UNIVERSITY of L'AQUILA

Department of Information Engineering, Computer Science, and Mathematics

Master Thesis

in

Mathematical Engineering

Stability of a stratified fluid over an oscillating flat wall

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Abstract

Retinal detachment often occurs when the vitreous gel, a thick gel that fills the center of the eye, shrinks and separates from the retina. This process is called a posterior vitreous detachment (PVD); it normally appears with aging and can be harmless. Sometimes, though, PVD can tear the retina. This happens where the vitreous gel is strongly attached to the retina. As the vitreous gel shrinks, it might pull so hard that the retina tears. The tear allows fluid to flow under the retina and this may cause the retina to detach. Vitrectomy is a surgical procedure employed to treat retinal breaks and retinal detachment. It consists in the replacement of the whole vitreous humor, which is substituted with tamponade fluids. Often, silicone oils are used as tamponade fluids. Since the oil is hydrophobic a thin layer of aqueous always forms between the oil and the retina. In some cases the oil-water interface turns out to be unstable and the oil forms an emulsion. In this case the oil has to be removed from the vitreous chamber. The conditions under which this happens are still unclear. In the thesis we presented a preliminary model of the stability of the oil-aqueous interface. We modeled two layers of different immiscible fluids set in motion by harmonic oscillation of the wall.
We found an analytical solution of the basic flow and a numerical solution of the eigenvalue problem given by the Orr-Sommerfeld equations in terms of stream functions together with boundary conditions. The analysis was made for values of parameters that are relevant for the retinal detachment problem. That led us to use so-called quasi-steady approach.

The results showed us that instability is possible for the certain values of parameters. Also, we concluded the the surface tension has a strong stabilizing effect on short perturbations. The viscosity also slightly influences the stability of the system.

In addition to that the possibilities for further research are discussed.

Chapter 1

Introduction

1.1 Eye anatomy

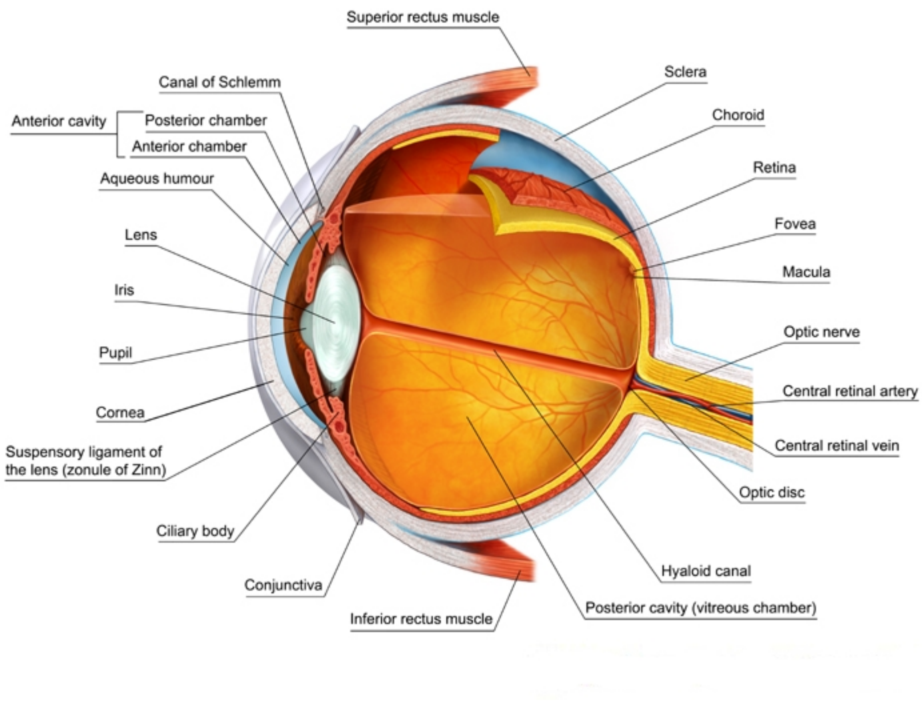

Figure 1.1: Eye anatomy

The human eye is the organ which gives us the sense of sight, allowing us to observe and learn more about the surrounding world than we do with any of the other four senses. We use our eyes in almost every activity we perform, whether reading, working, watching television, writing a letter, driving a car, and in countless other ways. Most people probably would agree that sight is the sense they value more than all the rest.

The eye allows us to see and interpret the shapes, colors, and dimensions of objects in the world by processing the light they reflect or emit. The eye is able to detect bright light or dim light, but it cannot sense objects when light is absent.

The eye is often compared to a camera. Both gather light and then transforms it into a picture. Both also have lenses to focus the incoming light. A camera uses the film to create a picture, whereas the eye uses a specialized layer of cells, called the retina, to produce an image.

Below we provide a short description of the main parts of the human eye.
Conjuctiva The conjunctiva is a thin, clear layer of skin covering the front of the eye, including the sclera and the inside of the eyelids. The conjunctiva avoids that bacteria and foreign material from getting behind the eye.
Sclera The white part of your eye that you see when you look at yourself in the mirror is the front part of the sclera. The sclera is, a tough, leather-like tissue, and constitutes the eye wall everywhere except the front part. Just like an eggshell surrounds an egg and gives an egg its shape, the sclera surrounds the eye and gives the eye its shape.
The sclera is also attached to the extraocular muscles, which allow the eye to perform rotations.
Cornea The cornea is a clear layer at the front and center of the eye. In fact, the cornea is so clear that you may not even realize it is there. It is located just in front of the iris, which is the colored part of the eye. The main purpose of the cornea is to help focus light as it enters the eye. If you wear contact lenses, the contact lens rests on your cornea.
Anterior Chamber The space in the eye that is behind the cornea and in front of the iris. The anterior chamber is filled with a watery fluid known as the aqueous humor, or aqueous. Produced by a structure alongside the lens called the ciliary body, the aqueous passes first into the posterior chamber (between the lens and iris) and then flows forward through the pupil into the anterior chamber of the eye.
Aqueous humor. In medicine, humor refers to a fluid (or semifluid) substance. Thus, the aqueous humor is the fluid normally present in the front and rear chambers of the eye. It is a clear, watery fluid that flows between and nourishes the lens and the cornea.
Anterior Chamber Angle/Trabecular Meshwork The anterior chamber angle and the trabecular meshwork are located where the cornea meets the iris. The trabecular meshwork is important because it is the site where the aqueous humor drains out of the eye. If the aqueous humor cannot properly drain out of the eye, the optic nerve damage can occur which leads to the vision loss, a condition known as glaucoma.
Posterior Chamber The posterior chamber is the fluid-filled space immediately behind the iris but in front of the lens.
Lens The lens is a clear, flexible structure that is located just behind the iris and the pupil. A ring of muscular tissue, called the ciliary body, surrounds the lens. The lens helps to control fine focusing of light.
Vitreous cavity The vitreous cavity is located behind the lens and in front of the retina. It is filled with a gel-like fluid, called the vitreous humor. The vitreous humor helps maintain the shape of the eye.
Retina/Macula/Choroid The retina acts like the film in a camera to create an image. To do this, the retina, a specialized layer of cells, converts light signals into nerve signals. After light signals are converted into nerve signals, the retina sends these signals to the optic nerve, which carries the signals to the brain. There, the brain processes the image.
The retina is primarily made up of 2 distinct types of cells: rods and cones. Rods are more sensitive to light; therefore, they allow you to see in low light situations but do not allow colour vision. Cones, on the other hand, allow you to see color but require more light.
The macula is located in the central part of the retina. It is the area of the retina that is responsible for giving you sharp central vision.
The choroid is a layer of tissue that separates the retina and the sclera. It is mostly made up of blood vessels. The choroid helps nourish the retina.
Optic nerve The optic nerve, a bundle of over 1 million nerve fibers, is responsible for transmitting nerve signals from the eye to the brain. These nerve signals contain information on an image to be processed by the brain. The front surface of the optic nerve, which is visible on the retina, is called the optic disk.

Extraocular muscles Six extraocular muscles are attached to each eye to help move the eye left and right, up and down, and diagonally. [8]
Process of vision

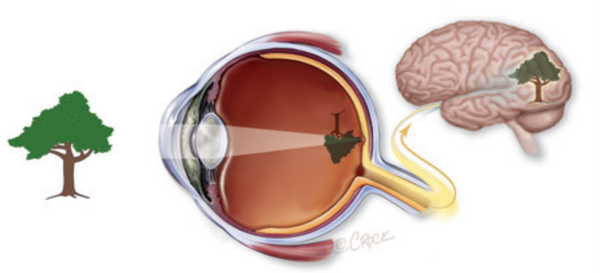

Figure 1.2: Healthy vision

Light waves from an object (such as the tree in the fig. (1.1)) enter the eye first through the cornea, which is the clear dome at the front of the eye. It is like a window that allows light to enter the eye. The light then progresses through the pupil, the circular opening in the center of the colored iris.
Fluctuations in the intensity of incoming light change the size of the pupil. As the light entering the eye becomes brighter, the pupil will constrict (getting smaller), due to the pupillary light response. As the entering light becomes dimmer, the pupil will dilate (getting larger).
Initially, light waves are bent or converged first by the cornea, and then further by the crystalline lens (located immediately behind the iris and the pupil), to a nodal point (N) located immediately behind the back surface of the lens. At that point, the image becomes reversed (turned backwards) and inverted (turned upside-down).
The light continues through the vitreous humor, the clear gel that makes up about 80% of the eyes volume, and then, ideally, back to a clear focus on the retina, behind the vitreous. The small central area of the retina is the macula, which provides the best vision of any location in the retina. If the eye is considered to be a type of camera (albeit, an extremely complex one), the retina is equivalent to the film inside of the camera, registering the tiny photons of light interacting with it.
Within the layers of the retina, light impulses are changed into electrical signals. Then they are sent through the optic nerve, along the visual pathway, to the occipital cortex at the posterior (back) of the brain. Here, the electrical signals are interpreted or seen by the brain as a visual image.
Actually, then, we do not see with our eyes but, rather, with our brains. Our eyes merely are the beginning of the visual process. [11]

1.2 Movements of the eye

We move our eyes using the extra ocular muscles (see Fig. 1.3) There are six muscles attached to the sclera that control the movements of the eye:

- medial rectus - moves eye towards nose
- lateral rectus - moves eye away from nose
- superior rectus - raises eye
- inferior rectus - lowers eye
- superior oblique- rotates eye
- inferior oblique - rotates eye

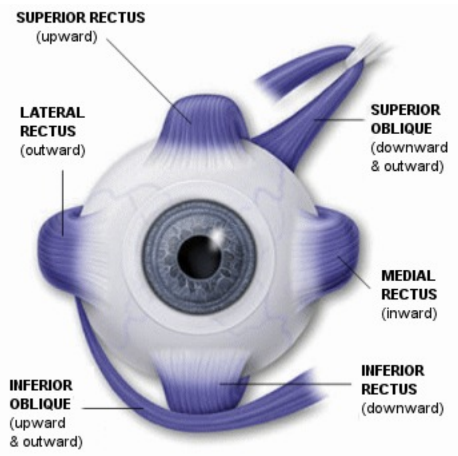

Figure 1.3: Extra ocular muscles

Conjugate eye movements are those that preserve the angular relationship between the right and left eyes. For example, when you move both eyes left and then right, a conjugate movement is made. Up and down movements and combinations of vertical and lateral movements also fall into the conjugate category.

Vergence eye movements are ones where the angle between they eyes changes.

Saccades or saccadic eye movements are very fast rotations from one eye position to another. To make a saccade or a series of saccades pick two objects as some distance from each other and look first at one then at the other. Because saccades are so very fast it may be difficult to see the eye movements as discrete jumps.

Smooth pursuit movements. They are called pursuit because this type of eye movement is made when the eyes follow an object.

1.3 Fluid mechanics of the eye

Figure 1.4: Aqueous humor flow

There are a number of processes within the eye in which fluid flow is important. The most evident of these are production, circulation, and drainage of aqueous humor. It is secreted with the flow rate of 2 to 2.5 $\mu \cdot L \cdot min^{-1}$. And due to such a small flow rates, the flow of aqueous humor is creeping and inertia can be neglected. The aqueous humor can be treated as an Newtonian with viscosity nearly identical to that of saline. [14] It flows into and fills the posterior chamber (see fig. (1.3)), then passes anteriorly through the pupil and enters the anterior

chamber, where it circulates while bathing the iris and the inner surface of the cornea Eventually the aqueous humor drains from the eye via specialized tissues located in the angle of the anterior chamber, where the iris, cornea, and sclera meet. These specialized tissues have a significant hydrodynamic flow resistance, and the drainage of the aqueous humor out of the eye therefore requires that there be a positive pressure within the eye itself, the so-called intraocular pressure. [14]
The flow of the aqueous humor performs two important physiological functions:

- The positive pressure that it generates stabilizes the otherwise flaccid eye, ensuring accurate positioning of the optical elements of the eye and hence clarity of vision.
- Aqueous humor supplies nutrients and removes waste products from the avascular lens and the central cornea, without which the cells in these tissues would die.

The majority of the ocular globe is filled by a clear, colorless, gel-like material known as vitreous humor, which occupies the vitreous chamber of the eye. Vitreous humor has complex viscoelastic properties , and although there have been several attempts to characterize its properties experimentally, its rheology is not fully understood. It is known that the vitreous humor becomes progressively liquefied with age. In approximately 25% 30% of subjects, liquefaction can lead to a process in which the retina detaches, risking loss of sight.[14]

With the notable exception of the lens, central cornea, and the vitreous humor, the eye is richly supplied by a complex network of blood vessels, leading to many interesting physiological problems associated with the regulation of blood flow in the network. [14]

Fluid mechanics of the vitreous humor

The vitreous cavity has an approximately spherical shape and contains the vitreous humor, which is subject to mechanical forces as a result of the motion of the eyeball (due primarily to the motion of the head and rotation of the eyeball within the socket). Deformation of the vitreous chamber, due for example to a head impact, lens movement during focusing or pulsation of retinal blood vessels, also gives rise to forces. However, in the absence of any deformation, purely translational motion does not result in any relative motion of the humor within the vitreous chamber because the accelerations involved can be balanced by a pressure gradient, whereas rotational motion does induce the relative motion of the humor. The fastest motions occur when the vitreous humor is liquefied, which can be the case either following the process of liquefaction or following vitrectomy, a surgical procedure in which some vitreous humor is removed and replaced with another fluid, often silicone oil or a gas bubble. In this case, the fluid filling the vitreous chamber is approximately Newtonian.

1.4 Retinal detachment

The vitreous humor is filled with a clear gel called vitreous that is attached to the retina.Tiny clumps of gel or cells inside the vitreous might cast shadows on the retina, and one may sometimes see small dots, specks, strings or clouds moving in your field of vision. These are called floaters. Floaters can be seen when looking at a plain, light background, like a blank wall or blue sky.
During the aging process the vitreous may shrink and pull on the retina. When this happens, the patient notices flashing lights, lightning streaks or the sensation of seeing stars. These are called flashes.

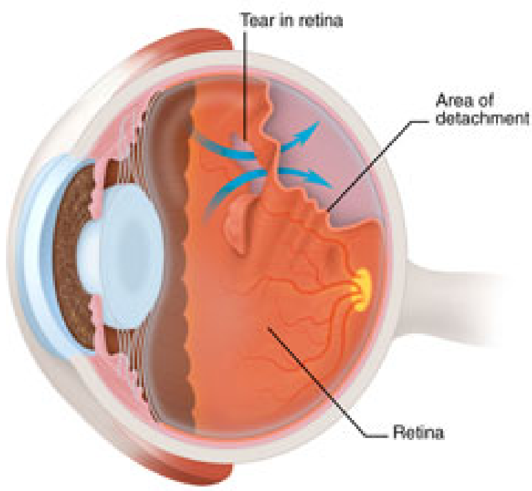

Figure 1.5: Retinal detachment

Usually, the vitreous moves away from the retina without causing problems. But sometimes the vitreous pulls hard enough to tear the retina in one or more places. Fluid may pass through a retinal tear, lifting the retina off the back of the eye much as wallpaper can peel off a wall. When the retina is pulled away from the back of the eye like this, it is called a retinal detachment.
The retina does not work when it is detached and vision becomes blurry. A retinal detachment is a very serious problem that almost always causes blindness unless it is treated surgically.
Retinal detachment often begins when the vitreous gel, a thick gel that fills the center of the eye, shrinks and separates from the retina. Called a posterior vitreous detachment (PVD), this is a normal part of aging and can be harmless. Sometimes, though, PVD can tear the retina. This happens where the vitreous gel is strongly attached to the retina. As the vitreous gel shrinks, it pulls so hard that the retina tears. The tear allows fluid to collect under the retina and may cause the retina to detach.
Other things that can lead to retinal detachment are an eye or head injury, nearsightedness (myopia), eye disease, and conditions such as diabetes.
Unfortunately, most cases of retinal detachment cannot be prevented. But seeing your eye doctor regularly, wearing protective helmets and eyeglasses, and treating diabetes may help protecting your vision.
Many people have symptoms of a posterior vitreous detachment, or PVD, before they have symptoms of retinal detachment. When the vitreous gel shrinks and separates from the retina, it causes floaters and flashes. Floaters are spots, specks, and lines that float through your field of vision. Flashes are brief sparkles or lightning streaks that are most easily seen when your eyes are closed. They often appear at the edges of your visual field. Floaters and flashes do not always mean that you will have a retinal detachment. But they may be a warning sign, so it is best to be checked by a doctor.
Sometimes a retinal detachment happens without warning. The first sign of detachment may be a shadow across part of your vision that does not go away. Or you may have new and sudden loss of side (peripheral) vision that gets worse over time.
To diagnose retinal detachment, your doctor will examine your eyes and ask you questions about any symptoms you have.
If you have symptoms of retinal detachment, your doctor will use a lighted magnifying tool called an ophthalmoscope to examine your retina. With this tool, your doctor can see holes, tears, or retinal detachment.
Warning signs of retinal detachment:

- Flashing lights.
- Sudden appearance of new floaters.
- Shadows on the side or periphery of your vision.
- Gray curtain moving across your field of vision.

1.5 Surgical procedures for treating the retinal detachment

Surgery is the only treatment for retinal detachment. The goals of surgery are:

- To reattach the retina. See an illustration of a detached retina.
- To prevent or reverse vision loss.

The most common methods of repairing a retinal detachment include:

- *Scleral buckling surgery.* This is the most common way to repair a detached retina. Your eye doctor (ophthalmologist) places a piece of silicone sponge, rubber, or semi-hard plastic on the outer layer of your eye and sews it in place. This relieves traction on the retina, preventing tears from getting worse, and it supports the layers of the retina.
- *Pneumatic retinopexy.* In this procedure, your eye doctor injects a gas bubble into the middle of the eyeball. The gas bubble floats to the detached area and presses lightly against the detached retina, flattening it so that the fluid below it can be reabsorbed. The eye doctor then uses a freezing probe (cryopexy) or laser beam (photocoagulation) to seal the tear in the retina.
- *Vitrectomy.* This is the removal of the vitreous gel from the eye. Vitrectomy gives your eye doctor better access to the retina and other tissues. It allows him or her to peel scar tissue off the retina, repair holes, close very large tears, and directly flatten a retinal detachment.

  There are number of risks associated with vitrectomy surgery. A few of them have been listed below:

  - Infection.
  - Increase in pressure inside the eye.
  - Post vitrectomy surgery may give rise to corneal edema. In this condition, fluid build up takes place within the clear covering of the eye. As a result, the pressure on the eye is increased and the patient may get blurred vision. In some cases, it may cause damage to the surrounding tissues.
  - Bleeding inside the eye. This happens particularly in those cases where the vitreous gel is removed due to bleeding into it. In many such cases, bleeding tend to recur within the vitreous cavity or frontal part of the eye and causes severe damage to the eye.
  - When vitrectomy is conducted on elderly above the age of 50 years or so, then the chances of cataract formation increases manifolds.
- *Laser photocoagulation,* in which an intense beam of light travels through the eye and makes tiny burns around the tear in the retina. The burns form scars that prevent fluid from getting under the retina.
- *Cryopexy (freezing),* in which your eye doctor uses a probe to freeze and seal the retina around the tear.

Factors that may make surgery more difficult include:

- Glaucoma.
- Pupils that will not get larger.
- Infection inside or outside the eye.
- Scarring from previous surgery.
- Bleeding (hemorrhage) in the vitreous gel.
- Scars on or cloudiness in the cornea.
- Clouding of the lens (cataract). [2]

1.6 Vitreous substitutes

As it was said before the vitrectomy is a surgery that treats retinal breaks and retinal detachment. It consists of replacement of the whole vitreous humor , which is substituted with tamponade fluids. This vitreous substitutes can be classified into short-term tamponade and longer-term tamponade. The former are intended to remain in the vitreous chamber for a limited time sufficient for retina reattachment to occur, and to be subsequently removed. The later are designed to be left in a vitreous chamber for much longer time. Materials that form an interface with the aqueous environment of the eye can be effective in closing retinal breaks and holding the neural retina in place against the retinal pigment epithelium. [12]

1.6.1 Classification of the substitutes

The vitreous substitutes may be classified according to its composition or function.

- Classification according to the composition of vitreous replacements:
  - Replacement with natural vitreous
  - Artificial vitreous substitutes
    - Gases
      - Air
      - Perfluorocarbon gases
    - Liquids
      - Aqueous solutions: water and balanced salt solutions
      - Silicone oil and its derivatives: silicone oil, fluorosilicone oil and silicone-fluorosilicone copolymer
      - Perfluorocarbon liquids: Perfluoro-n-octane, perfluorohydrophenanthrene, perfluorotributy-lamine, perfluorohexyloctane, perfluoropolyether and others
      - Semifluorinated alkanes
    - Gels. These are mainly polymeric materials
      - Semisynthetic polymers: methylated collagen, sodium hyaluronidate, hyaluronic acid-collagen mixture, and sodium carboxymethylcellulose
      - Synthetic polymers: poly(1-vinyl-2-pyrrolidinone), polyvinylalcohol, polyacrylamide, poly(glyceryl-methacrylate), poly(2-hydroxyethyl acrylate), poly(methyl acrylamidoglycolate methyl ether)
- Functional classification. This classification is based on the main roles that vitreous substitutes play in the treatment of various pathologies as:
  - Short intraoperative procedures: gases and perfluorocarbon liquids.
  - Longer-term retinal tamponade: silicone oil.
  - Permanent vitreous substitute: no material currently available.
  - Sustained intravitreal drug delivery: polymeric hydrogels. [12]

1.6.2 The requirements to the vitreous replacements

The ideal substitute is required to fulfill the following criteria:

- It should be clear, transparent and colorless.

- Its density and refractive index (a characteristic that describes how the radiation propagates through the medium) should be close to the natural vitreous.
- It should be storable and sterilizable, chemically and biologically inert, and non-toxic, hence should not trigger any undesirable biological responses.
- It should be biocompatible with the vitreous humor itself and with the adjacent tissues, and should not affect their physiological functions.
- It should not be adsorbable or biodegradable, in order to be maintained in the vitreous cavity for a period as long as possible.
- It should preferably lasting viscoelastic properties to avoid its own drainage through retinal breaks, to push back the retina to proper position until secure adhesion is achieved, and to prevent retinal detachment.
- It should have a high surface tension in order to assure proper tamponade of the retina against the choroids
- It must allow the transport of necessary metabolites and proteins inside the vitreous.
- Preferably, it should be injectable through a small-gauge needle and all the above properties should be retained after injection. Alternatively, it should be implantable through a small incision.
- Preferably, it should be a hydrophilic material with high equilibrium water content. [12]

1.6.3 Performance of silicone oil derivatives as vitreous substitutes

Silicone oils are used in the vitreous surgeries since 1960. They have a suitable properties of stability, transparency and causing a high interfacial surface energy with the aqueous humor and retina. Usually silicone oil is injected into the vitreous cavity after the chamber has been filled with air. As the silicone oil is injected, the air is drained through a second needle inserted through a sclerotomy.

Advantages of using silicone oils

- The high interfacial surface energy of silicone oil at the tamponade/aqueous/retina interface ensures the closure of the retinal breaks and reduces subretinal leakage.
- The lower specific gravity of silicone oil as compared to aqueous solutions causes it to float upon residual fluid and thus helps in retinal tamponade in case of superior breaks.

The success rate of using the silicone oils is about 70%.

Disadvantages of using silicone oils

- The hydrophobic nature of silicone oil leads to a poor contact with the retina and aqueous fluids, which inhibits the total filling of the vitreous cavity which is required for the effective closure of retinal breaks.
- Since silicone oil has a lower density, it leads to reduced or absent tamponade effect in inferior retinal tears.

- The persistence of silicone oil over long periods (6 months) leads to life-threatening complications of cataract, glaucoma and keratopathy and these complications therefore necessitate its removal. [12]
- The emulsification of the silicone oil might occur.

Crisp et al. in 1986 [4] have compared the emulsification potential(*emulsify* means combining two liquids togeter which normally don't mix easily) of silicone oil of different viscosities and molecular composition.

For the treatments of retinal detachment the liquid silicone dimethylpolysiloxane is used. It is a polymer composed of repetitive units with the certain molecular weight. The oil's viscosity increases when the total number of units (chain length) increase as well. The intermediate viscosity can be created by using polymers with the different molecular weights.The silicon oils with the high viscosity are more homogeneous with the uniform chain length.

Crisp et al. showed the importance of both viscosity and low-molecular weight components in the prevention of silicone oil emulsification. 4 mL of silicone oil and 7 mL aqueous phase containing the surface-active agents were used. The aqueous phase was prepared using 0.013% benzalkonium chloride (cationic surface active agent) and varying quantities of bovine serum. These were combined in varying proportions in order to derive four different aqueous mixtures:

- a solution containing only 0.013% benzalkonium chloride,
- a 10% solution of bovine serum together with 0.013% benzalkonium chloride,
- a 40% solution of bovine serum together with 0.013% benzalkonium chloride,
- 100% of bovine serum.

Results show the following: as oil viscosity decreased, the number of oil-in-water droplets increased. In addition to that the four factors in emulsion formation were pointed out:

- surface tension;
- presence of surface-active agents (emulsifiers);
- viscosity;
- molecular entanglements.

Surface tension refers to the forces of attraction operating within liquids in contact with gas. When two immiscible liquids are in contact with one another the interfacial tension occurs. Surface tension and the consequent surface free energy cause liquids to assume the smallest surface area-volume ratio. Creation of new interfaces involves increasing the surface area - volume ratio which to leads to emulsion formation (fig. 1.6)
This phenomenon requires the input of energy into the system. The dispersion of oil within the water increases the surface area. And this greatly increases the surface free-energy of the system. At such a hight energy level the system in *thermodynamically unstable.* By separation into the bulk phases the surface free-energy decreases to its minimum value leading to the smallest possible surface-volume ratio. Which means that within the eye there are must be present agents that reduce the interfacial surface tension so that the emulsified oil is stable thermodynamically.

The results in [4] show that even in the presence of surface-active agents, highly viscous oils and those with the shorter chain lengths removed resist emulsification better then low-molecular weight oils or polymers containing low molecular weight.

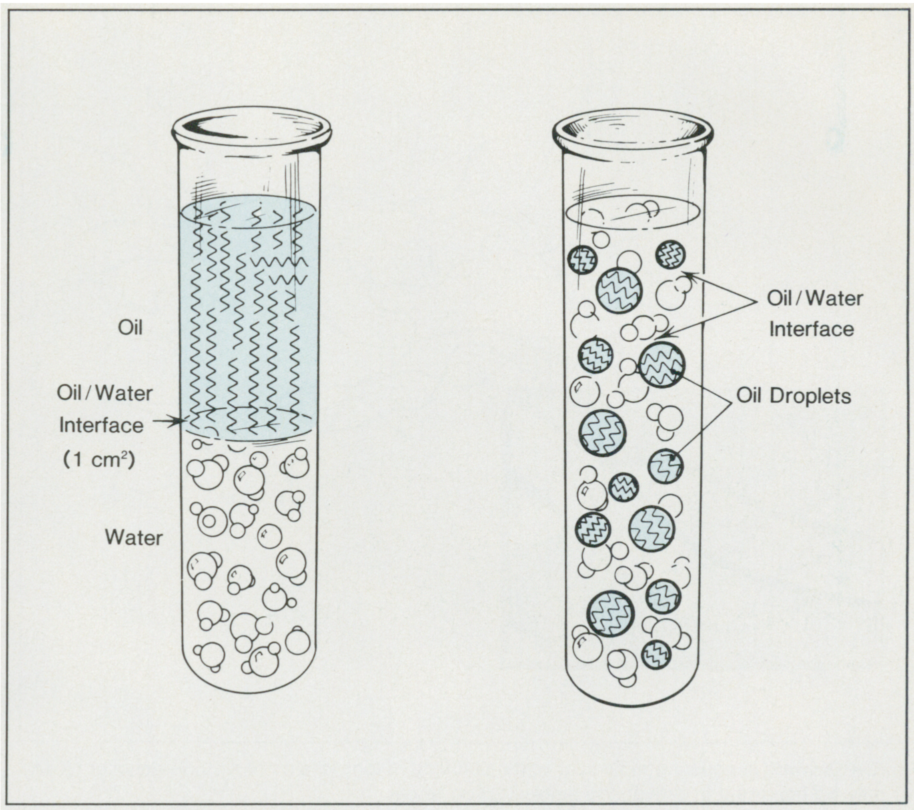

Figure 1.6: **Left:** silicone oil in bulk phase. **Right:** greatly expanded area of oil/water interface with silicone oil emulsified ([4])

Chapter 2

Hydrodynamic Instability

2.1 Introduction

Hydrodynamics stability theory considers the response of laminar flow due to perturbations of small amplitude. A flow is defines *stable* if it turns to its laminar state and *unstable* if it changes into a different state. Hydrodynamic stability can be studied in different ways:

- *Natural phenomena and laboratory experiments.* Observations of nature and experiments are the primary means of study. All theoretical investigations need to be related, directly or indirectly, to understanding these observations.

- *Numerical experiments.* Computational fluid dynamics has become increasingly important in hydrodynamic stability since 1980s, as numerical analysis has improved and computers have become faster and gained more memory, so that the Navier - Stokes equations may be integrated accurately for more complex flows. Indeed, computational fluid dynamics has now reached a stage where it can rival laboratory investigation of hydrodynamic stability by simulating controlled experiments.

- *Linear and weakly nonlinear theory.* Linearization for small perturbations of a given basic flow is the primary method to be used in the theory of hydro- dynamic stability, and it was the method used much more than any other until the 1960s. It remains the foundation of the theory. However, weakly nonlinear theory, which builds on the linear theory by treating the leading nonlinear effects of small perturbations, began in the nineteenth century, and has been intensively developed since the 1960s.

- *Qualitative theory of bifurcation and chaos.* The qualitative theory of dynamical systems, as well as weakly nonlinear analysis, provides a useful conceptual framework to interpret laboratory and numerical experiments.

- *Strongly nonlinear theory.*There are various mathematically rigorous methods, notably Serrins theorem and Liapounovs direct method, which give detailed results for arbitrarily large perturbations of specific flows. These results are usually bounds giving sufficient conditions for stability of a flow or bounds for flow quantities.[7]

2.2 Turbulence and stability

Turbulent flow is inherently more energetic compared to its laminar counterpart and it is of engineering interest to reduce its effects. For example,

in aircraft design where this may cause structural damage to the body of the aircraft, or in the prototype fusion power plants where turbulent flow in the plasma causes difficulty in plasma containment. Trying to understand turbulence has been an important engineering problem as well as a mathematical one. [10]
The essential problems of hydrodynamic stability were recognized and formulated in nineteenth century, notably by Helmotz, Kelvin, Rayleigh and Reynolds [7]. In his experiment Reynolds brought out the main classification of the flows (laminar, transitional and turbulent). Into a flow through a glass tube he injected a dye to observe the nature of the flow. If the velocities were sufficiently small the flow followed a straight line path apart a slight blurring due to the diffusion. When the velocities were increased the dye was blurred and seemed to fill the entire pipe. In this way the laminar, transitional and turbulent flows were observed. And Reynolds has concluded that the smooth flow breaks down when the ration $\frac{V \cdot a}{\nu}$ exceeds a certain critical value, where

- V is the maximum velocity of the water in pipe;
- a is the radius of the pipe;
- ν is the kinematic viscosity of water.

This dimensionless number $\frac{V \cdot a}{\nu}$ is called *Reynolds number.*

2.3 Stability of parallel flows

It is known that the motion of incompressible Newtonian fluid is governed by Navier-Stokes equations (3.1) and continuity equation (3.2):

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \boldsymbol{f} + \frac{1}{\rho}\nabla p - \nu \nabla^2 \boldsymbol{u} = 0, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3.2}$$

where $\boldsymbol{u}$ is the flow velocity, $\boldsymbol{f}$ is the force acting on the flow, ρ is the density, ν is the viscosity.

For the two-dimensional flow of inviscid fluid between two flat plates the basic equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.3a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.3b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.3c}$$

The parallel shear steady flow in Cartesian coordinates is given by:

$$\boldsymbol{u_0} = [U(y), 0, 0]. \tag{3.4}$$

The pressure is constant p_0.

Considering two-dimensional disturbances, the flow is decomposed as:

$$\boldsymbol{u} = [U(y) + u_1, v_1, 0], \tag{3.5a}$$
$$u_1 = u_1(x, y, t), \tag{3.5b}$$
$$v_1 = v_1(x, y, t), \tag{3.5c}$$
$$p = p_0 + p_1. \tag{3.5d}$$

The perturbation terms u_1, v_1 and p_1 are assumed to be small. After substituting 3.5 into 3.3 and linearizing the system is defined by:

$$\frac{\partial u_1}{\partial t} + U\frac{\partial u_1}{\partial x} + v_1 U' = -\frac{1}{\rho}\frac{\partial p_1}{\partial x}, \tag{3.6a}$$
$$\frac{\partial v_1}{\partial t} + U\frac{\partial v_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial y}, \tag{3.6b}$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{3.6c}$$

with prime denoting the derivative with respect to y.

The above equations have coefficients that depend on y alone. The modes can be explored of the form:

$$u_1 = \hat{u}(y)\exp[i(kx - \omega t)], \tag{3.7a}$$
$$v_1 = \hat{v}(y)\exp[i(kx - \omega t)], \tag{3.7b}$$
$$p_1 = \hat{p}(y)\exp[i(kx - \omega t)] \tag{3.7c}$$
$$\tag{3.7d}$$

In this way the equations are given by:

$$-i(\omega - Uk)\hat{u} + U'\hat{v} = -\frac{ik}{\rho}\hat{p}, \tag{3.8a}$$
$$-i(\omega - Uk)\hat{v} = -\frac{1}{\rho}\hat{p}', \tag{3.8b}$$
$$ik\hat{u} + \hat{v}' = 0. \tag{3.8c}$$

Finally, eliminating $\hat{p}$ and $\hat{u}$, we have:

$$\hat{v}'' + \Big(\frac{kU''}{\omega - Uk} - k^2\Big)\hat{v} = 0, \tag{3.9}$$

with $\hat{v} = 0$ on the boundaries $y = -L$ and $y = L$.

Multiplying last equation on complex conjugate to $\hat{v}$ - $\tilde{v}$ and integrating over the whole domain, one gets:

$$\int_{-L}^{L} \tilde{v}\hat{v}''\,\mathrm{d}y + \int_{-L}^{L}\Big(\frac{kU''}{\omega - Uk} - k^2\Big)|\hat{v}|^2\,\mathrm{d}y = 0, \tag{3.10}$$

The first term equals to 0 due to the boundary conditions. Substituting $\omega = \omega_r + i\omega_i$ we have:

$$\omega_i k \int_{-L}^{L} \frac{U''|\hat{v}|^2}{|\omega - Uk|^2}\,\mathrm{d}y = 0 \tag{3.11}$$

If there is at least one which had $\omega_i > 0$ corresponding to the exponential growth of the amplitude with time. It implies that U'' changes sign somewhere in the domain so that the integral is 0. This leads to the following: ***Rayleigh's inflection point theorem***.*A necessary condition for the linear instability of an inviscid shear flow $U(y)$ is that $U''(y)$ should change the sign somewhere in the flow.*

Note that the presence of an inflection point in the velocity profile is a *necessary* condition for stability for infinitesimal disturbances; there is no claim here that any velocity profile with an inflection point is unstable. To pronounce confidently that a system is stable one need to know the fate of finite-amplitude, as well as infinitesimal disturbances.[3]

2.4 Review of related previous works

A lot of existing work on stability of unsteady flows is based on the assumption of quasi-steadiness which means that the stability of the unsteady flow is determined by whether or not it is stable for all the velocity distributions if each of these distribution is assumed to persist. If the frequency of the primary flow is much less then the reference velocity divided by the reference length, it can be shown that the approach of quasi-steadiness can predict stability or instability over time intervals small compared with the period. If the instability was predicted, the slow variation of the primary flow with time may not affect the conclusion. But many flows predicted to be stable by the approach of quasi-steadiness may turn out to be unstable in the long run [15].

Chia-Shun Yih has investigated in his paper [15] the stability of the layer of viscous fluid with the free surface which is set in motion by the lower boundary moving harmonically. He has considered the stability of the primary flow which was completely unsteady. Yih has studied in [15] the stability of the long waves by a perturbation method. This was the firs time when this approach had been applied to the problem of stability of the unsteady flows. Since the primary flow is the time-periodic the extension of the Floquet theorem was applied together with the expansion in terms of the wave number due to the long-wave approach.

In 1970 Li has studied the stability of the two layers of liquid set in a motion by the lower boundary in his paper [9]. The investigation was made for the two superposed fluids with different viscosities and densities and with the free surface on top. The purpose was to see how an interface, which is a second surface of discontinuity in density and viscosity, will affect the stability of a single layer of fluid studied by Yih in [15] [9].

In his paper Li also has adopted the extended Floquet theorem in order to solve the Orr-Sommerfeld equation with the time-periodic coefficients. He has concluded that it is the interfacial mode which governs the instability of the flow when the Froude number is less then 3. This means that the energy required to distort the interface is less then that required to distort the free surface. The interfacial mode reduces to a neutral mode with equal densities and viscosities. For the Froude number greater then 3 there would be a competition between the interfacial and free-surface modes in governing the stability or instability of the system.

Chapter 3

Formulation of the problem

The first step in studying problems caused by using the silicone oils in the vitreous chamber, in particular, the instability of the oil-water interface and the onset of oil bubble formation, is investigating the stability of a two-layered stratified fluid over a flat wall set in a motion by harmonic oscillations.

3.1 Primary flow

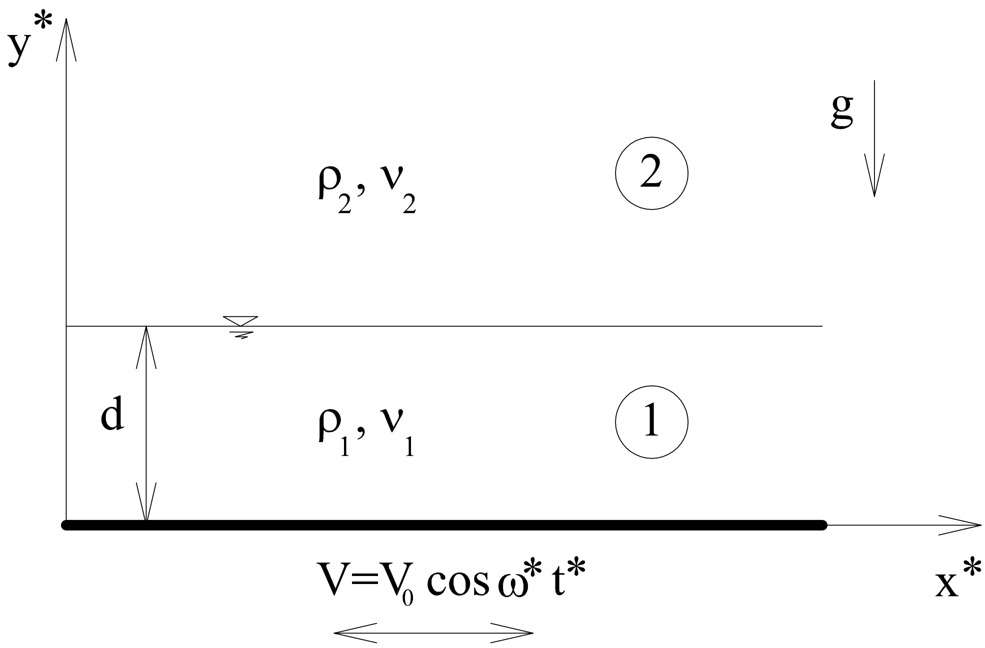

Figure 3.1: Definition sketch

Consider two inviscid fluids occupying the region of space $0 \leq y < d$ and $y > d$, and with densities ρ_1, ρ_2 and the viscosities μ_1 and μ_2 for the lower and the upper fluid respectively. The flow is induced by periodic horizontal motion of the rigid wall located at $y = 0$ (see figure 3.1). The oscillation is described by the equation:

$$u_w^* = V_0 \cos(\omega^* t^*) = \frac{V_0}{2}(e^{i\omega^* t^*} + c.c.), \tag{1.1}$$

where V_0 is the amplitude of the oscillations, t is time and ω^* is the frequency of oscillations. The motion of the two fluids is governed by the Navires-Stokes and continuity equations. Assuming a 2D problem and that the velocity in two fluids only has x-component, which is function of y and t we obtain:

$$\frac{\partial U_1^*}{\partial t^*} = -\frac{1}{\rho_1}\frac{\partial P_1^*}{\partial x^*} + \nu_1 \frac{\partial^2 U_1^*}{\partial y^{*2}}, \tag{1.2a}$$

$$0 = \frac{1}{\rho_1}\frac{\partial P_1^*}{\partial y^*} - g, \tag{1.2b}$$

$$\frac{\partial U_2^*}{\partial t^*} = -\frac{1}{\rho_2}\frac{\partial P_2^*}{\partial x^*} + \nu_2 \frac{\partial^2 U_2^*}{\partial y^{*2}}, \tag{1.2c}$$

$$0 = \frac{1}{\rho_2}\frac{\partial P_2^*}{\partial y^*} - g, \tag{1.2d}$$

$$\tag{1.2e}$$

where g denotes gravity in the direction by the coordinate y. In order to make the problem dimensionless the following rescaling is used:

$$(U_1^*, U_2^*) = V_0 \cdot (U_1, U_2), \tag{1.3a}$$

$$(x^*, y^*) = d \cdot (x, y), \tag{1.3b}$$

$$(P_1^*, P_2^*) = \rho_1 \cdot V_0^2 \cdot (P_1, P_2), \tag{1.3c}$$

$$t^* = \frac{d}{V_0} t, \tag{1.3d}$$

$$\omega^* = \frac{V_0}{d}\omega, \tag{1.3e}$$

$$m = \frac{\mu_2}{\mu_1}, \tag{1.3f}$$

$$\gamma = \frac{\rho_2}{\rho_1}. \tag{1.3g}$$

In dimensionless form the system becomes:

$$\frac{\partial U_1}{\partial t} = \frac{1}{R}\frac{\partial^2 U_1}{\partial y^2}, \tag{1.4a}$$

$$\frac{\partial P_1}{\partial y} = Fr^{-2}, \tag{1.4b}$$

$$\frac{\partial U_2}{\partial t} = \frac{m}{\gamma}\frac{1}{R}\frac{\partial^2 U_2}{\partial y^2}, \tag{1.4c}$$

$$\frac{\partial P_2}{\partial y} = \gamma Fr^{-2}, \tag{1.4d}$$

where $R = V_0 d/\nu_1$ is the Reynolds number and $Fr = \frac{V_0}{\sqrt{gd}}$ is the Froude number.

The pressure has hydrostatic distribution and it is given by:

$$P_1 = Fr^{-2} y + const \tag{1.5a}$$

$$P_2 = \gamma Fr^{-2} y + const \tag{1.5b}$$

The velocities U_1 and U_2 have to satisfy the boundary conditions:

$$U_1(0, t) = \cos(\omega t), \tag{1.6a}$$

$$U_1(1, t) = U_2(1, t), \tag{1.6b}$$

$$U_2(+\infty, t) = 0 \tag{1.6c}$$

$$\frac{\partial}{\partial y} U_1(1, t) = m \frac{\partial}{\partial y} U_2(1, t). \tag{1.6d}$$

$$\tag{1.6e}$$

The solution of the system is given by:

$$U_1 = [c_1 e^{-ay} + c_2 e^{ay}]e^{i\omega t} + c.c., \tag{1.7a}$$

$$U_2 = c_3 e^{-by} e^{i\omega t} + c.c., \tag{1.7b}$$

where

$$a = \frac{i+1}{\sqrt{\frac{2}{\omega R}}}, \tag{1.8a}$$

$$b = \frac{i+1}{\sqrt{\frac{2m}{\omega R \gamma}}}, \tag{1.8b}$$

$$c_1 = \frac{\frac{-e^{a-b}}{2}[mb+a]}{e^{-a-b}[mb-a] - e^{a-b}[mb+a]}, \tag{1.8c}$$

$$c_2 = \frac{\frac{e^{-a-b}}{2}[mb-a]}{e^{-a-b}[mb-a] - e^{a-b}[mb+a]}. \tag{1.8d}$$

$$c_3 = \frac{-a}{e^{-a-b}[mb-a] - e^{a-b}[mb+a]} \tag{1.8e}$$

$$\tag{1.8f}$$

Thus, we have an the analytical solution for our basic flow. In Figures (3.2), (3.3) and (3.2) the basic flow is plotted together with its first derivative for different values of the input parameters. On the first two plots (3.2 and 3.3) a single fluid is assumed ($m = \gamma = 1.0, S = 0$), while on the last one (3.4) the upper fluid has a viscosity 10 times larger then the lower one ($\gamma = 10$).

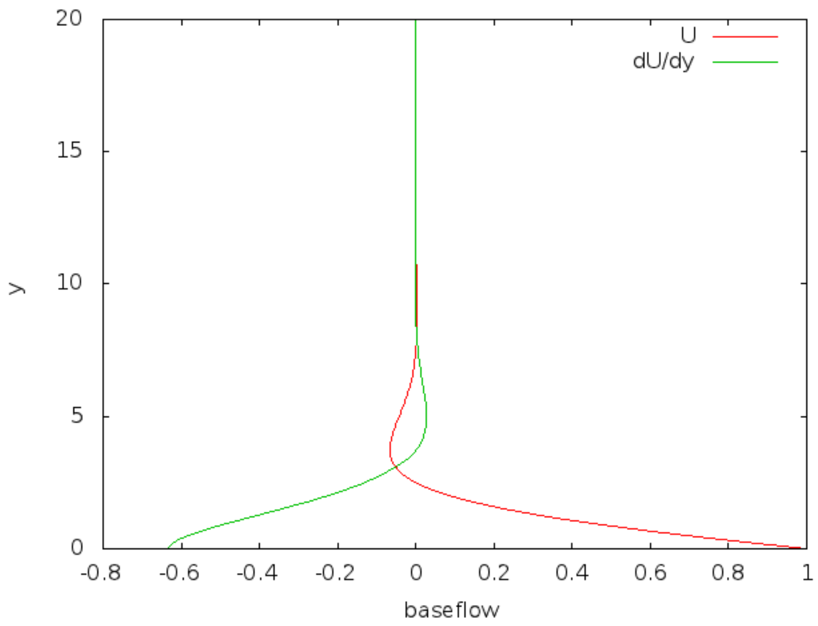

Figure 3.2: $Re = 80, m = 1, \gamma = 1, \omega = 0.01, Fr = 2, t = 0$

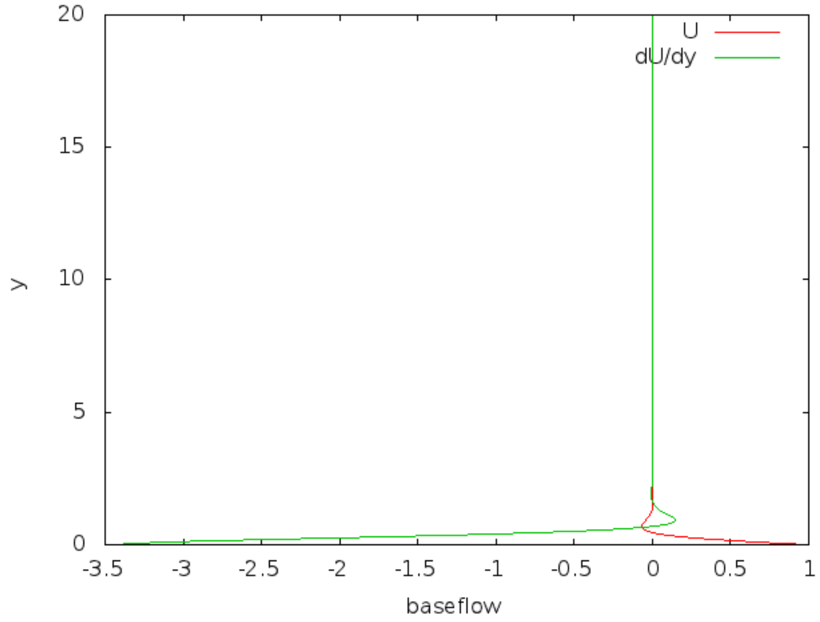

Figure 3.3: $Re = 30, m = 1, \gamma = 1, \omega = 0.9, Fr = 2.t = 0$

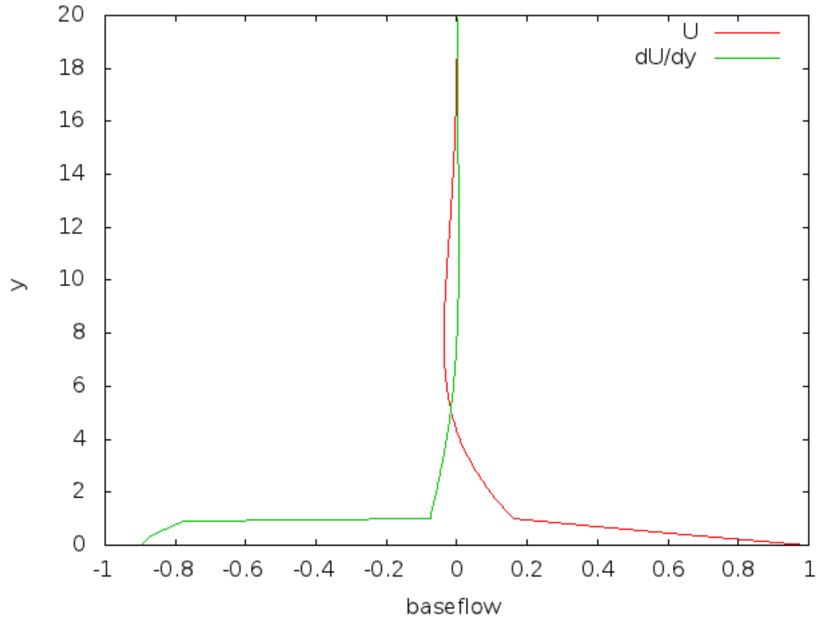

Figure 3.4: $Re = 100, m = 10, \gamma = 1, \omega = 0.01, Fr = 2, t = 0$

3.2 Analysis of the value of the parameters

We now determine realistic ranges of the dimensionless parameters introduced in the previous section. This is need to be done in order to choose the best strategy.

3.2.1 Saccadic movement of the eye

A quick rotation of the eye to redirect the sight from one target to another is called the **saccadic movement**. The basic features of saccadic eye movements are:

1. A very high initial angular acceleration (up to 30000° s^{-2});
2. A somewhat less intense deceleration that is nevertheless capable of inducing a very efficient stop of the movement;
3. A peak angular velocity that rises in proportion to the saccade amplitude up to a saturation value ranging between 400 and 600° s^{-1} .

   Saccade amplitude ranges from 0.05° (microsaccades) to 8090° , which is the physical limit for the orbit. Obviously, very large saccades are normally accompanied by head rotations.

In order to describe the saccade movement the following parameters are used:

- the saccade amplitude A;
- the saccade duration D;
- the peak angular velocity Ω_p;
- the acceleration time t_p. It is the time required to reach the peak velocity starting from the rest.

Becker (1989) reports that the relationship between saccade duration and amplitude is very well described by the following linear law:

$$D = D_0 + dA, \tag{2.9}$$

in the range:

- $5° < A < 50°$, with
- $d \approx 0.0025\ s \cdot deg^{-1}$ and
- $0.02\ s < D_0 < 0.03\ s$

The average angular velocity is defined as $\bar{\Omega} = A/D$. And the ratio $\Omega_p/\bar{\Omega}$ is found to be approximately constant and equal to 1.6 A. The small amplitude sacads (less then 10°) follow an almost symmetrical time law, the acceleration time is equal to $0.45D$. The dimensionless acceleration time t_p/D varies linearly with increasing saccade amplitudes, to the value $t_p/D \approx 0.25$ for saccades of 50 .
In our model we describe a sequence of saccades as a periodic motion with the period of $2 \cdot D$. With this we can estimate the velocity of the movement:

$$V_0 = \frac{A \cdot \pi \cdot \omega \cdot R}{180}, \tag{2.10}$$

Where ω is the frequency which is equal to:

$$\omega = \frac{2\pi}{T}, \tag{2.11}$$

and R is an average radius of the eye equals to $12 \cdot 10^{-3} m$. From those data the relation between dimensional values of the velocity and the frequency

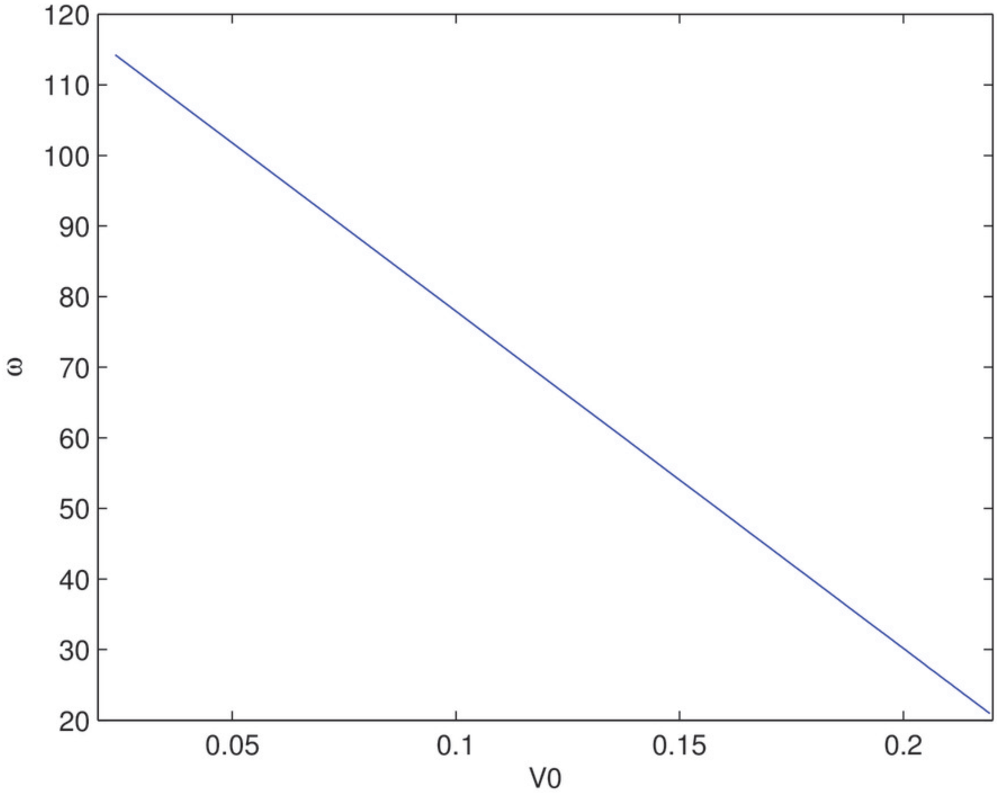

Figure 3.5: The relation between velocity of the movement and the frequency (dimensional values)

can be determined, and is reported in figure 3.5: The viscosity of the aqueous (fluid 1) is given by:

$$\nu_1 = 10^{-6} m^2 \cdot s \tag{2.12}$$

We assumed the thickness of the layer of the aqueous humor to be equal to:

$$d = 5 \cdot 10^{-5} \tag{2.13}$$

which gives us the estimate of the Reynolds number. Taking into account scaling for the frequency, the relation between two dimensionless parameters - Reynolds number and frequency - can be determined and is reported in figure 3.6: This relation will be taken into account for further analysis.

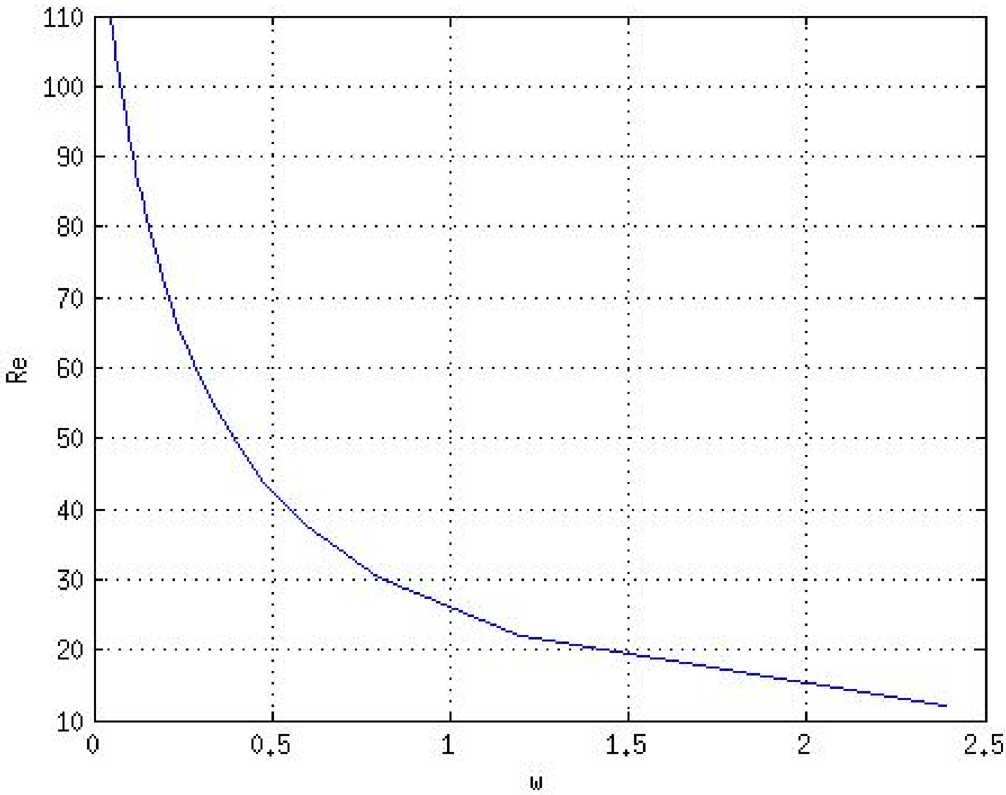

Figure 3.6: The relation between the frequency (dimensionless) and the Reynolds number.

3.3 The differential system governing the stability

In order to derive the stability conditions of the system we will use a linear stability analysis and study the time evolution of infinitesimally small perturbations. If their amplitude grows instability occurs. The S quires theorem states that for a steady parallel shear flow, the flow first becomes unstable to 2D perturbations:

Squire's theorem: *Given Re_L as the critical Reynolds number for the onset of linear instability for some given α, β the Reynolds number Re_C*

below which no exponential instabilities exist for any wave numbers satisfies:

$$Re_C \equiv \min_{\alpha,\beta} Re_L(\alpha,\beta) = \min_{\alpha} Re_L(\alpha,0) \tag{3.14}$$

The Squire's theorem can be also valid for the *quasi-steady approach*([6]). Let u_i and v_i denote, respectively, the velocity components in the x and y directions and p_i denote the pressure. The index i is taken to be 1 for the lower fluid and 2 for the upper fluid.

$$u_i = U_i + u_i', \quad v_i = v_i', \quad p_i = P_i + p_i' \tag{3.15}$$

where the small letters with a prime refer to the perturbation quantities. Substituting (3.15) into the equation of continuity and assuming that term involving the disturbance of quadratic order and higher are small and thus negligible, the system is becoming:

$$\frac{\partial u_i'}{\partial x} + \frac{\partial v_i'}{\partial y} = 0 \tag{3.16}$$

which allows us to define a stream function $\hat{\psi}_i$, such as

$$u_i' = (\hat{\psi}_i)_y, \; v_i' = -(\hat{\psi}_i)_x \tag{3.17}$$

where the subscripts x and y denote the partial differentiation. The system of the equations governing a given problem in perturbed terms is given by:

$$\frac{\partial u_1'}{\partial t} + U_1\frac{\partial u_1'}{\partial x} + v_1'\frac{\partial U_1}{\partial y} = -\frac{\partial p_1'}{\partial x} + \frac{1}{R}\left[\frac{\partial^2 u_1'}{\partial x^2} + \frac{\partial^2 u_1'}{\partial y^2}\right] \tag{3.18a}$$

$$\frac{\partial v_1'}{\partial t} + U_1\frac{\partial v_1'}{\partial x} = -\frac{\partial p_1'}{\partial y} + \frac{1}{R}\left[\frac{\partial^2 v_1'}{\partial x^2} + \frac{\partial^2 v_1'}{\partial y^2}\right] \tag{3.18b}$$

$$\frac{\partial u_2'}{\partial t} + U_2\frac{\partial u_2'}{\partial x} + v_2'\frac{\partial U_2}{\partial y} = -\frac{1}{\gamma}\frac{\partial p_2'}{\partial x} + \frac{m}{R}\left[\frac{\partial^2 u_2'}{\partial x^2} + \frac{\partial^2 u_2'}{\partial y^2}\right] \tag{3.18c}$$

$$\frac{\partial v_2'}{\partial t} + U_2\frac{\partial v_2'}{\partial x} = -\frac{1}{\gamma}\frac{\partial p_2'}{\partial y} + \frac{m}{R}\left[\frac{\partial^2 v_2'}{\partial x^2} + \frac{\partial^2 v_2'}{\partial y^2}\right] \tag{3.18d}$$

(3.18e)

Where quadratic terms in the perturbations have been neglected. The above equations together with the boundary conditions admit a solution of the form

$$\hat{\psi}_i = \psi_i(y,t)e^{i\alpha x} + c.c. \tag{3.19a}$$

$$p_i' = f_i(y,t)e^{i\alpha x} + c.c. \tag{3.19b}$$

where α is the dimensionless wavenumber. Substituting the stream function into the system it we are going to end up with the well-known Orr-Sommerfeld equations:

$$\psi_1'''' - 2\alpha^2\psi_1'' + \alpha^4\psi_1 = R\left[\left(\frac{\partial}{\partial t} + i\alpha U_1\right)(\psi_1'' - \alpha^2\psi_1) - i\alpha\psi_1\frac{\partial^2 U_1}{\partial y^2}\right], \tag{3.20a}$$

$$\psi_2'''' - 2\alpha^2\psi_2'' + \alpha^4\psi_2 = \frac{R}{m}\left[\left(\frac{\partial}{\partial t} + i\alpha U_2\right)(\psi_2'' - \alpha^2\psi_2) - i\alpha\psi_2\frac{\partial^2 U_1}{\partial y^2}\right], \tag{3.20b}$$

in which primes on ψ_1 and ψ_2 indicate the differentiation with rescect to y.

The boundary conditions are:

$$\psi_1(0,t) = 0 \tag{3.21a}$$
$$\psi_1'(0,t) = 0 \tag{3.21b}$$
$$\psi_2(+\infty,t) = 0 \tag{3.21c}$$
$$\psi_2'(+\infty,t) = 0 \tag{3.21d}$$

which express the conditions of adherence of the fluid to the lower rigid boundary, and the condition of vanishing velocity at infinity. Let $\hat{\eta}$ denote the dimensionless perturbation of the interface position, measured in units of d. We impose

$$\hat{\eta}(x,t) = \eta(t)e^{i\alpha x} \tag{3.22}$$

The equation of the boundary is given by $\frac{DF}{Dt} = 0$, which can be written as

$$F = y - \hat{\eta}(x,t) = y - \eta(t)e^{i\alpha x} = 0 \tag{3.23}$$

The kinematic boundary condition imposes:

$$\left(\frac{\partial}{\partial t} + U_1(1,t)\frac{\partial}{\partial x}\right)\hat{\eta} = v_1' = -(\hat{\psi_1})_x, \tag{3.24}$$

which becomes:

$$\left(\frac{\partial}{\partial t} + i\alpha U_1(1,t)\right)\eta = -i\alpha\psi_1(1,t). \tag{3.25}$$

The continuity of normal and tangential components of the velocity at the interface demand:

$$\psi_1(1,t) = \psi_2(1,t), \tag{3.26}$$
$$\psi_1'(1,t) + \eta(t)\frac{\partial}{\partial y}U_1(1,t) = \psi_2'(1,t) + \eta(t)\frac{\partial}{\partial y}U_2(1,t). \tag{3.27}$$

The continuity of shear stress at the interface is expressed by

$$\psi_1''(1,t)+\alpha^2\psi_1(1,t)+\eta(t)\frac{\partial^2}{\partial y^2}U_1(1,t) = m\gamma\left(\psi_2''(1,t)+\alpha^2\psi_2(1,t)+\eta(t)\frac{\partial^2}{\partial y^2}U_2(1,t)\right) \tag{3.28}$$

And the continuity of the normal stress at the interface is given by:

$$\begin{aligned} -i\alpha R\left(-U_1\psi_1' + \frac{i}{\alpha}\frac{\partial}{\partial t}\psi_1' + \psi_1\frac{\partial U_1}{\partial y}\right) - (\psi_1''' - \alpha^2\psi_1') \\ +i\alpha\gamma R\left(-U_2\psi_2' + \frac{i}{\alpha}\frac{\partial}{\partial t}\psi_2' + \psi_2\frac{\partial U_2}{\partial y}\right) \\ +m\gamma(\psi_2''' - \alpha^2\psi_2') + 2\alpha^2\psi_1' - 2\alpha^2 m\gamma\psi_2' = i\alpha R((1-\gamma)Fr^{-2} + \alpha^2 S)\eta \end{aligned} \tag{3.29}$$

In the above equations all variables are evaluated at $y = 1$, and

$$S = \frac{T}{\rho_1 d V_0^2} \tag{3.30}$$

where T is the surface tension.

3.4 Quasi-steadiness

Consider the Orr-Sommerfeld equation has to be solved with linear homogeneous boundary conditions. U_1 and U_2 are assumed to be a function of

y with at least some of their coefficients containing $\cos(\omega t)$ or $\sin(\omega t)$. In the following we assume $\omega << 1$, which justifies the quasi-steady approach as shown in the following. Denoting ωt by t' the following expansions are assumed for ψ_1 and ψ_2:

$$\psi_1 = e^{-i\sigma t}\{\phi_0(y,t') + \omega\phi_1(y,t') + \omega^2\phi_2(y,t') + \ldots\} \quad (4.31a)$$
$$\psi_2 = e^{-i\sigma t}\{\chi_0(y,t') + \omega\chi_1(y,t') + \omega^2\chi_2(y,t') + \ldots\} \quad (4.31b)$$
$$\eta(t) = e^{-i\sigma t}\{\eta_0(t') + \omega\eta_1(t') + \omega^2\eta_2(t') + \ldots\} \quad (4.31c)$$
$$\sigma = \sigma_0 + \omega\sigma_1 + \omega^2\sigma_2 + \ldots \quad (4.31d)$$

Expansions are then substituted into the governing equations and the boundary conditions, and the terms of equal power in ω are sorted out. The system of equations for the 0th order is given by:

$$\phi_0'''' - 2\alpha^2\phi_0'' + \alpha^4\phi_0 = R\left[(-i\sigma_0 + i\alpha U_1)(\phi_0'' - \alpha^2\phi_0) - i\alpha\phi_0\frac{\partial^2}{\partial y^2}U_1\right], \quad (4.32a)$$

$$\chi_0'''' - 2\alpha^2\chi_0'' + \alpha^4\chi_0 = \frac{R}{m}\left[(-i\sigma_0 + i\alpha U_2)(\chi_0'' - \alpha^2\chi_0) - i\alpha\chi_0\frac{\partial^2}{\partial y^2}U_2\right], \quad (4.32b)$$

where all the accents mean the differentiation with respect to y and the boundary conditions at the interface ($y = 1$) are:

$$(-\sigma_0 + \alpha U_1)\eta_0 = -\alpha\phi_0, \quad (4.33)$$

$$\phi_0 = \chi_0, \quad (4.34)$$

$$\phi_0' + \eta_0\frac{\partial}{\partial y}U_1 = \chi_0' + \eta_0\frac{\partial}{\partial y}U_2, \quad (4.35)$$

$$\phi_0'' + \alpha^2\phi_0 + \eta_0\frac{\partial^2}{\partial y^2}U_1 = m\gamma\left(\chi_0'' + \alpha^2\chi_0 + \eta_0\frac{\partial^2}{\partial y^2}U_2\right), \quad (4.36)$$

$$\begin{aligned} -i\alpha R\left(-U_1\phi_0' + \frac{1}{\alpha}\sigma_0\phi_0' + \phi_0\frac{\partial U_1}{\partial y}\right) - (\phi_0''' - \alpha^2\phi_0'), \\ +i\alpha\gamma R\left(-U_2\chi_0' + \frac{1}{\alpha}\sigma_0\chi_0' + \chi_0\frac{\partial U_2}{\partial y}\right) + \\ m\gamma(\chi_0''' - \alpha^2\chi_0') + 2\alpha^2\phi_0' - 2\alpha^2 m\gamma\chi_0' = i\alpha R((\gamma-1)Fr^{-2} + \alpha^2 S)\eta, \end{aligned} \quad (4.37)$$

and on the boundaries ($y = 0$, $y = +\infty$):

$$\phi_0(0,t') = 0, \quad (4.38)$$

$$\phi_0'(0,t') = 0, \quad (4.39)$$

$$\chi_0(+\infty,t') = 0, \quad (4.40)$$

$$\chi_0'(+\infty,t') = 0, \quad (4.41)$$

In the same way the equations (3.20) can be expressed for ϕ_1, χ_1 etc. Note that σ_0, ϕ_0 and χ_0 are determined with t' in U. Hence σ_0 is a function of t' and ϕ_0 and χ_0 contain t'. The expansion (4.31) is convergent for any finite ω. If ω is very small (which is our case according to the previous analysis), the quasi-steadiness approach gives good results. Otherwise one may use more terms in the expansion.

The Orr-Sommerfeld equations (4.32) together with the boundary conditions can be considered as the eigenvalue problem

$$\boldsymbol{A}\hat{\boldsymbol{v}} = \sigma_0 \boldsymbol{B}\hat{\boldsymbol{v}}, \tag{4.42}$$

where

$$\hat{\boldsymbol{v}} = \begin{pmatrix} \phi_0 \\ \chi_0 \\ \eta_0 \end{pmatrix} \tag{4.43}$$

end $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined by equations 4.32, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39, 4.40, 4.41.

Chapter 4

Numerical approach

4.1 Finite difference scheme

In order to solve the Orr-Sommerfeld equations (4.32) together with the boundary conditions (4.33)-(4.41), a second-order finite-difference scheme was used.

The domain $y \in [0; +\infty]$ is discretized uniformly with a constant step h.

As mentioned in the section 3.4, our problem is given by:

$$\boldsymbol{A}\hat{\boldsymbol{v}} = \sigma_0 \boldsymbol{B}\hat{\boldsymbol{v}} \tag{1.1}$$

where

$$\hat{\boldsymbol{v}} = \begin{pmatrix} \phi_0 \\ \eta_0 \\ \chi_0 \end{pmatrix} \tag{1.2}$$

For equation (4.32) defined inside a domain the central schemes as applied in order to get the approximation of the 2nd and 4th derivatives:

$$f_i^{''} = \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2}, \tag{1.3}$$

$$f_i^{(IV)} = \frac{f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+1}}{h^4}, \tag{1.4}$$

where $f_i = f(y_i)$. Four equations for the discretization the boundary conditions (4.38), (4.39), (4.40), (4.41) are required:

$$f_0 = 0, \tag{1.5}$$

$$f_0' = \frac{-3f_0 + 4f_1 - f_2}{2h}, \tag{1.6}$$

$$f_{N+1}' = \frac{3f_{N+1} - 4f_N + f_{N-1}}{2h}, \tag{1.7}$$

$$f_{N+1} = 0. \tag{1.8}$$

Obviously for the boundary conditions at the interface the central differences cannot be applied. In this case the second-order backward- and the forward-difference formulas are used for the approximation of the derivatives of ϕ_0 and χ_0 respectively. Denoting by I the index for the interface, the schemes are given by:

- First derivative:

* Backward-difference scheme:

$$f'_I = \frac{f_{I-2} - 4f_{I-1} + 3f_I}{2h} + O(h^2) \tag{1.9}$$

* Forward-difference scheme:

$$f'_I = \frac{-3f_I + 4f_{I+1} - f_{I+2}}{2h} + O(h^2) \tag{1.10}$$

– Second derivative:

* Backward-difference scheme:

$$f''_I = \frac{-f_{I-3} + 4f_{I-2} - 5f_{I-1} + 2f_I}{h^2} + O(h^2) \tag{1.11}$$

* Forward-difference scheme:

$$f''_I = \frac{2f_I - 5f_{I+1} + 4f_{I+2} - f_{I+3}}{h^2} + O(h^2) \tag{1.12}$$

– Third derivative:

* Backward-difference scheme:

$$f'''_I = \frac{3f_{I-4} - 14f_{I-3} + 24f_{I-2} - 18f_{I-1} + 5f_I}{2h^3} + O(h^2) \tag{1.13}$$

* Forward-difference scheme:

$$f'''_I = \frac{-5f_I + 18f_{I+1} - 24f_{I+2} + 14f_{I+3} - 3f_{I+4}}{2h^3} + O(h^2) \tag{1.14}$$

4.2 Discretization

The domain is discretized with a constant step h on N intervals:

$$y = [0, \dots, I, \dots, N] \tag{2.15}$$

, where I denotes the interface ($y = 1$). Since the vector $\hat{\boldsymbol{v}}$ in (1.1) is defined as :

$$\hat{\boldsymbol{v}} = \begin{pmatrix} \phi_0 \\ \eta_0 \\ \chi_0 \end{pmatrix},$$

one more point that corresponds to the displacement (η_0) should be added:

$$y' = [0, \dots, I-1, I, I+1, \dots, N, N+1]. \tag{2.16}$$

This can be summarized as :

– Lower fluid:
  * Stream function: ϕ_0
  * Domain: $[0, \dots, I]$
– Upper fluid:
  * Stream function: χ_0
  * Domain: $[I, I+2, \dots, N+1]$
– Displacement η_0 defined at $I+1$

The Orr-Sommerfeld equations (4.32) are defined on the nodes:

$$[2, \dots, I-2] \cup [I+3] \cup [I+4, \dots, N-1]$$

and the boundary conditions are defined on :

$$[0, 1] \cup [I-1, I, I+1, I+2] \cup [N, N+1]$$

In this way we end up with a system of $(N+1)$ equations with $(N+1)$ unknowns.

4.3 Parameters of the system

The input parameters for our system are the next :

- Initial time (t_{in});
- Number of points in wall normal direction (N);
- Maximum value of y ($y_{\max}$);
- Wave number (α);
- Reynolds number(Re);
- Frequency of the wall oscillations (ω);
- The ration between viscosities (m);
- The ration between densities (γ);
- Surface tension (S);
- Froude number(Fr).

Having this there are other parameters needed to be defined:

- The index I that corresponds to $y = 1$;
- The dimensionless wall normal coordinates:

$$y = [0, \ldots, N + 1]$$

  with $y(I+1) = y(I)$ since $I+1$ corresponds to the displacement eta_0 as it was mentioned before.
- Distribution in time. First the final time should be defined in a way:

$$t_{fin} = \frac{2\pi}{\omega}$$

  And together with the number of discrete points the distribution is defined.
- Band width ($bwidth$) of the matrices A and B. This parameter serves to simplify the process of building matrices. First, we build them as a rectangular with the width $[-bwidth \ldots bwidth]$. And then putting zeros the matrices are going to have the size $(N+1) \times (N+1)$. The parameter $bwidth$ is defined by the largest number of nodes required for the approximation of the derivative. From the section 4.1 one can see that the $bwidth$ is defined by the approximation of the third derivatives on the interface (equations 1.13 and 1.14) and it equals to 5.
- The base flow velocity together with its firs and second derivatives calculated in the nodes.

4.4 Eigenvalues

In order to solve the given eigenvalue problem Lapack routine ZGGEV ([1]) was used. It is solving the generalized eigenvalue problem

$$[\boldsymbol{A} + \sigma_0 \boldsymbol{B}]\hat{\boldsymbol{v}} = 0 \tag{4.17}$$

for a pair of N-by-N matrices ($\boldsymbol{A}$, $\boldsymbol{B}$), and optionally, calculates the left or/and the right generalized eigenvectors.

A *generalized eigenvalue* for a pair $(\boldsymbol{A}, (B)$ is a scalar σ_0 or a ration $\alpha/\beta = \sigma_0$, such that $[\boldsymbol{A} - \sigma_0 \boldsymbol{B}]$ is singular. It is usually represented as the pair (α, β), as there is a reasonable interpretation for $\beta = 0$, and even for both being zero.

The right generalized eigenvector $\boldsymbol{v}(j)$ corresponding to the generalized eigenvalue $\sigma_0(j)$ of $(\boldsymbol{A}, \boldsymbol{B})$ satisfies

$$\boldsymbol{A} \cdot \boldsymbol{v}(j) = \sigma_0(j) \cdot \boldsymbol{B} \cdot \boldsymbol{v}(j) \tag{4.18}$$

The left generalized eigenvector $\boldsymbol{u}(j)$ corresponding to the generalized eigenvalue $\sigma_0(j)$ of $(\boldsymbol{A}, \boldsymbol{B})$ satisfies

$$\boldsymbol{u}(j)^{**}H \cdot \boldsymbol{A} = \sigma_0(j) \cdot \boldsymbol{u}(j)^{**}H \cdot \boldsymbol{B} \tag{4.19}$$

where $\boldsymbol{u}(j)^{**}H$ is the conjugate-transpose of $\boldsymbol{u}(j)$.

In addition to that in order to find the eigenvalues for a certain range of parameters we used *inverse iteration algorithm.* It is implemented in the following way: given an initial estimate λ_0 for the eigenvalue and $\boldsymbol{x}_0$ for its respective eigenvector we set:

1. $\lambda \leftarrow \lambda_0$, $\boldsymbol{x} \leftarrow \boldsymbol{x}_0$
2. $\boldsymbol{x}' \leftarrow (\boldsymbol{A} - \lambda \boldsymbol{I})^{-1} \boldsymbol{x}_0$
3. $\lambda' \leftarrow \lambda + \frac{\boldsymbol{xp}}{\boldsymbol{x'p}}$ where $\boldsymbol{p}$ can be, for example, conjugate to $\boldsymbol{x}$
4. $\boldsymbol{x}' \leftarrow \frac{\boldsymbol{x}'}{||\boldsymbol{x}'||}$
5. if $||\lambda' - \lambda|| < tol$ then stop
   else $\lambda \leftarrow \lambda_0$, $\boldsymbol{x} \leftarrow \boldsymbol{x_0}$, goto 1.

The inversion in 2 is performed using LU decomposition.

Chapter 5

Results

5.1 Single fluid analysis

Assuming the dimensionless parameters both m and γ denoting the ratio between viscosities and densities equal to 1 and setting $S = 0$ (no surface tension), we obtain the stability problem for a single fluid, i.e. the stability of a Stokes layer. This has been studied by several authors, see for instance [13] for an early contribution, and [5] for a recent review. Note, however, that our findings are not directly comparable with the existing literature as they are based on a quasi steady approach. The basic flow in this case given by:

$$U_1 = [c_1 e^{-ay} + c_2 e^{ay}] e^{i\omega t} + c.c., \tag{1.1a}$$

$$U_2 = c_3 e^{-by} e^{i\omega t} + c.c., \tag{1.1b}$$

$$U_1(0,t) = \cos(\omega t), \tag{1.2a}$$

$$U_1(1,t) = U_2(1,t), \tag{1.2b}$$

$$U_2(+\infty,t) = 0, \tag{1.2c}$$

$$\frac{\partial}{\partial y} U_1(1,t) = \frac{\partial}{\partial y} U_2(1,t), \tag{1.2d}$$

$$\tag{1.2e}$$

The system of equation governing the stability is given by:

$$\phi_0'''' - 2\alpha^2 \phi_0'' + \alpha^4 \phi_0 = R\left[(-i\sigma_0 + i\alpha U_1)(\phi_0'' - \alpha^2 \phi_0) - i\alpha\phi_0 \frac{\partial^2}{\partial y^2} U_1\right], \tag{1.3a}$$

$$\psi_0'''' - 2\alpha^2 \psi_0'' + \alpha^4 \psi_0 = R\left[(-i\sigma_0 + i\alpha U_2)(\psi_0'' - \alpha^2 \psi_-) - i\alpha\psi_0 \frac{\partial^2 U_1}{\partial y^2}\right], \tag{1.3b}$$

with the boundary conditions:

$$\phi_0(0,t') = 0, \tag{1.4}$$

$$\phi_0'(0,t') = 0, \tag{1.5}$$

$$\chi_0(+\infty,t') = 0 \tag{1.6}$$

$$\chi_0'(+\infty,t') = 0. \tag{1.7}$$

And the boundary conditions at the $y = 1$ are given by:

$$(-\sigma_0 + \alpha U_1)\eta_0 = -\alpha\phi_0, \tag{1.8}$$

$$\phi_0 = \chi_0, \tag{1.9}$$

$$\phi_0' = \chi_0', \tag{1.10}$$

$$\phi_0'' = \chi_0'', \tag{1.11}$$

$$\phi_0''' = \chi_0'''. \tag{1.12}$$

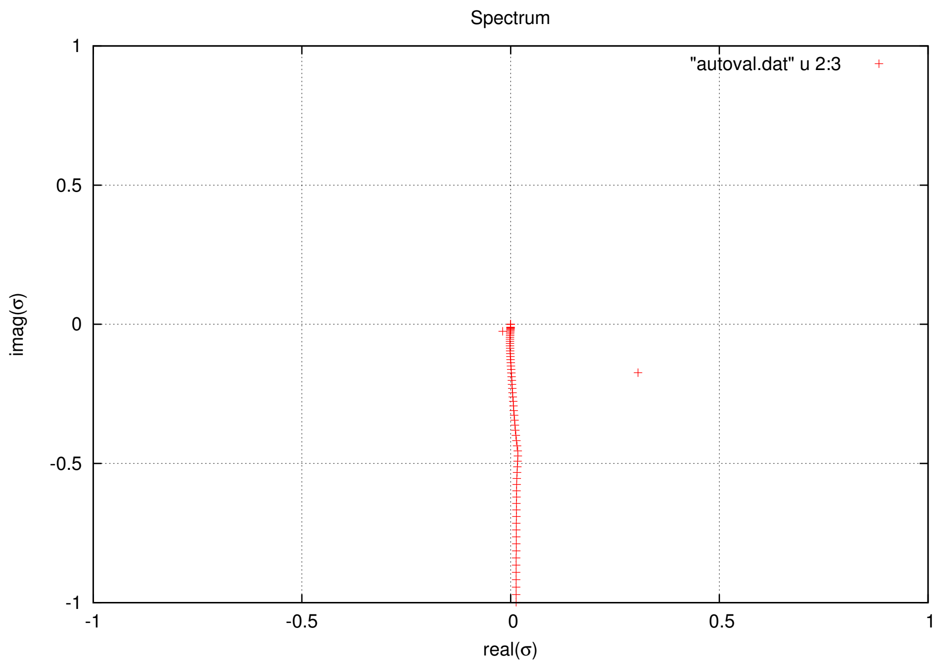

Figure 5.1: Spectrum of the eigenvalues for the single-fluid layer

As we can see, the boundary conditions at the interface for the single fluid layer impose the continuity of stream function together with its first, second and third derivatives.

On figures 5.2 and 5.3 we reported different eigenfunctions of the single fluid system.

The analysis of the single fluid layer (with $m = 1$, $\gamma = 1, S = 0$) showed that this system would be always stable under the conditions that we have imposed up to values of Reynolds number equal to 5000.

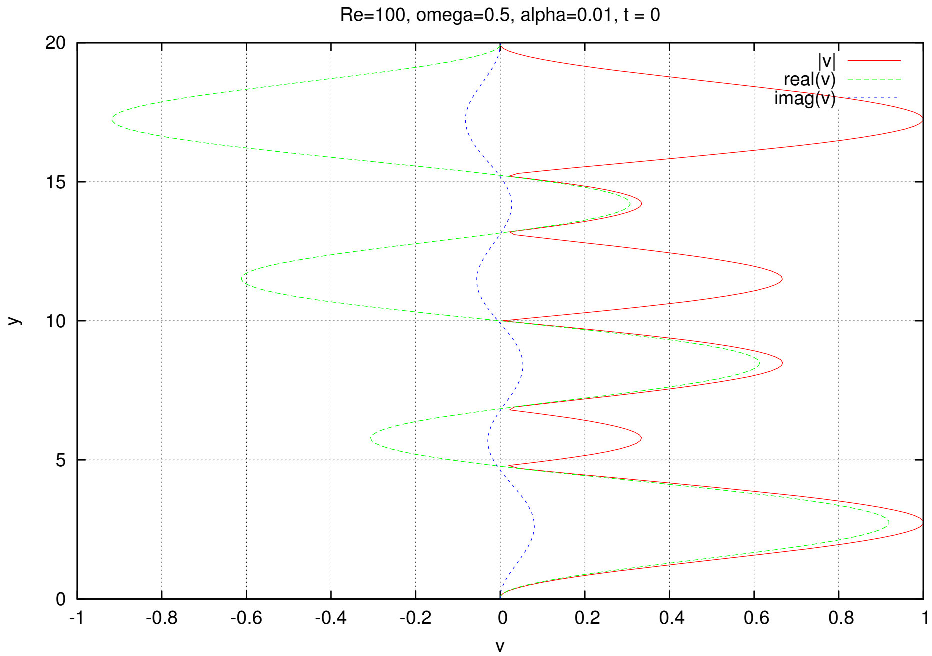

Figure 5.2: Real, imaginary and absolute values of the eigenfunction of the system

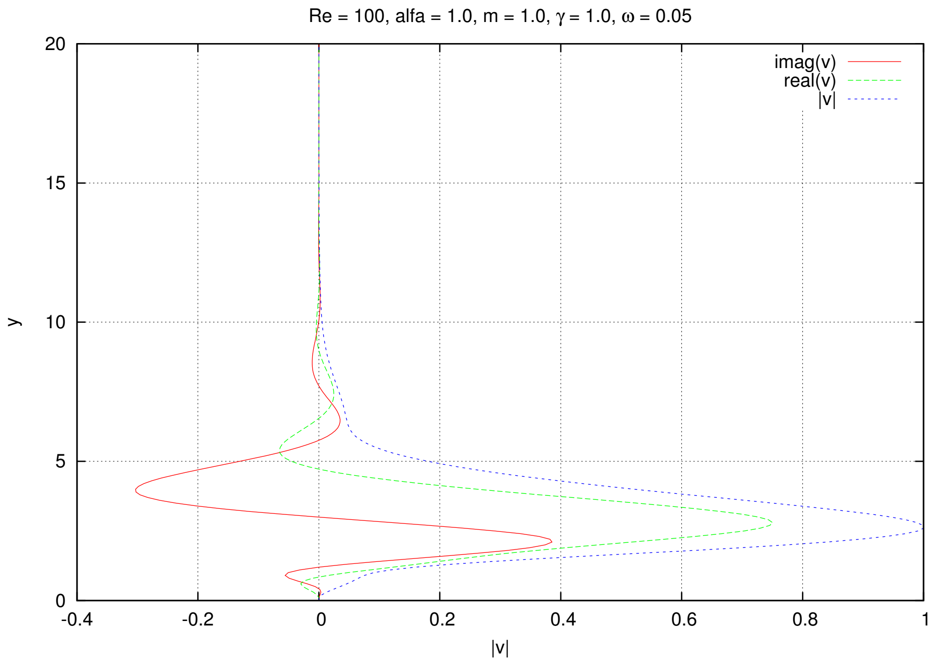

Figure 5.3: Real, imaginary and absolute values of the eigenfunction of the system

5.2 Results for realistic values of the parameters

We fixed all the parameters according to the analysis described in section 3.2. α can be fixed arbitrary. However, there is an upper bound for the length of the imposed perturbations, as they need to be shorter than the circumference of the big circle of the eye globe. As an example $\alpha = 8\cdot 10^{-3}$. In figures 5.4-5.7 the solution of the problem is reported in a plane $t - \alpha$ for different values of the surface tension parameter S. In those figures contour lines of the imaginary part of the eigenvalues are shown. The contour lines corresponding to $Im(\sigma_0) = 0$ separates stable and unstable manifolds.

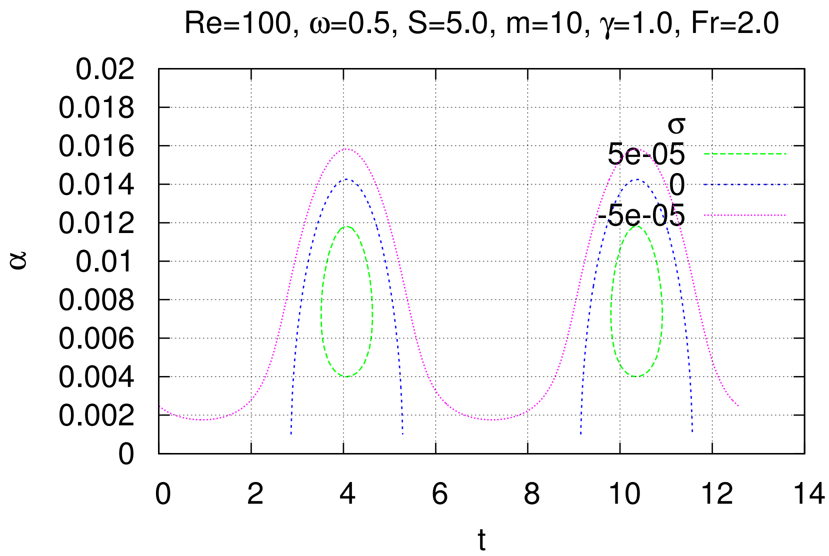

Figure 5.4: Contour lines of the growth rate; $S = 5.0$

As we can see in figures 5.4 - 5.7 increasing the surface tension system tends to be more stable. We report on the dependence of the growth rate on the wave number at fix time corresponding to the most unstable time (figure 5.8).

The same analysis was made for the viscosity. It is reported on figures 5.9 - 5.12

From figures 5.4-5.13 we conclude the following. As the ratio between viscosities m increases the system goes to unstable regime. In the limit $m \to \infty$ the solution is that of a Couette flow. This results differ from [4]. Because in [4] authors claim that with the increasing viscosity of silicone oil the system is stabilizing. However, in addition to that in [4] together with changing viscosity the surface tension changes as well.

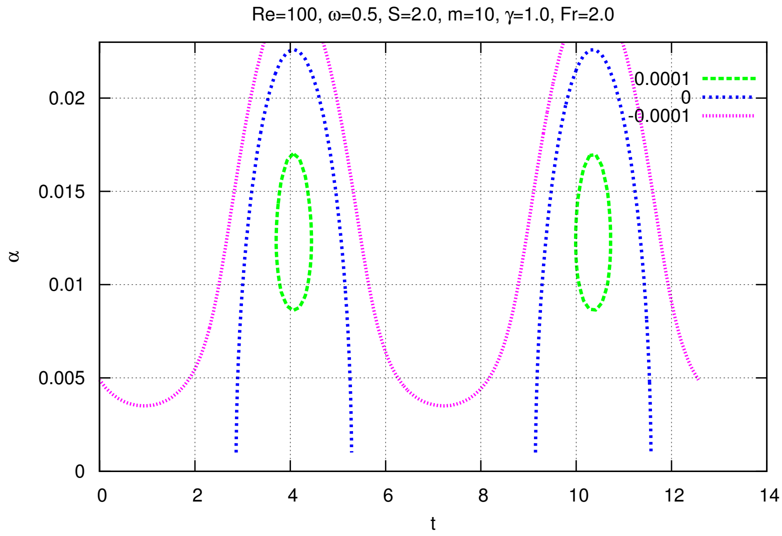

Figure 5.5: Contour lines of the growth rate; $S = 2.0$

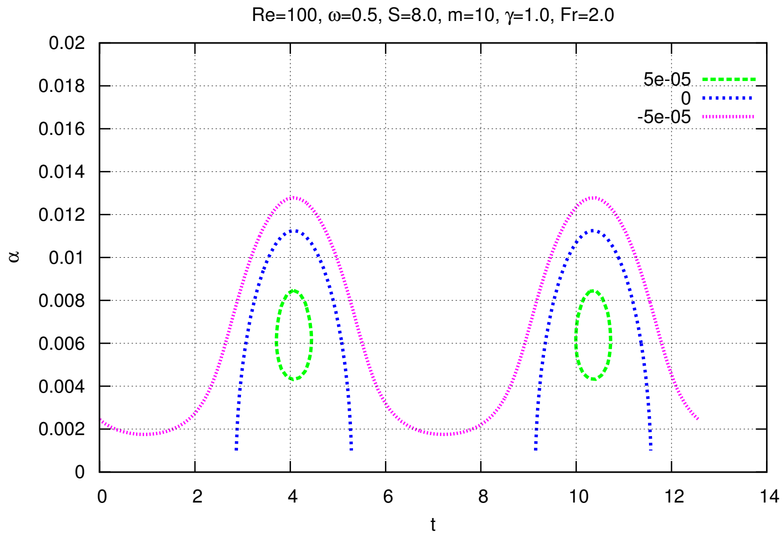

Figure 5.6: Contour lines of the imaginary part of eigenvalues; $S = 8.0$

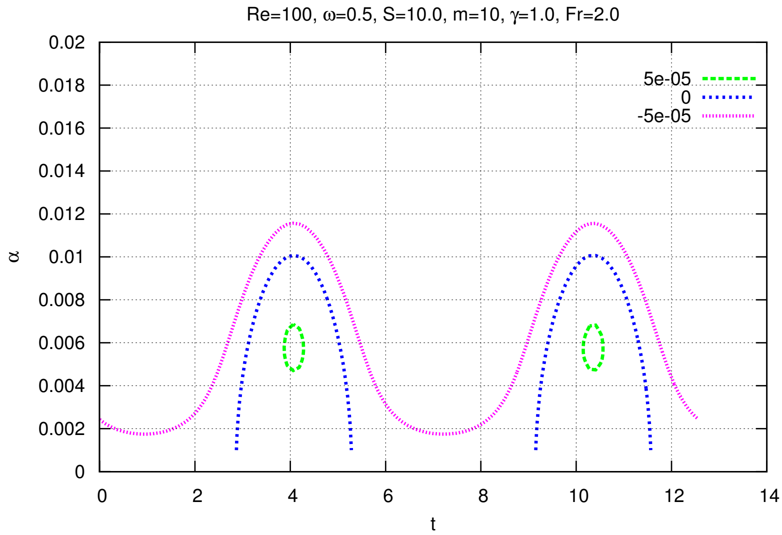

Figure 5.7: Contour lines of the growth rate of eigenvalues; $S = 10.0$

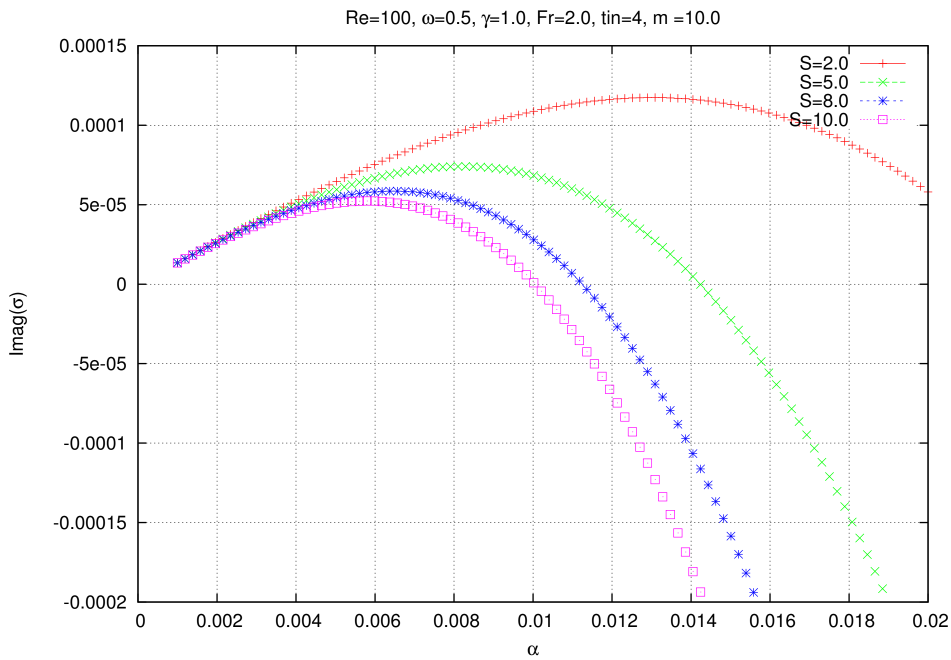

Figure 5.8: Cross section

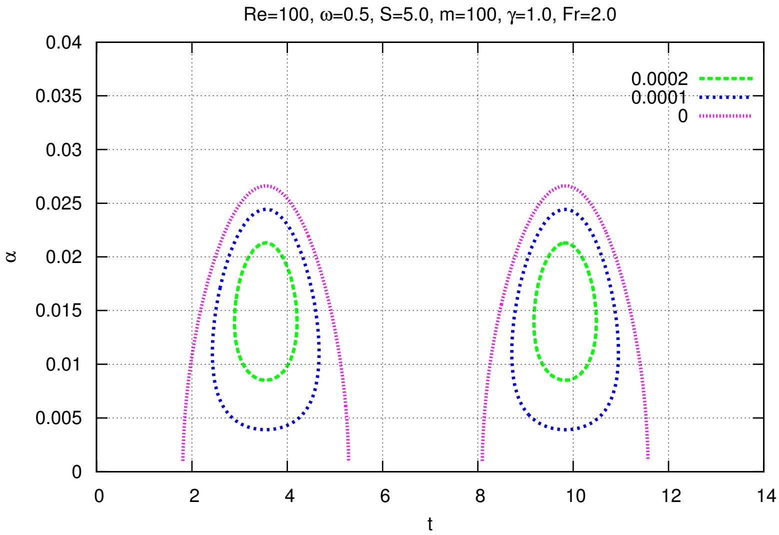

Figure 5.9: Contour lines of the growth rate; $m = 100.0$

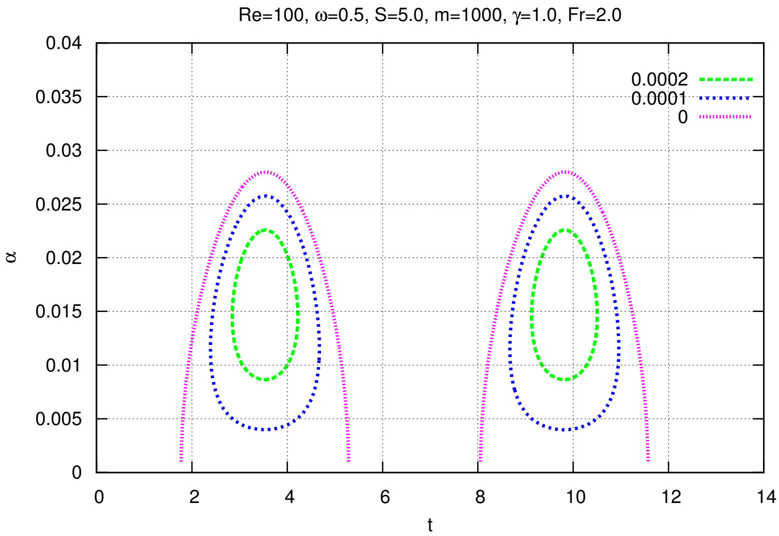

Figure 5.10: Contour lines of the growth rate; $m = 1000$

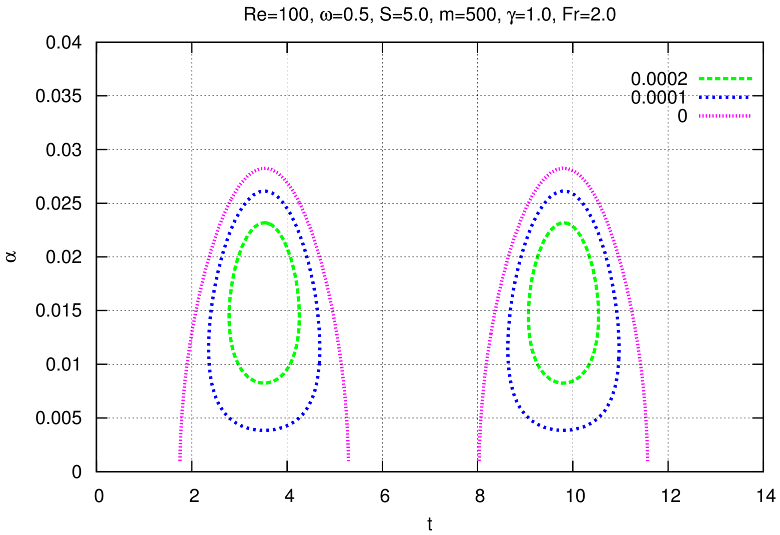

Figure 5.11: Contour lines of the growth rate; $m = 500$

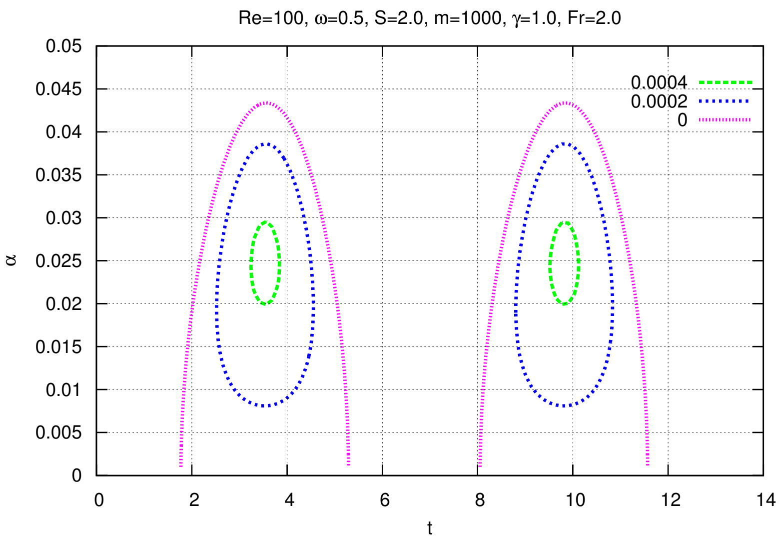

Figure 5.12: Contour lines of the growth rate; $S = 2.0, m = 500.0$

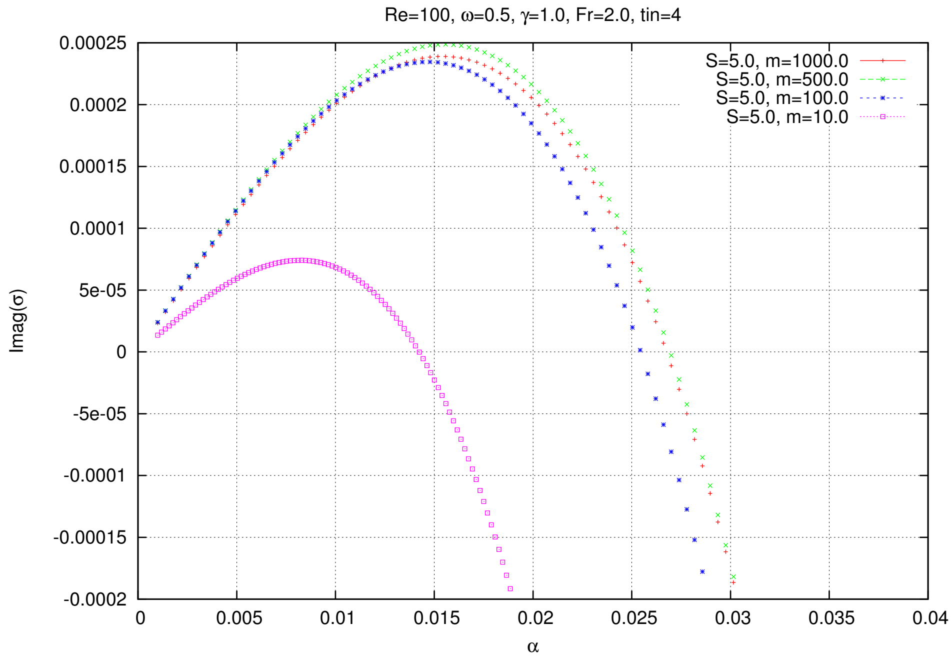

Figure 5.13: Cross section

Chapter 6

Conclusions

In this work the stability of two immiscible fluids set in motion by a flat wall oscillation has been studied. This model can be considered as a first step in studying problems related to the employment of silicone oils in order to treat retinal detachments, since the instability of the oil-aqueous interface might lead to the generation of oil bubbles and, eventually, to emulsion.

The problem is governed by the Navier-Stokes and continuity equations, subjected to the following boundary conditions: no-slip boundary condition at the rigid wall, condition of vanishing velocity at the infinity, kinematic and dynamic conditions at the interface. The problem for the basic flow was solved analytically.

The quasi-steady approach was used due to the fact the scaled frequency of oscillation is sufficiently small. Together with this approach the linear stability theory was applied. With this technique we studied the evolution in time of small perturbations of the interface.

The stability of the system is governed by the Orr-Sommerfeld equations and boundary conditions. The second-order finite-difference scheme was used for solving the generalized resulting eigenvalue problem. The results are given in terms of eigenvalues and eigenfunctions withing certain values of input parameters.

For values of the controlling parameters that are realistic for the physical problem under consideration we found that the system is linearly unstable, at least for long wave length disturbances. Whether instability will actually manifest itself in the real case of the human eye depends on a number of factors that need to be considered in future work. However, this work demonstrates that formation of oil droplets in the eye potentially can be caused by an instability process induced by eye rotations and can be explained on purely mechanical grounds.

Chapter 7

Future developments

There are several possibilities to the extend this work to make it closer to the real case:

- More realistic geometry, which can be studied gradually:
  * including effect of wall curvature;
  * considering a 2D circular model;
  * considering a 3D domain including the effect of gravity.
- Non-modal stability analysis. The motivation for using tools of non-modal analysis is given by the fact that not always the linear stability analysis is in agreement with experimental results. That might occur when the superposition of the decaying modes gives rise to a short-term transient growth due to the non-orthogonality of the eigenfunctions of the systeAuthentication Required

  m.
- Inclusion of the effect of wall roughness. If the layer of the fluid is very thin then the presence of ripples on the wall might affect greatly stability conditions;
- Use of the Floquet theory. This analysis might be applied to study the stability of periodic solutions when the quasi-steady approach is not valid;
- Numerical simulations of the fully non-linear equations.

Appendices

.1 Global Solution

```
COMPLEX FUNCTION Glob
!
WRITE 'Compute spectrum using ZGGEV'
BuildMats(A,B)
! zggev solves [A-LAMBDA*B]*q=0
CHAR JOBVL='N'
CHAR JOBVR='V'
INTEGER N=neq*(ny+2)
INTEGER INFO,OPTLWORK=3*N
ARRAY(1..N,1..N) OF COMPLEX AZ=0,BZ=0,VL=0,VR=0
ARRAY(1..N,1..N) OF REAL VecR=0,VecI=0
ARRAY(1..N)      OF COMPLEX LAMBDA=0,ALPHA=0,BETA=0
ARRAY(1..N)      OF REAL lamr=0,lami=0
ARRAY(0..ny)     OF REAL x=0, z=0,d1=0, d2=0, d3=0
!
! Band matrix to square matrix
LOOP FOR nj=1 TO N AND j=-bwidth TO bwidth
  IF nj+j>0 AND nj+j<N+1 THEN
    AZ(nj,nj+j)=A(nj-1,j)
    BZ(nj,nj+j)=B(nj-1,j)
  END IF
REPEAT LOOP
FILE MATRIX_A=CREATE("matrix_a.dat")
LOOP FOR i=1 TO N
DO WRITE TO MATRIX_A i, AZ(i,j) FOR j=1 TO N; WRITE TO MATRIX_A;
WRITE TO MATRIX_A
REPEAT LOOP
FILE MATRIX_B=CREATE("matrix_b.dat")
LOOP FOR i=1 TO N
DO WRITE TO MATRIX_B i, BZ(i,j) FOR j=1 TO N; WRITE TO MATRIX_B;
WRITE TO MATRIX_B
REPEAT LOOP
CLOSE MATRIX_B
! my equation is [A+LAMBDA*B]*q=0, so I need to change sign of B
BZ=-BZ
! ------------------------------------------------------------------
! Solve the generalized eigenvalue problem
ZGGEV(JOBVL,JOBVR,AZ,BZ,ALPHA,BETA,VL,VR,INFO,OPTLWORK)
LOOP FOR k=1 TO N
   IF ABS(BETA(k))>1.E-15 THEN LAMBDA(k)=ALPHA(k)/BETA(k)
REPEAT LOOP
! ------------------------------------------------------------------
! Save the results
FILE AVAL=CREATE('autoval.dat')
FILE AVEC=CREATE('autovec.dat')
FILE TEMP=CREATE('temp.dat')
LOOP FOR j=1 TO ny1+1
WRITE TO AVAL j,LAMBDA(j).REAL,LAMBDA(j).IMAG
DO WRITE TO AVEC j,f(k):15.8,ABS(VR(j,neq*k+1))/MAXABS(VR(j,*)),
REAL(VR(j,neq*k+1)),IMAG(VR(j,neq*k+1)) FOR k=0 TO ny1
DO WRITE TO AVEC j,f(k):15.8,ABS(VR(j,neq*k+1))/MAXABS(VR(j,*)),
REAL(VR(j,neq*k+1)),IMAG(VR(j,neq*k+1)) FOR k=ny1+2 TO ny+1
REPEAT LOOP
LOOP FOR j=ny1+3 TO N
WRITE TO AVAL j,LAMBDA(j).REAL,LAMBDA(j).IMAG
DO WRITE TO AVEC j,f(k):15.8,ABS(VR(j,neq*k+1))/MAXABS(VR(j,*)),
REAL(VR(j,neq*k+1)),IMAG(VR(j,neq*k+1)) FOR k=0 TO ny1
DO WRITE TO AVEC j,f(k):15.8,
ABS(VR(j,neq*k+1))/MAXABS(VR(j,*)),
REAL(VR(j,neq*k+1)),IMAG(VR(j,neq*k+1)) FOR k=ny1+2 TO ny+1
WRITE TO AVEC; WRITE TO AVEC
REPEAT LOOP
LOOP FOR k=0 TO ny1
x(k)=f(k)
z(k)=ABS(VR(4,neq*k+1))/MAXABS(VR(4,*))
REPEAT LOOP
h=x(2)-x(1)
LOOP FOR k=ny1+2 TO ny+1
x(k-1) = f(k)
z(k-1) = ABS(VR(4,neq*k+1))/MAXABS(VR(4,*))
REPEAT LOOP
LOOP FOR k=2 TO ny-2
d1(k) = (z(k+1) - z(k-1))/2/h
d2(k) = (z(k-1)-2*z(k)+z(k+1))/h/h
d3(k) = (z(k+2)-2*z(k+1)+2*z(k-1)-z(k-2))/2/h/h/h
REPEAT LOOP
DO WRITE TO TEMP x(k), z(k), d1(k), d2(k), d3(k) FOR k=0 TO ny
```

```
WRITE TO TEMP
CLOSE TEMP
COMPLEX maxim=1-1.E+05*I
INTEGER imax=0
LOOP FOR j=1 TO N
IF ABS(LAMBDA(j))>1.E-13 THEN
IF IMAG(LAMBDA(j))>IMAG(maxim) THEN
maxim=LAMBDA(j)
imax=j
END IF
END IF
REPEAT LOOP
RETURN maxim
FILE MAXVAL=CREATE('maxval.dat')
WRITE TO MAXVAL maxim
WRITE TO MAXVAL
CLOSE MAXVAL
END Glob
```

.2 Main

```
! Main program
!
! Jan Pralits
! V1.0 2012-05-28
!
USE rbmat
USE cbmat
!USE rtchecks
!USE gnuplot
USE Lapack
!-------------------read from file----------------------------------------------------------------
! tin          : initial time
! nt           : number of discrete points in time
! nt0          : parameter for parabolic distribution in time
! np           : baseflow is computed np periods
! ny           : number of points in wall normal direction
! ny0          : parameter for parabolic distribution in space
! ymax         : maximum value of y
! alfa         : streamwise wave number
! beta         : spanwise wave number
! Re           : Reynolds number
! omega        : frequency of wall oscillation
! mval         : viscosity ratio v2/v1
! sigma0       : initial guess of eigenvalue for inverse
iteration computation
!
COMPLEX sigma0
REAL Re, tin, ymax, mval, tfin, Re1, Re2, alfa1, alfa2
REAL alfa, beta, omega, S, Fr, gamma
INTEGER nt, nt0, ny, ny0, ny1, nalf, np, glob_on, nRe, nalfa, cycle
FILE INPUT=OPEN("occhio.in")
DO WHILE READ BY NAME FROM INPUT Re OR tin OR nt OR nt0 OR np OR ny OR ny1 OR ny0
OR ymax OR mval OR alfa OR beta OR sigma0 OR omega OR Fr OR
gamma OR glob_on OR S OR Re1 OR Re2 OR alfa1 OR alfa2 OR nRe OR nalfa OR cycle
CLOSE(INPUT)
WRITE BY NAME Re,tin,nt,nt0,np,ny,ny0,ymax,mval,alfa,beta,sigma0,omega,
Fr, gamma, S, glob_on
!------------- define function to find ny1 --------------------------------------------------
! ny1 is the index of the interface at which y=1
INTEGER FUNCTION findny1(REAL yy(*))
INTEGER j=0
  DO j=j+1 WHILE yy(j)<1
  IF ABS(yy(j)-1)>1.E-10 THEN
     WRITE "*** Change ny because the interface is not y=1"; STOP
  ELSE
     RETURN j
  END IF
END findny1
!------------- define dimensionless wall normal coordinate ----------------------------------
REAL y(0..ny+1)=0
REAL f(0..ny+1)=0
DO y(i)=ymax*i/ny FOR ALL i
ny1=findny1(y)
DO y(i)=ymax*(i-1)/ny FOR i=ny1+1 TO ny+1
y(ny1+1)=y(ny1)
f=y
!
```

```
WRITE BY NAME ny1, f, y
!------------- define distribution in time ---------------- -----------------------------------------
REAL t,dt
tfin=2*PI*np/omega
! parabolic distribution
REAL FUNCTION tt(INTEGER it)=tin+(tfin-tin)*(it-1)/(nt-1)
!
!---------------- define some stuff ------------------------------------------------------------
!
! tfin      : final time 2*pi in multiples of integer np (tfin = 2*pi*np)
! y         : dimensionless wall normal coordinate, y=ydim/d0star
! neq       : number of equations
! nstate    : number of state variables
! bwidth    : band width of matrix
! U,U1,U2   : streamwise meanflow velocities, first derivative, second derivative
! A,B,T     : matrices T = B*sigma + A, sigma is the eigevalue
! psi       : eigenvector of states
!
neq = 1
nstate = 1
INTEGER bwidth=5
ARRAY(0..ny+1) OF REAL U,U1,U2
ARRAY(0..neq*ny+1,-bwidth..bwidth)  OF  COMPLEX A, B, T
ARRAY(0..ny+1) OF COMPLEX psi, d0zeta, d0zstar, psistar, uvel
COMPLEX sigma
REAL U11,U12,U21,U22

!-----------------------------------------------------------------------------------------------
! coefficients for second order finite differences
USE Deriv4th
! subroutines to set up matrices of the OS equations
USE BuildMats
!---------------- define some stuff ------------------------------------------------------------
! subroutine that solves U for the two fluid flow over an oscillating plate
USE BaseFlow
!-----------------------------------------------------------------------------------------------
! phase is fixed to zero for v-velocity at the nphase point above the wall
nphase=5
!-------------------------------------------------------------------
! compute coefficients of derivatives
SetDerivatives
! eigenvalue solver using inverse iterations
USE eigen
!-----------------------------------------------------------------------------------------------
! MAIN program ---------------------------------------------------------------------------------
!-----------------------------------------------------------------------------------------------
! ny1 is the index at y=1 and should be evaluated with an INLINE FUNCTION
! create files
FILE growth=CREATE("growth.dat")
FILE bf=CREATE("baseflow.dat")
FILE ev=CREATE("eigenvector.dat")
ARRAY(1..nt) OF COMPLEX v
INTEGER imax
COMPLEX maxval
! initial guess of eigenvector
DO psi(i)=y(i)*EXP(-y(i)) FOR ALL i;
 psistar=psi
!
USE GlobalSolution
COMPLEX maxeigenval=0
!Global solver
IF glob_on=1 THEN
t=tin
BaseFlow(t,Re,mval,omega,gamma)
!U=0;U1=0;U2=0;U11=0;U22=0;U12=0;U21=0
maxeigenval=Glob();
WRITE BY NAME maxeigenval,Re,alfa
FILE MAXVAL=CREATE('maxval.dat')
WRITE TO MAXVAL maxeigenval
CLOSE(MAXVAL)
END IF
!------------------------------
!
! main loop
COMPLEX sigmastart=sigma0
IF cycle=1 THEN
!LOOP FOR ii=1 TO nRe
!Re=Re2+(Re1-Re2)*(ii-1)/(nRe-1)
sigma=sigmastart
LOOP FOR jj=1 TO nalfa
```

```
alfa = alfa1+(alfa2-alfa1)*(jj-1)/(nalfa-1)
LOOP FOR it=1 TO nt
IF nt=1 THEN t=tin; ELSE t=tt(it)
BaseFlow(t,Re,mval,omega,gamma)
eigenval
IF jj=1 THEN sigmastart=sigma
WRITE TO growth it,t,REAL(sigma),IMAG(sigma),alfa,beta,Re
DO WRITE TO bf y(j),U(j),U1(j),U2(j) FOR j=0 TO ny1
DO WRITE TO bf y(j),U(j),U1(j),U2(j) FOR j=ny1+2 TO ny;
WRITE TO bf; WRITE TO bf
DO WRITE TO ev y(j),ABS(psi(j)),REAL(psi(j)),IMAG(psi(j)) FOR j=0 TO ny;
WRITE TO ev; WRITE TO ev
REPEAT LOOP
!REPEAT LOOP
WRITE TO growth
REPEAT LOOP
END IF
!
! close files
CLOSE(growth)
CLOSE(bf)
CLOSE(ev)
!-------------------------------------------------------------------------------------------
! end MAIN program --------------------------------------------------------------------------
!-------------------------------------------------------------------------------------------
```

.3 Base Flow

```
SUBROUTINE BaseFlow(REAL tme,Rey,mm,Omega,Gamma)
!
!
U=0; U1=0; U2=0
ARRAY(0..ny+1) OF COMPLEX Utmp1=0, Utmp2=0,
dUtmp1=0, dUtmp2=0, ddUtmp1=0, ddUtmp2=0
! a, b coefficients
a=(I+1)/SQRT(2/(Rey*Omega))
b=(I+1)/SQRT(2*mm/(Rey*Omega))
! c coefficients ------------------
mbpa=a+b*mm*Gamma
mbma=a-b*mm*Gamma
epab=EXP(a-b)
emab=EXP(-a-b)
denom=epab*mbpa+emab*mbma
c1=0.5*epab*mbpa/denom
c2=0.5*emab*mbma/denom
c3=a/denom
! --------------------------------
! U velocity
! y=[0,1]
DO Utmp1(j)=(c1*EXP(-a*y(j))+c2*EXP(a*y(j)))*EXP(I*Omega*tme) FOR j=0 TO ny1
DO U(j)=2*REAL[Utmp1(j)] FOR j=0 TO ny1
! y=[1,ymax]
DO Utmp2(j)=c3*EXP(-b*y(j))*EXP(I*Omega*tme) FOR j=ny1 TO ny+1
DO U(j)=2*REAL[Utmp2(j)] FOR j=ny1 TO ny+1
! dU/dy
! y=[0,1]
DO dUtmp1(j)=(-a*c1*EXP(-a*y(j))+a*c2*EXP(a*y(j)))*EXP(I*Omega*tme)
FOR j=0 TO ny1
DO U1(j)=2*REAL[dUtmp1(j)] FOR j=0 TO ny1
! y=[1,ymax]
DO dUtmp2(j)=-b*c3*EXP(-b*y(j))*EXP(I*Omega*tme) FOR j=ny1 TO ny+1
DO U1(j)=2*REAL[dUtmp2(j)] FOR j=ny1 TO ny+1
U1(ny1+1)=U1(ny1)
! d2U/dy2
! y=[0,1]
DO ddUtmp1(j)=(a^2*c1*EXP(-a*y(j))+a^2*c2*EXP(a*y(j)))*EXP(I*Omega*tme)
FOR j=0 TO ny1
DO U2(j)=2*REAL[ddUtmp1(j)] FOR j=0 TO ny1
! y=[1,ymax]
DO ddUtmp2(j)=b^2*c3*EXP(-b*y(j))*EXP(I*Omega*tme) FOR j=ny1 TO ny+1
DO U2(j)=2*REAL[ddUtmp2(j)] FOR j=ny1 TO ny+1
U11=2*REAL[dUtmp1(ny1)]
U21=2*REAL[dUtmp2(ny1)]
U12=2*REAL[ddUtmp1(ny1)]
U22=2*REAL[ddUtmp2(ny1)]
FILE MATRIX=CREATE("base.dat")
LOOP FOR i=0 TO ny+1
WRITE TO MATRIX i, y(i), U(i), U1(i), U2(i); WRITE TO MATRIX
```

```
REPEAT LOOP
WRITE TO MATRIX U11, U21, U12, U22
CLOSE MATRIX
END BaseFlow
```

.4 Eigenvalue approximation

```
REAL FUNCTION Energy(COMPLEX Psi(*))
  RESULT=MAXABS(Psi)^2
END Energy

SUBROUTINE eigenval
   BuildMats(A,B)
  WRITE "--------------------------------------------------------------"
  WRITE "t =",t," Re =",Re," alpha =",alfa," beta =",beta
  WRITE "--------------------------------------------------------------"
  DO
    T=A+sigma*B; LUdecomp T
    d0zeta=B*psi; psi=T\d0zeta
    d0zstar=psistar/T; psistar=d0zstar*B
    dsigma=-(d0zstar*d0zeta)/(psistar*psi)
    sigma=sigma+dsigma
    !WRITE BY NAME sigma
    psi=psi/(Energy(psi))**0.5
    psi=psi*EXP(-I*ARG(psi(nphase)))
    s=psistar*psi
    d0zstar=d0zstar/s
    psistar=psistar/s
    WRITE REAL(sigma):15.8,IMAG(sigma):15.8,ABS(dsigma):15.8
  WHILE ABS(dsigma)>1.E-5
END eigenvalx
```

.5 Derivatives

```
STRUCTURE[ARRAY(-5..5) OF REAL d0,di0,d1l,d1r,di1r, d2l, d2r, d3l, d3r, d4, di4,
d1,d2,d0m,d1m,d0p,d1p,dw1,di1l,di02,di03,di1r2] derivatives(0..ny+1)
! forward difference used at the wall
ARRAY(0..2) OF REAL d0f, d1f, d0fp, d1fp

SUBROUTINE SetDerivatives
!
!   set up coefficients for 2nd order central difference
!   forward difference used at the wall
!
h=ymax/ny
h4=h^4
h3 = h^3
h2 = h^2
hym=h
hyp=h

LOOP FOR i=0 TO ny+1 WITH derivatives(i)

       ! h=[y(i+1)-y(i-1)]/2

        d0(-5)=0;d0(-4)=0; d0(-3)=0; d0(-2)=0; d0(-1)=0; d0(0)=1; d0(1)=0;
        d0(2)=0; d0(3)=0; d0(4)=0; d0(5)=0
        di0(-5)=0;di0(-4)=0; di0(-3)=0; di0(-2)=0; di0(-1)=0; di0(0)=0;
        di0(1)=1; di0(2)=0; di0(3)=0; di0(4)=0; di0(5)=0
        di02(-5)=0;di02(-4)=0;di02(-3)=0;di02(-2)=1;di02(-1)=0;di02(0)=0;
        di02(1)=0;di02(2)=0;di02(3)=0;di02(4)=0;di02(5)=0
        di03(-5)=0;di03(-4)=0;di03(-3)=0;di03(-2)=0;di03(-1)=1;di03(0)=0;
        di03(1)=0;di03(2)=0;di03(3)=0;di03(4)=0;di03(5)=0
        d1l(-5)=0;d1l(-4)=0; d1l(-3)=0; d1l(-2)=1/2/h; d1l(-1)=-2/h;
        d1l(0)=3/2/h; d1l(1)=0; d1l(2)=0; d1l(3)=0; d1l(4)=0; d1l(5)=0
        d1r(-5)=0;d1r(-4)=0; d1r(-3)=0; d1r(-2)=0; d1r(-1)=1/2/h;
        d1r(0)=-2/h; d1r(1)=3/2/h; d1r(2)=0; d1r(3)=0; d1r(4)=0; d1r(5)=0
        di1r(-5)=0;di1r(-4)=0; di1r(-3)=0; di1r(-2)=0; di1r(-1)=-3/2/h;
        di1r(0)=0; di1r(1)=2/h; di1r(2)=-1/2/h; di1r(3)=0; di1r(4)=0; di1r(5)=0
        di1r2(-5)=0;di1r2(-4)=0; di1r2(-3)=0; di1r2(-2)=0; di1r2(-1)=0;
        di1r2(0)=-3/2/h; di1r2(1)=0; di1r2(2)=2/h; di1r2(3)=-1/2/h;
        di1r(4)=0; di1r2(5)=0; di1l(-5)=0;di1l(-4)=0; di1l(-3)=1/2/h;
        di1l(-2)=-2/h; di1l(-1)=3/2/h;
        di1l(0)=0; di1l(1)=0; di1l(2)=0; di1l(3)=0; di1l(4)=0; di1l(5)=0
        d2(-5)=0;d2(-4)=0; d2(-3)=0; d2(-2)=0; d2(-1)=1/h2; d2(0)=-2/h2;
```

```
        d2(1)=1/h2; d2(2)=0; d2(3)=0; d2(4)=0; d2(5)=0
        d2l(-5)=-1/h2;d2l(-4)=4/h2; d2l(-3)=-5/h2; d2l(-2)=2/h2; d2l(-1)=0;
        d2l(0)=0; d2l(1)=0; d2l(2)=0; d2l(3)=0; d2l(4)=0; d2l(5)=0;
        d2r(-5)=0;d2r(-4)=0; d2r(-3)=0; d2r(-2)=2/h2; d2r(-1)=0; d2r(0)=-5h2;
        d2r(1)=4/h2; d2r(2)=-1/h2; d2r(3)=0; d2r(4)=0; d2r(5)=0
        d3l(-5)=0;d3l(-4)=3/2/h3; d3l(-3)=-7/h3; d3l(-2)=12/h3; d3l(-1)=-9/h3;
        d3l(0)=5/2/h3; d3l(1)=0; d3l(2)=0; d3l(3)=0; d3l(4)=0; d3l(5)=0
        d3r(-5)=0;d3r(-4)=0; d3r(-3)=0; d3r(-2)=0; d3r(-1)=0; d3r(0)=-5/2/h3;
        d3r(1)=0; d3r(2)=9/h3; d3r(3)=-12/h3; d3r(4)=7/h3; d3r(5)=-3/2/h3
        d4(-5)=0;d4(-4)=0; d4(-3)=0; d4(-2)=1/h4; d4(-1)=-4/h4; d4(0)=6/h4;
        d4(1)=-4/h4; d4(2)=1/h4; d4(3)=0; d4(4)=0; d4(5)=0
        di4(-5)=0;di4(-4)=0; di4(-3)=1/h4; di4(-2)=0; di4(-1)=-4/h4;
        di4(0)=6/h4; di4(1)=-4/h4; di4(2)=1/h4; di4(3)=0; di4(4)=0; di4(5)=0
        dw1(-5)=0;dw1(-4)=0; dw1(-3)=0; dw1(-2)=0; dw1(-1)=-1/h; dw1(0)=1/h;
        dw1(1)=0; dw1(2)=0; dw1(3)=0; dw1(4)=0; dw1(5)=0
        d1m(-2)=0        ; d1m(-1)=-1/hym            ;
        d1m(0)=1/hym             ; d1m(1)=0          ;
        d1m(2)=0;d1m(-5)=0;d1m(-4)=0;d1m(-3)=0;d1m(3)=0;d1m(4)=0;d1m(5)=0

        d1p(-2)=0        ; d1p(-1)=0                 ; d1p(0)=-1/hyp            ;
        d1p(1)=1/hyp     ; d1p(2)=0; d1p(-5)=0;d1p(-4)=0;d1p(-3)=0;d1p(3)=0;
        d1p(4)=0;d1p(5)=0
REPEAT LOOP
!(
WITH derivatives(0) :
        hyp=y(1)-y(0)
        d0p(-1)=0        ; d0p(0)=1/2                ; d0p(1)=1/2
        d1p(-1)=0        ; d1p(0)=-1/hyp             ; d1p(1)=1/hyp
        d0(-1)=0         ; d0(0)=1                   ; d0(1)=0
END WITH

WITH derivatives(ny) :
        hym=y(ny)-y(ny-1)
        d0m(-1)=1/2      ; d0m(0)=1/2                ; d0m(1)=0
        d1m(-1)=-1/hym   ; d1m(0)=1/hym              ; d1m(1)=0
        d0(-1)=0         ; d0(0)=1                   ; d0(1)=0
END WITH
!)

!diffx1=y(1)-y(0)
!diffx2=y(2)-y(1)
!bx=(diffx1+diffx2)/(diffx1*diffx2)
!cx=-(diffx1/diffx2)/(diffx1+diffx2)
!ax=-(bx+cx)

!d0f(0)=1                ; d0f(1)=0                 ; d0f(2)=0
!d1f(0)=ax               ; d1f(1)=bx                ; d1f(2)=cx
!d0fp(0)=1/2             ; d0fp(1)=1/2              ; d0fp(2)=0
!d1fp(0)=-1/[y(1)-y(0)]  ; d1fp(1)=1/[y(1)-y(0)]    ; d1fp(2)=0

END SetDerivatives
```

.6 Building Matrices

```
SUBROUTINE BuildMats(COMPLEX AA^(*,*), BB^(*,*))
!
AA=0
BB=0
!nstate=1
!neq=1
ARRAY(1..nstate) OF REAL vv
vv=0    ; vv(1)=1
INTEGER i,j,k,n
! neq=number of equations (1), i=index of equation (1..neq), j=index of
primitive variable (1..nstate)
! n=index in f (0..ny+1), k=index of block (-5..5)
p==neq*n+i-1      !index in direction of rows
q==k*neq-i+j      !index in direction of columns, wrt diagonal where
diagonal has value 0
!p==n
!q==k
!
LOOP   FOR n=2  TO ny1-2 WITH derivatives(n)
LOOP   FOR k=-2 TO 2
LOOP   FOR j=1  TO nstate

     v=d0(k)*vv(j); vyy=d2(k)*vv(j); vyyyy=d4(k)*vv(j)
     a2=alfa*alfa;  b2=beta*beta;  k2=a2+b2
```

```
!      i=1; WRITE BY NAME p,q,i,j,k; READ
!-----------------------------------------------------------------------------------------------------------
! Matrix A:
!-----------------------------------------------------------------------------------------------------------
! v-momentum equation
      i=1;   BB(p,q) = I*Re*alfa*(vyy-k2*v)
!-----------------------------------------------------------------------------------------------------------
! Matrix B:
!-----------------------------------------------------------------------------------------------------------
! v-momentum equation
      i=1;   AA(p,q) = [vyyyy -2*vyy*k2 +k2*k2*v] - I*alfa*Re*[ (vyy-k2*v)*
      U(n+k) - d0(k)*U2(n+k) ]
!-----------------------------------------------------------------------------------------------------------

REPEAT LOOP
REPEAT LOOP
REPEAT LOOP
!
!
!Orr-Sommerfield equation next to the boundary:
n=ny1+3
WITH derivatives(n)
LOOP FOR k=-3 TO 2
j=1
v=d0(k)*vv(j); vyy=d2(k); vyyyy=di4(k)*vv(j)
a2=alfa*alfa; b2=beta*beta; k2=a2+b2
!-----------------------------------------------------------------------------------------------------------
! Matrix A:
!-----------------------------------------------------------------------------------------------------------
! v-momentum equation
      i=1;   BB(p,q) = I*Re*alfa*(vyy-k2*v)/mval
!-----------------------------------------------------------------------------------------------------------
! Matrix B:
!-----------------------------------------------------------------------------------------------------------
! v-momentum equation
      i=1;   AA(p,q) = [vyyyy -2*vyy*k2 +k2*k2*v] - I*alfa*Re*[ (vyy-k2*v)
      *U(n+k) - d0(k)*U2(n+k) ]/mval
!-----------------------------------------------------------------------------------------------------------


REPEAT LOOP
END WITH

LOOP   FOR n=ny1+4  TO ny-1 WITH derivatives(n)
LOOP   FOR k=-2 TO 2
LOOP   FOR j=1  TO nstate

!      Uyy=SUM d0(jj)*U2(n+jj) FOR jj=-1 TO 1
      v=d0(k)*vv(j); vyy=d2(k)*vv(j); vyyyy=d4(k)*vv(j)
      a2=alfa*alfa; b2=beta*beta; k2=a2+b2


!      i=1; WRITE BY NAME p,q,i,j,k; READ
!-----------------------------------------------------------------------------------------------------------
! Matrix A:
!-----------------------------------------------------------------------------------------------------------
! v-momentum equation
      i=1;   BB(p,q) = I*Re*alfa*(vyy-k2*v)/mval
!-----------------------------------------------------------------------------------------------------------
! Matrix B:
!-----------------------------------------------------------------------------------------------------------
! v-momentum equation
      i=1;   AA(p,q) = [vyyyy -2*vyy*k2 +k2*k2*v] - I*alfa*Re
      *[ (vyy-k2*v)*U(n+k) - d0(k)*U2(n+k) ]/mval
!-----------------------------------------------------------------------------------------------------------

REPEAT LOOP
REPEAT LOOP
REPEAT LOOP
!
!--------------------------------
! boundary condition at the wall
!--------------------------------
!(
n=0
      i=1; j=1; k=0;  AA(p,q)=1   ! v=0  at y=0

n=1
i=1
```

```
j=1
WITH derivatives(n):
                              ! dv/dy=0 at y=0
      LOOP FOR k=-5 TO 5
         AA(p,q) = dw1(k)*vv(j)
      REPEAT LOOP
END WITH
!)
n=0
      i=1; j=1; k=0;
      AA(p,q)=1    ! v=0  at y=0
n=1
WITH derivatives(n):
      i=1                                   ! dv/dy=0 at y=0
      LOOP FOR j=1 TO neq AND k=-1 TO 1
      AA(p,q) = d1m(k)*vv(j)
      REPEAT LOOP
END WITH
!--------------------------------
! boundary condition at infinity
!--------------------------------
!(
n=ny+1
i=1;j=1;
LOOP FOR k=-5 TO 5
A(p,q)=0
REPEAT LOOP
k=0; A(p,q)=1
!)
n=ny+1
      i=1;j=1;k=0;        AA(p,q)=1
! dv/dy=0 at y=ymax
(written in the point n=ny-1)
n=ny
WITH derivatives(n)
      i=1
      LOOP FOR j=1 TO neq AND k=-1 TO 1
         AA(p,q) = d1p(k)*vv(j)
      REPEAT LOOP
END WITH

!--------------------------------------------
!(
n=ny
i=1
j=1
WITH derivatives(n):
      LOOP FOR k=-5 TO 5
         AA(p,q) = d1r(k)*vv(j)
      REPEAT LOOP
END WITH
!)
!--------------------------------
! boundary condition at the interfase
!--------------------------------
n=ny1-1
i=1;j=1;k=1;              AA(p,q)=1
i=1;j=1;k=2;              AA(p,q)=U(ny1)
i=1;j=1;k=2;              BB(p,q)=-1

!--------------------------------
n=ny1+1
j=1
i=1
WITH derivatives(n):
LOOP   FOR k=-5 TO 5
v1y=di1l(k)*vv(j);v2y=di1r(k)*vv(j)
eta=d0(k)*vv(j)
AA(p,q)=v1y-v2y+eta*U11-eta*U21
BB(p,q)=0
REPEAT LOOP
END WITH
!--------------------------------
n=ny1+2
j=1
i=1
a2=alfa*alfa
WITH derivatives(n):
LOOP FOR k=-5 TO 5
v1yy=d2l(k)*vv(j); v2yy=d2r(k)*vv(j)
```

```
v1=di02(k)*vv(j); v2=di02(k)*vv(j)
eta=di03(k)*vv(j)
AA(p,q)=v1yy+a2*v1+eta*U12-mval*gamma*(v2yy+a2*v2+eta*U22)
BB(p,q)=0
REPEAT LOOP
END WITH
!-------------------------------
n=ny1
i=1
j=1
WITH derivatives(n):
LOOP FOR k=-5 TO 5
v1=d0(k)*vv(j); v2=d0(k)*vv(j); eta=di0(k)
v1y=d1l(k)*vv(j); v2y=di1r2(k)*vv(j)
v1yyy=d3l(k)*vv(j); v2yyy=d3r(k)*vv(j)
AA(p,q)=-I*alfa*Re*(-U(ny1)*v1y+v1*U11)-(v1yyy-a2*v1y)+I*alfa*Re
*gamma*(-U(ny1)*v2y+v2*U21)+mval*gamma*(v2yyy-a2*v2y)+2*a2*v1y-
2*a2*mval*gamma*v2y-I*alfa*Re*((gamma-1.0)*Fr^(-2)-a2*S)*eta
BB(p,q)=I*Re*alfa*(gamma*v2y-v1y)

REPEAT LOOP
!k=1; A(p,q)=0
END WITH
FILE MATRIX=CREATE("matrix.dat")
LOOP FOR i=0 TO ny+1
DO WRITE TO MATRIX i, AA(i,j) FOR j=-5 TO 5;
WRITE TO MATRIX; WRITE TO MATRIX
REPEAT LOOP
CLOSE MATRIX

END BuildMats
```

.7 Examples of code that produces plots

```
#set termina png
#set output "single1.png"
set title "Re=100, omega=0.01, alpha=0.5"
set title "spectrum"
set xlabel "c_r"
set ylabel "c_i"
set grid
plot [][]\
"autovec.dat" i 350 u 3:2 w l title "|v|",\
"autovec.dat" i 200 u 4:2 w l title "real(v)",\
"autovec.dat" i 200 u 5:2 w l title "imag(v)"
pause -1 "hit return"

#set terminal postscript eps enhanced color
#set output "t_alpha.eps"
#set title "Re=100,
{/Symbol w}=0.5, S=5.0, m=1000, {/Symbol g}=1.0, Fr=2.0"
set cntrparam levels 40
set contour
set view map
unset surface
set grid
set xlabel "t"
set ylabel "{/Symbol a}"
set zlabel "Imag(sigma)"
splot [][][]\
"growth.dat" u 2:5:4 w l title ""
pause -1 "hit return"
```

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