field
Perturbation of the steady base-flow equations
Perturbation of the *structure* of the LNSE
Perturbation of the *coefficient* of the LNSE
Eigenvalue Sensitivity "Wavemaker"
Eigenvalue sensitivity to base-flow modifications
Total Sensitivity "passive control"

Sensitivity to perturbation modifiction

- Perturb the eigen problem with a localised feedback

$$\sigma'\mathbf{u}' + \mathbf{L}\{\mathbf{U}_b, Re\}\mathbf{u}' + \nabla p' = \delta(x - x_0, y - y_0)\delta\mathbf{C}_0 \cdot \mathbf{u}'$$
$$\nabla \cdot \mathbf{u}' = 0$$

- Expand $\mathbf{u}' = \mathbf{u} + \delta\mathbf{u}$, $p' = p + \delta p'$ and $\sigma' = \sigma + \delta\sigma$, insert into the equations and apply the Lagrange identity. The eigenvalue drift $\delta\sigma$ is then written

$$\boxed{\delta\sigma = \mathbf{S}_p(x_0, y_0) : \delta\mathbf{C}_0 \qquad \mathbf{S}_p(x,y) = \frac{\mathbf{u}(x,y)\ \mathbf{f}^+(x,y)}{\int_{\mathcal{D}} \mathbf{f}^+ \cdot \mathbf{u}\ \mathrm{d}^2\mathbf{x}}}$$

where $\mathbf{S}_p(x,y)$ is the sensitivity, and $\mathbf{f}^+$ is the adjoint velocity eigenvector.

Sensitivity to base-flow modification

- Perturb the base flow equations with a feedback from velocity to force ($\delta \mathbf{C}_b \cdot \mathbf{U_b}$), linearise and use the Lagrange identity to get the eigenvalue drift $\delta\sigma$ and the sensitivity $\mathbf{S}_b$

$$\delta\sigma = \mathbf{S}_b(x_0, y_0) : \delta\mathbf{C}_b \qquad \mathbf{S}_b(x,y) = \frac{\mathbf{U}_b(x,y)\, \mathbf{f}_b^+(x,y)}{\int_{\mathcal{D}} \mathbf{f}^+ \cdot \mathbf{u}\, \mathrm{d}^2\mathbf{x}}$$

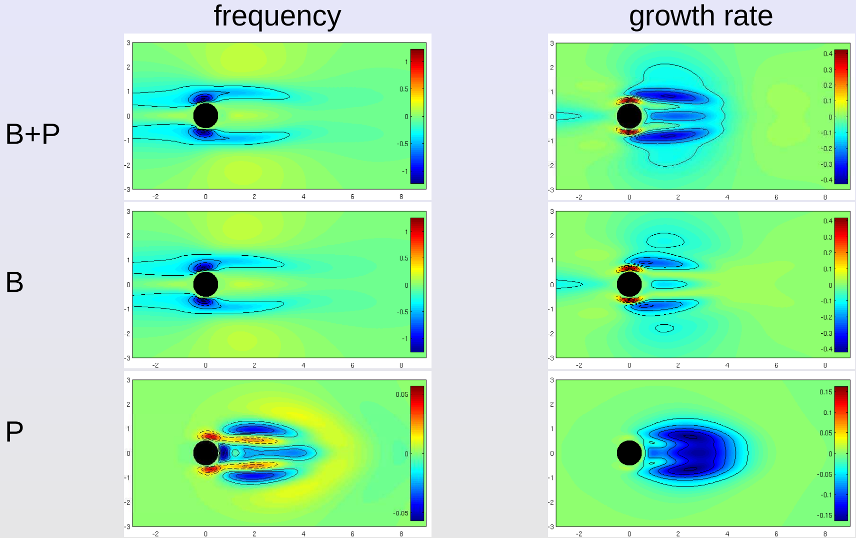
- Here $\mathbf{U}_b$ is the steady base flow velocity field, $\mathbf{f}^+$ and $\mathbf{u}$ are the direct and adjoint velocity eigenvectors while $\mathbf{f}_b^+$ satisfies the forced adjoint base flow equations

$$\mathbf{L}^+\{\mathbf{U}_b, Re\}\mathbf{f}^+ + \nabla m^+ = \delta\mathbf{C}^+(\mathbf{f}^+, \mathbf{u})$$
$$\nabla \cdot \mathbf{f}^+ = 0$$

where $$\delta\mathbf{C}^+(\mathbf{f}^+, \mathbf{u}) = \mathbf{u} \cdot \nabla\mathbf{f}^+ - \nabla\mathbf{u} \cdot \mathbf{f}^+,$$

Results: stationary base flow $Re = 50$

Note comparison with Marquet et al. (2008) for B



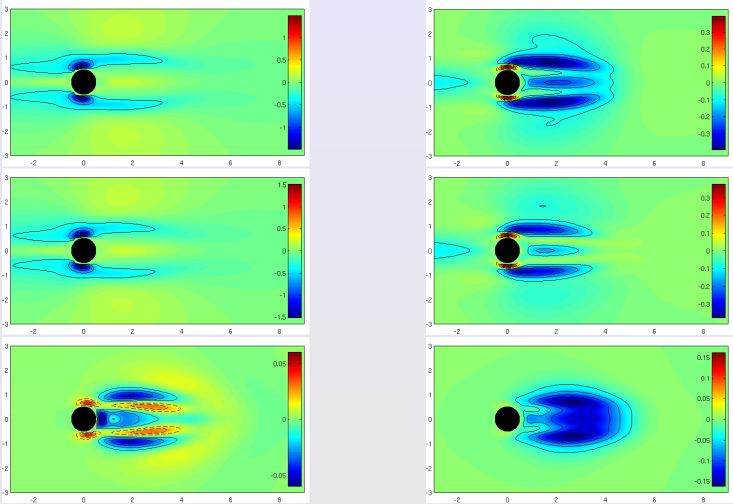
Results: stationary base flow $Re = 60$

frequency | growth rate

B+P

B

P



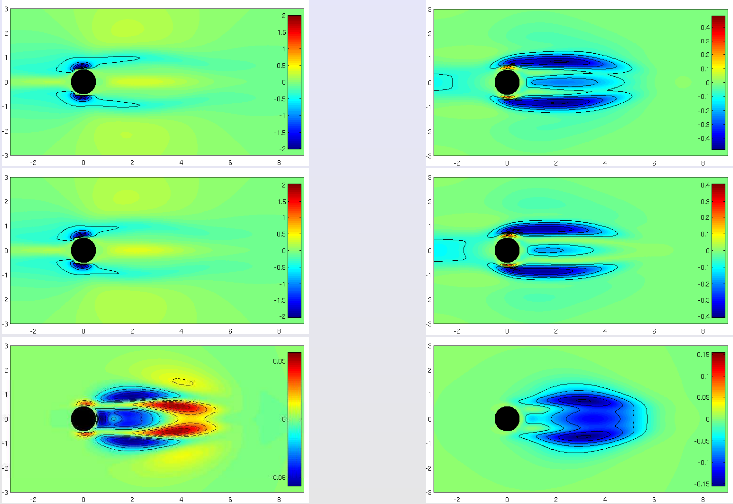
Results: stationary base flow $Re = 80$

frequency | growth rate

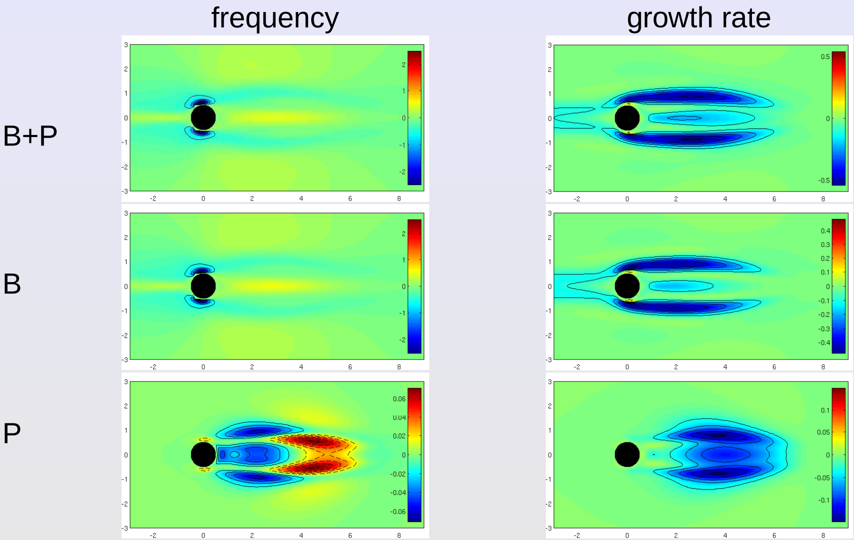
B+P

B

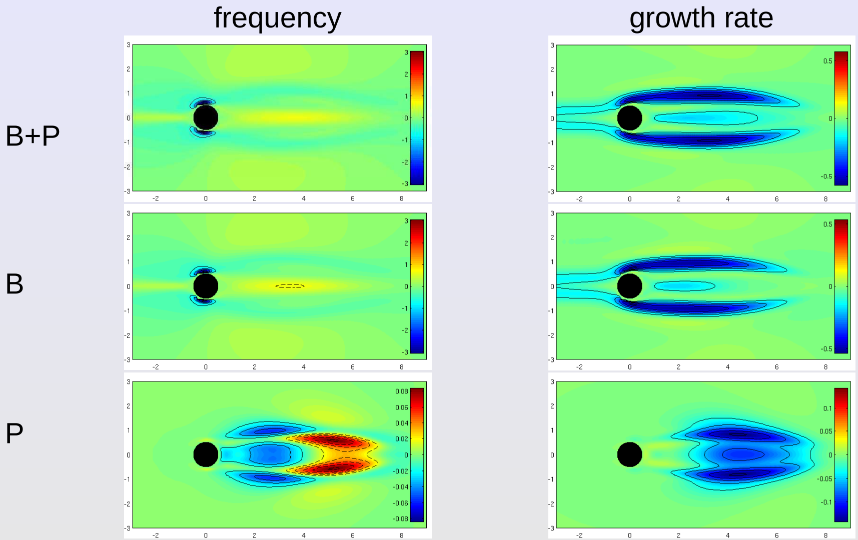
P



Results: stationary base flow $Re = 100$



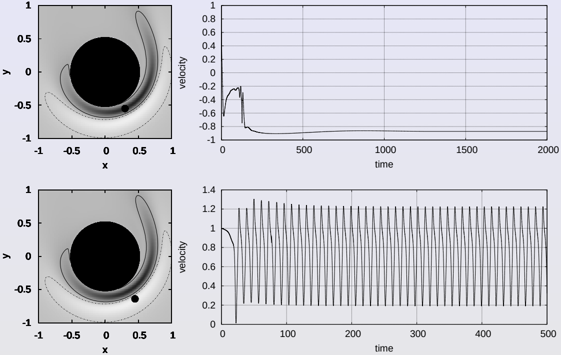
Results: stationary base flow $Re = 120$



Application: Control

Flow around the rotating cylinder, control of 2^{nd} shedding mode.

streak-100-5-zoom.avi



Pralits, Brandt & Giannetti (2010)

Analysis of finite-amplitude vortex shedding

- Linear analysis is only valid in proximity of the neutral curve ($Re_c \approx 47$). When the vortex shedding sets in, one may wonder where the wavemaker of the nonlinear periodic oscillation resides.
- We[2] investigate the finite-amplitude vortex shedding in order to assess how unsteadiness and saturation modify the linear sensitivity results.
- The quantity that enable us to do so is the spatial distribution of the sensitivity of the limit-cycle frequency and amplitude to a structural perturbation of the problem.

[2] *Luchini, Giannetti & Pralits*, **AIAA-2008-4227** (2008)

Problem formulation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \frac{1}{Re} \Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

$$A = \frac{1}{T} \int (\mathbf{u} - \bar{\mathbf{u}}) \cdot (\mathbf{u} - \bar{\mathbf{u}}) \, \mathrm{d}^3\mathbf{x} \, \mathrm{d}t$$

$$\mathbf{u}(t+T) = \mathbf{u}(t) \, , \; p(t+T) = p(t)$$

VorticityRe50.avi

Structural perturbation

Now give a structural perturbation **h** to the problem

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re} \Delta \mathbf{u} = \mathbf{h}$$

$$\nabla \cdot \mathbf{u} = 0$$

in the form of a bulk force depending on the local velocity

$$\underbrace{\mathbf{h}(\mathbf{u})}_{\text{force}} = \underbrace{\delta \mathbf{C}}_{\text{tensor coefficient}} \underbrace{\mathbf{u}}_{\text{velocity}}$$

If the perturbation is small, the new solution will remain periodic but with a different period (and a real frequency, in contrast with the corresponding linear problem whose frequency will in general become complex).

Summary of how to obtain the Sensitivity[3]

1. Make the period appear in the equations by rescaling the time as $0 < \tau = t/T < 1$ ▸ Step 1
2. Linearise the equations around the periodic undisturbed nonlinear solution $\mathbf{u_0}$ with period T_0. Perform a Floquet analysis of the resulting linear forced equations ▸ Step 2
3. Use the adjoint Floquet mode and the Lagrange identity to derive the compatibility condition which guarantees the existence of the solution of the original inhomogeneous linear problem ▸ Step 3

[3] for details see *Luchini, Giannetti & Pralits*, *AIAA conference paper*, **ID AIAA-2008-4227** (2008)

Adjoint equations

$$-\frac{1}{T_0}\frac{\partial \mathbf{f}^+}{\partial \tau} - \nabla \cdot (\mathbf{u_0}\mathbf{f}^+) + \nabla \mathbf{u_0} \cdot \mathbf{f}^+ - \nabla m^+ - \frac{1}{Re}\Delta \mathbf{f}^+ = \frac{2}{T_0}(\mathbf{u} - \bar{\mathbf{u}})$$

$$\nabla \cdot \mathbf{f}^+ = 0$$

The solution of these forced equations is given by

$$\mathbf{f}^+ = \underbrace{\mathbf{f}_p^+}_{\text{particular}} + \underbrace{\epsilon \mathbf{f}_0^+}_{\text{homogeneous}}$$

$$\delta A = \int (\mathbf{f}_p^+ + \epsilon \mathbf{f}_0^+) \cdot \left(\frac{\delta T}{T_0^2}\frac{\partial \mathbf{u_0}}{\partial \tau} + \delta \mathbf{C}\mathbf{u} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t$$

Note that δA does not depend on the value of ϵ since the forced direct equation has only a periodic solution if

$$\int \mathbf{f}_0^+ \cdot \left(\frac{\delta T}{T_0^2}\frac{\partial \mathbf{u_0}}{\partial \tau} + \delta \mathbf{C}\mathbf{u} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0.$$

Sensitivity with respect to limit-cycle frequency

Since $\delta\omega/\omega_0 = -\delta T/T_0$

$$\mathbf{S}_F(x,y) = \frac{\delta\omega}{\delta\mathbf{C}} = \frac{\omega_0}{N}\int_0^{T_0}\mathbf{u_0}(x,y)\mathbf{f}_0^+(x,y)\,\mathrm{d}t$$

$$\mathbf{S}_A(x,y) = \frac{\delta A}{\delta\mathbf{C}} = \int_0^{T_0}\mathbf{u_0}(x,y)\mathbf{f}^+(x,y)\,\mathrm{d}t$$

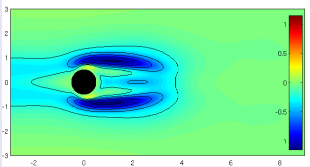
where $\mathbf{f}^+ = \mathbf{f}_p^+ + \epsilon\mathbf{f}_0^+$,

$$N = \int_0^{T_0}\mathbf{f}_0^+\cdot\frac{1}{T_0}\frac{\partial\mathbf{u_0}}{\partial t}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t, \qquad \epsilon = -\frac{1}{N}\int\mathbf{f}^+\cdot\frac{\partial\mathbf{u_0}}{\partial\tau}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t$$

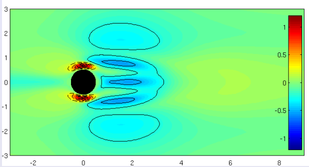
Results: Limit Cycle $Re = 50$ (Total)

LC

frequency

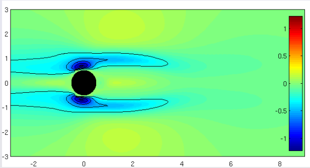


amplitude

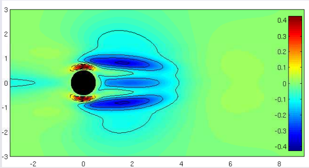


SF

frequency



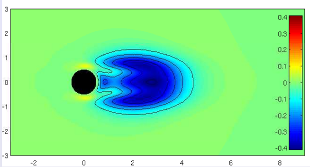
growth rate



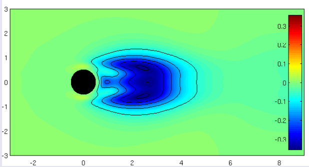
Results: Limit Cycle $Re = 50$ (“perturbation”)

LC

frequency

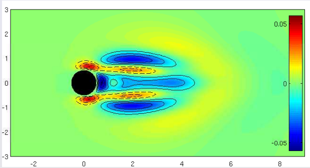


amplitude

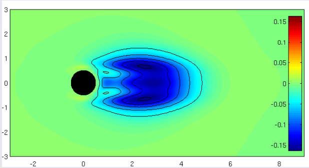


SF

frequency



growth rate



Results: Limit Cycle Frequency (Total)

Re

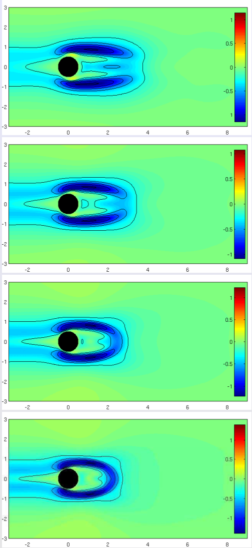
50

60

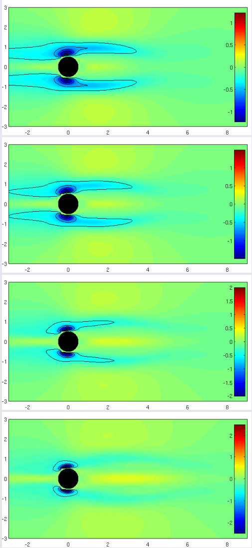
80

100

LC

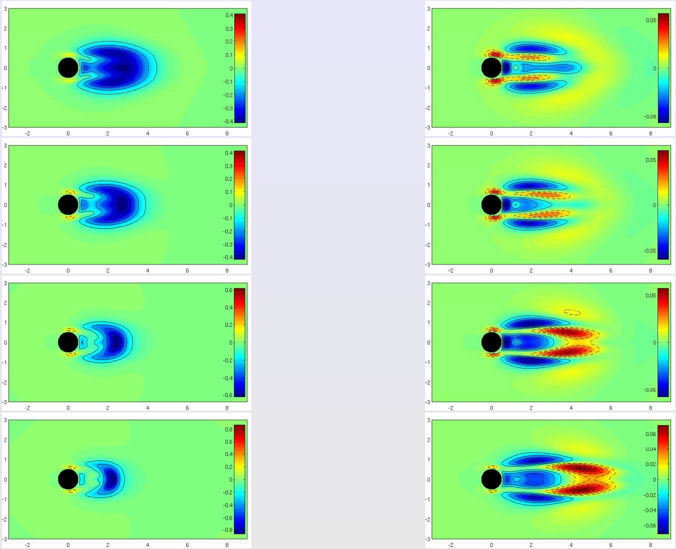


SF



Results: Limit Cycle Frequency (“perturbation”)

Re	LC	SF
50		
60		
80		
100		



Results: Limit Cycle Amplitude (Total)

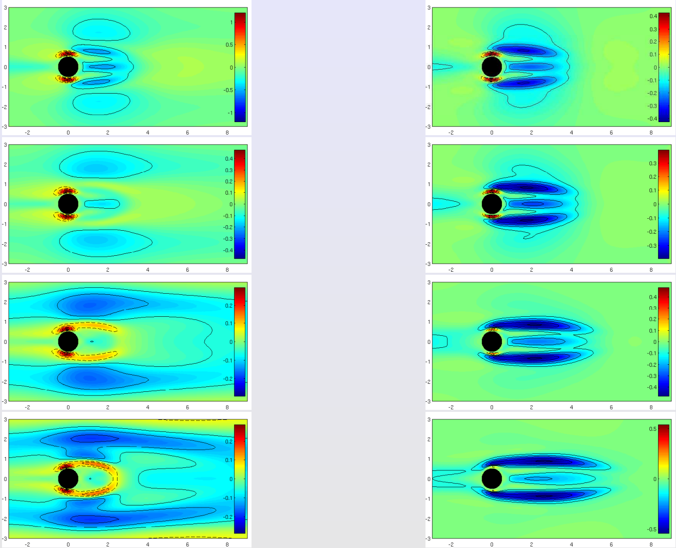
Re LC SF

50

60

80

100



Results: Limit Cycle Amplitude (“perturbation”)

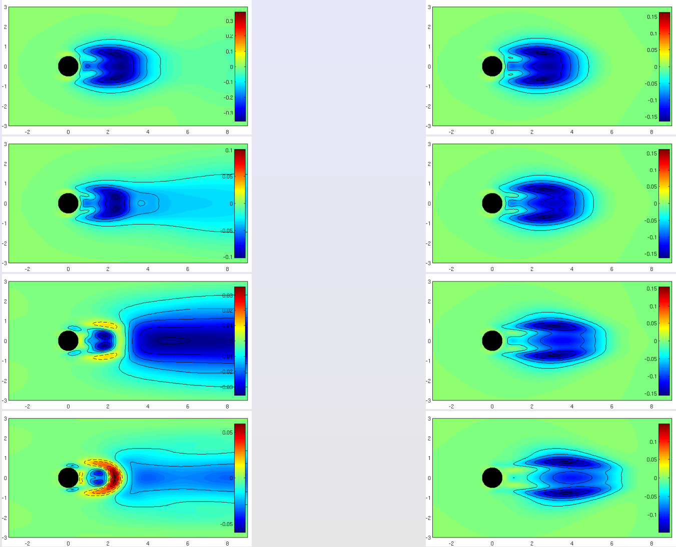
Re | LC | SF

50

60

80

100



Conclusions

- The idea of Localised structural sensitivity has been extended to supercritical flows.
- The theory has been devloped for periodic flows using floquet analysis.
- Expressions for the sensitivity have been derived both for the frequency and amplitude.
- A comparisons with the case of stationary base flow show similarities close to the first bifurcation.

- To be done: derive coefficients for the Stuart-Landau amplitude equation accounting for structural forcing.

Appendix

Appendix Appendix

Step 1: making the period appear in the equations

Define an amplitude

$$A = \frac{1}{T}\int_0^T (\mathbf{u} - \bar{\mathbf{u}}) \cdot (\mathbf{u} - \bar{\mathbf{u}})\, \mathrm{d}t .$$

and a scaled time, $\tau = \frac{t}{T}$, then the equations can be rewritten as

$$\frac{1}{T}\frac{\partial \mathbf{u}}{\partial \tau} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \frac{1}{Re}\Delta \mathbf{u} + \mathbf{h}$$

$$\nabla \cdot \mathbf{u} = 0$$

where T is an additional unknown and the period in the variable τ is constant and equal to 1.

Step 2: linearization

$$\mathbf{u}(\tau) = \mathbf{u_0}(\tau) + \delta\mathbf{u}(\tau), \quad p = p_0(\tau) + \delta p(\tau)$$

$$T = T_0 + \delta T, \quad A = A_0 + \delta A$$

Upon linearization with respect to $\delta\mathbf{u}$, δp and δT:

$$\frac{1}{T_0}\frac{\partial \delta\mathbf{u}}{\partial \tau} + \mathbf{u_0}\cdot\nabla\delta\mathbf{u} + \delta\mathbf{u}\cdot\nabla\mathbf{u_0} + \nabla\delta p - \frac{1}{Re}\Delta\delta\mathbf{u} = \frac{\delta T}{T_0^2}\frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h}$$

$$\nabla\cdot\delta\mathbf{u} = 0$$

$$\delta A = \int_0^1 2(\mathbf{u} - \bar{\mathbf{u}})\cdot\delta\mathbf{u}\,\mathrm{d}\tau.$$

- The resulting perturbation will in general *not* be periodic, but modified by the Floquet exponent.
- The condition that a constant period equals to 1 be maintained in τ implicitly determines the variation δT

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$$\nabla\cdot\delta\mathbf{u} = 0$$

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- The condition that a constant period equals to 1 be maintained in τ implicitly determines the variation δT

Step 3: adjoint equations

- If we just wanted to determine the variation of period for a specific form of structural perturbation we could solve the problem as stated above; but we can obtain a much more powerful result, *i.e.* the sensitivity of the period to an *arbitrary* structural perturbation with the aid of adjoint equations.
- Key to this approach is the observation that the limit cycle is non-unique, insofar as if $\mathbf{u_0}(\tau)$ is a periodic solution, $\mathbf{u_0}(\tau + \delta\tau)$ is as well.
- By Fredholm's alternative, the original inhomogeneous linear problem only has a solution if a compatibility condition is satisfied. This compatibility condition can be derived through a generalized Lagrange identity.

◂ Summary of the Procedure ◂ Step 2 ▸ Generalized Lagrange identity

Generalized Lagrange identity

$$\int \mathbf{f}^{+} \cdot \left(\frac{\delta T}{T_0^2} \frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \left[\mathbf{f}^{+} \cdot \left(\frac{1}{T_0} \frac{\partial \delta \mathbf{u}}{\partial \tau} + \mathbf{u_0} \cdot \nabla \delta \mathbf{u} + \delta \mathbf{u} \cdot \nabla \mathbf{u_0} + \nabla \delta p - \frac{1}{Re} \Delta \delta \mathbf{u} \right) + \right.$$
$$\left. + m^{+} \nabla \cdot \delta \mathbf{u} \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \left[\delta \mathbf{u} \cdot \left(-\frac{1}{T_0} \frac{\partial \mathbf{f}^{+}}{\partial \tau} - \nabla \cdot (\mathbf{u_0} \mathbf{f}^{+}) + \nabla \mathbf{u_0} \cdot \mathbf{f}^{+} - \nabla m^{+} - \frac{1}{Re} \Delta \mathbf{f}^{+} \right) + \right.$$
$$\left. - \delta p \nabla \cdot \mathbf{f}^{+} \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \delta \mathbf{u} \cdot 2(\mathbf{u} - \bar{\mathbf{u}}) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = \delta A$$

Generalized Lagrange identity

$$\int \mathbf{f}^{+} \cdot \left(\frac{\delta T}{T_0^2} \frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \left[\mathbf{f}^{+} \cdot \left(\frac{1}{T_0} \frac{\partial \delta \mathbf{u}}{\partial \tau} + \mathbf{u_0} \cdot \nabla \delta \mathbf{u} + \delta \mathbf{u} \cdot \nabla \mathbf{u_0} + \nabla \delta p - \frac{1}{Re} \Delta \delta \mathbf{u} \right) + \right.$$
$$\left. + m^{+} \nabla \cdot \delta \mathbf{u} \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

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Generalized Lagrange identity

$$\int \mathbf{f}^+ \cdot \left(\frac{\delta T}{T_0^2} \frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h} \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \left[\mathbf{f}^+ \cdot \left(\frac{1}{T_0} \frac{\partial \delta \mathbf{u}}{\partial \tau} + \mathbf{u_0} \cdot \nabla \delta \mathbf{u} + \delta \mathbf{u} \cdot \nabla \mathbf{u_0} + \nabla \delta p - \frac{1}{Re} \Delta \delta \mathbf{u} \right) + \right.$$
$$\left. + m^+ \nabla \cdot \delta \mathbf{u} \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \left[\delta \mathbf{u} \cdot \left(-\frac{1}{T_0} \frac{\partial \mathbf{f}^+}{\partial \tau} - \nabla \cdot (\mathbf{u_0} \mathbf{f}^+) + \nabla \mathbf{u_0} \cdot \mathbf{f}^+ - \nabla m^+ - \frac{1}{Re} \Delta \mathbf{f}^+ \right) + \right.$$
$$\left. - \delta p \nabla \cdot \mathbf{f}^+ \right] \mathrm{d}^3\mathbf{x}\, \mathrm{d}t =$$

$$= \int \delta \mathbf{u} \cdot 2(\mathbf{u} - \bar{\mathbf{u}}) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = \delta A$$

Generalized Lagrange identity

$$\int \mathbf{f}^+ \cdot \left(\frac{\delta T}{T_0^2} \frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h} \right) \mathrm{d}^3\mathbf{x}\,\mathrm{d}t =$$

$$= \int \left[\mathbf{f}^+ \cdot \left(\frac{1}{T_0} \frac{\partial \delta \mathbf{u}}{\partial \tau} + \mathbf{u_0} \cdot \nabla \delta \mathbf{u} + \delta \mathbf{u} \cdot \nabla \mathbf{u_0} + \nabla \delta p - \frac{1}{Re} \Delta \delta \mathbf{u} \right) + \right.$$
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$$\left. - \delta p \nabla \cdot \mathbf{f}^+ \right] \mathrm{d}^3\mathbf{x}\,\mathrm{d}t =$$

$$= \int \delta \mathbf{u} \cdot 2(\mathbf{u} - \bar{\mathbf{u}}) \mathrm{d}^3\mathbf{x}\,\mathrm{d}t = \delta A$$

Compatibility Conditions

$$-\frac{1}{T_0}\frac{\partial \mathbf{f}^+}{\partial \tau} - \nabla\cdot(\mathbf{u_0}\mathbf{f}^+) + \nabla\mathbf{u_0}\cdot\mathbf{f}^+ - \nabla m^+ - \frac{1}{Re}\Delta\mathbf{f}^+ = \frac{2}{T}(\mathbf{u}-\bar{\mathbf{u}})$$

$$\nabla\cdot\mathbf{f}^+ = 0$$

The solution of these forced equations is given by

$$\mathbf{f}^+ = \underbrace{\mathbf{f}_p^+}_{\text{particular solution}} + \underbrace{\epsilon\mathbf{f}_0^+}_{\text{homogeneous solution}}$$

$$\delta A = \int (\mathbf{f}_p^+ + \epsilon\mathbf{f}_0^+)\cdot\left(\frac{\delta T}{T_0^2}\frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}t$$

Note that δA does not depend on the value of ϵ since the forced direct equation has only a periodic solution if

$$\int \mathbf{f}_0^+\cdot\left(\frac{\delta T}{T_0^2}\frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{h}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}t = 0.$$

Sensitivity

From the last compatibility condition we easily obtain

$$\boxed{\frac{\delta T}{T} = -\frac{\displaystyle\int_0^T \mathbf{f}_0^+ \cdot \mathbf{h}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t = 0}{\displaystyle\int_0^T \mathbf{f}_0^+ \cdot \frac{\partial \mathbf{u_0}}{\partial \tau}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t}}$$

$$\boxed{\delta A = \int_0^T \mathbf{f}^+ \cdot \mathbf{h}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t}$$

where $\mathbf{f}^+ = \mathbf{f}_p^+ + \epsilon \mathbf{f}_0^+$ and

$$\epsilon = -\frac{\displaystyle\int_0^T \mathbf{f}^+ \cdot \frac{\partial \mathbf{u_0}}{\partial \tau}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t}{\displaystyle\int_0^T \mathbf{f}_0^+ \cdot \frac{\partial \mathbf{u_0}}{\partial \tau}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t}$$