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INGEGNERIA DELLE ACQUE E DELLA DIFESA DEL SUOLO

An analytical non linear model for equilibrium configurations of meandering rivers: the effect of suspended sediment transport

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Abstract

In the present thesis we formulate a new model able to predict the equilibrium configurations of both bed and flow field in meandering channels with sediment transport involving an appreciable fraction of suspended load. For this purpose we have extended a recently developed model (Nobile (2008)) including the convection-diffusion equation for sediment concentration and the related suspended sediment transport.

The three dimensional model developed is able to account for arbitrary, yet slow, variations of channel curvature. We tackle the problem by making use of an asymptotic theory according to which every variable is expanded in terms of small perturbative parameter δ which describes the curvature of the channel. The leading order and first order solutions are found. The computational effort needed in the present model is several order of magnitude smaller that the one which would be required in the context of a conventional three dimensional model.

Finally, we report some results about the evaluation of maximum scour in channels with constant curvature; this is a very remarkable issue in engineering. In order to make this prediction, we show the effect of suspended load effect on the transverse slope, compared to the case with bedload transport only. Suspended load effect is greater when sediments diameter is small.

Riassunto

Nella presente tesi viene formulato un nuovo modello in grado di predire le configurazioni di equilibrio del fondo e del campo di moto in alvei meandriformi soggetti a trasporto solido sia al fondo che in sospensione. A tal fine si è esteso un modello recentemente sviluppato (Nobile (2008)) includendo l'equazione di convezione-diffusione per la determinazione della distribuzione di concetrazione e del relativo trasporto solido di sedimenti in sospensione.

Il modello tridimensionale sviluppato è in grado di descrivere variazioni arbitrarie, seppur lente, della curvatura dell'asse dell'alveo. L'analisi è condotta con l'utilizzo di un metodo perturbativo, secondo cui tutte le variabili sono espanse in serie di potenze in termini di un parametro perturbativo δ che misura il grado di curvatura dell'alveo. Si è trovata la soluzione sia al primo che al secondo ordine di approssimazione. Il presente approccio consente di trovare l'assetto di equilibrio di un alveo meandriforme con uno sforzo computazionale diversi ordini di grandezza inferiore rispetto a quello che sarebbe richiesto nell'ambito di una modellazione numerica tridimensionale convenzionale.

Si riportano infine alcuni risultati preliminari relativi alla stima del massimo scavo che si realizza in alvei a curvatura costante, problema di grande rilevanza nella pratica ingegneristica. A tal proposito si mostra che l'effetto del trasporto in sospensione induce pendenze trasversali del fondo e dunque valori massimi dello scavo assai superiori rispetto al caso di trasporto solido di fondo. Tale effetto è tanto più rilevante quanto minore è il diametro dei sedimenti.

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Introduction

River meandering is a major topic in the field of morphodynamics. It has been the subject of extensive investigations in the recent past. The review paper of Seminara (2006), to which the reader is referred to for a broad overview of the subject, has outlined the main steps whereby progress has been made in this field. Investigations on the subject of bed topography in curved cohesionless channels have been proposed by several authors in the last decades. The various approaches appeared so far in the literature may roughly be classified into two main groups: (1) linear and weakly non linear theories which are essentially based on the assumption that perturbation of bottom elevation associated with the effect of curvature are sufficiently small compared with the unperturbed flow depth (Blondeaux & Seminara (1985), Seminara & Tubino (1986)) and strongly nonlinear numerical calculations which remove the latter restrictions at the expense of some numerical effort, (Nelson & Smith (1989)).
The advantage of theorical analysis over numerical approaches is usually their ability to provide insight on the basic mechanisms operating in the process under investigation. On the other hand, numerical work is usually superior in that it allows removal of the restrictions of linearity or weakly nonlinearity which may severely reduce the range of applicability of theoretical approaches. The importance of analytical work is also related to the need to provide to engineers simple tools able to predict the scour depth developing in river bends under given hydraulic conditions.
The purpose of the present paper is to derive a new theory for equilibrium configurations in meandering channels able to accounting for the effect of transport in suspension. It is important to note that the need to relax the linear constraint was recognized in the engineering literature, where a large effort was made to construct a rational framework, amenable to numerical treatment, in order to predict flow and bed topography in meandering channels with finite curvature and arbitrary width variations. These models serve the interests of river engineering, being fairly successful when applied to relatively short reaches of alluvial rivers and fairly short events. However, a more general interest towards the construction of sound analytical non linear models arises in the context of the fundamental research on the subject. The availability of such a model is also potentially suitable to investigate a number of important processes observed in meander evolution, which still await

to be understood. Another motivation to develop an analytical approach to non linear meanders involving a sufficiently modest computational effort, is related to investigations of long term meander evolution, a topic which has attracted the attention of both geomorphologists (Sun *et al.* (1996)) and engineers (Camporeale & Ridolfi (2006)). For such applications numerical models are not appropriate tools as the computational effort they require would be prohibitive. Researchers have then been forced to employ analytical linear models for flow and bed topography, allowing only for geometric nonlinearities arising from plan form evolution. The present model removes the latter restriction.
This idea is pursued by resorting to the use of perturbation methods. We set up an appropriate perturbation expansion for the solution of the problem of morphodynamics, valid in the general case of rivers with arbitrary distributions of channel curvature, the only constraint being that flow and bottom topography must be 'slowly varying' in both longitudinal and lateral directions and channel curvature must be 'sufficiently small'. The former assumption requires the channel to be 'wide' with channel alignment varying on a longitudinal scale much larger than channel width, while the latter assumption is satisfied provided the radius of curvature of channel axis is large compared with channel width. Both conditions are typically met in actual rivers but, in spite of the popularity enjoyed by linear models, neither of them implies that perturbations of bottom topography are necessarily 'small'.
Taking advantage of the slowly varying assumption, a suitable extension of the approach developed by Nobile (2008) to investigate bed deformations in slowly varying channels, with the additional contribution related to suspended transport is here developed. The latter approach allows for slow, yet finite, perturbations of flow and bed topography relative to a basic state consisting of a *locally and instantaneously* uniform flow, slowly varying in both the lateral and longitudinal directions. The only unknowns left for numerical computation are then flow depth, a slow function of longitudinal and lateral coordinates, and variation of the longitudinal free surface slope satisfying a strongly non linear differential equation subject to continuity constraints.
The present thesis is then organized as follows. In chapter 1 we briefly recall the main mechanism involved suspended transport, and recall the most relevant formulas to quantify it. In chapter 2 we formulate the 3D problem of flow in sinuous alluvial channels with a non cohesive bed. In the analysis, the direct effect of secondary flow on the transverse distribution of the main flow, leading to lateral transfer of longitudinal momentum is accounted for. This effect, which has been argued to be important by many authors (e.g. Nelson & Smith (1989), Imran & Parker (1999)), appears at the first order of approximation in the present scheme. In chapter 3 we report some

results evidencing the additional contribution driven by suspended sediment transport with respect to the case of bedload only. Some concluding remarks follows in the final chapter.

Chapter 1

Suspended sediment transport theory

Let us consider a turbulent flow. Particles with low longitudinal speed are entrained by the flow, discontinuously in time and space. Under specified conditions, when the emission is strong enough, sediment on the bottom are entrained in the center of the flow cell and are driven towards the external region. Some of these sediments do not reach the external because of their weight, so they quickly come back to the bed. When Shields's stress τ_* rises, the fraction of sediment able to reach the outer region tends to coincide with the whole. Far from the wall region, the main mechanisms which drive the direction of the sediment clouds are two: the transport by the fluid and sedimentation, driven by the excess of weight of the particles compared to the surrounding fluid. Sometimes, the sediment movement downward can be influenced by other sediments from the walls with the consequence to make the movement ascending. In suspended sediment transport most remarkable technical issues are:

i) Searching of threshold suspended sediment transport;

ii) Evaluation of volumetric concentration of transported sediment and of volumetric flux of suspended sediment transported by an uniform flow; Let us examine each of these points:

Different arguments (Bagnold, 1966) suggest the ratio W_s/u_τ between sedimentation velocity and friction velocity as the right parameter to define the threshold condition to have suspended transport. Another chance is using the well known *Rouse number*, that is basically similar to the previous. It can be interpreted as the ratio between a measure of the stability trend expressed by the tendency of the particles to settle, and a measure of the destabilizing tendency (by means of friction velocity, u_τ), which express the power of the fluid particles from the bed to entrain sediments. In this paper we will refer to Rouse' criterion, that can be expressed in the following form:

$$\frac{\sqrt{\tau_{*s}}}{\hat{W}_s} = f_s(R_p) \tag{1.1}$$

where $\hat{W}_s$ dimensionless sedimentation velocity, defined as:

$$\hat{W}_s = \frac{W_s}{\sqrt{(s-1)gd}} \tag{1.2}$$

In the following we report most remarkable empiric expression of the sedimentation velocity.
• Van Rijn (1989)

$$W_s = \frac{gd^2}{18\nu}(s-1) \qquad (d < 0,085mm) \tag{1.3a}$$

$$W_s = \frac{10\nu}{d}\left[\left(1+\frac{0,01(s-1)gd^3}{\nu^2}\right)^{0,5} - 1\right] \qquad (0,085 \leq d \leq 1()) \tag{1.3b}$$

$$W_s = 1,1[(s-1)gd]^{0,5} \qquad (d > 1mm) \tag{1.3c}$$

• Parker (1978)

$$\hat{W}_s = \exp\left[-1,181 + 0,966k - 0,1804k^2 + 0,003746k^3 + 0,0008782k^4\right] \tag{1.4}$$

Several closures for the function f_s are available in literature. According to Bagnold (1966), f_s has the simple form:

$$f_s = 1 \tag{1.5}$$

When Reynold's number R_p assume low value, and recalling the expression for W_s, (1.2) we get:

$$\tau_{*s} = \frac{R_p^2}{324} \tag{1.6}$$

Bagnold's criterion is reported in figure (1.1). A further closure for the function f_s has been proposed by Van Rijn (1984):

$$f_s = 4R_p^{-2/3} \qquad (1 \leq R_p^{2/3} \leq 10) \tag{1.7}$$

$$f_s = 0.4 \qquad (R_p^{2/3} \geq 10) \tag{1.8}$$

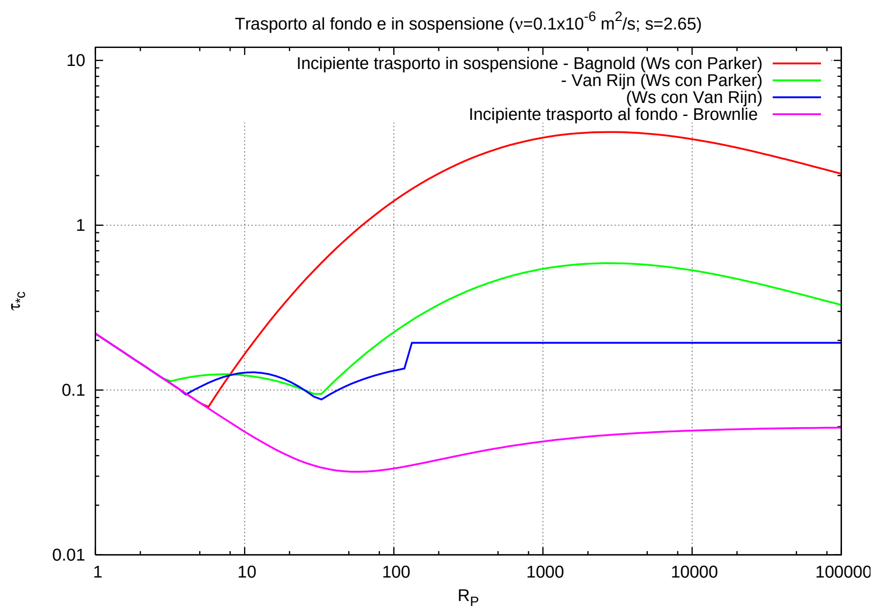

Figure 1.1: Sketch illustrating Shield's function(1936), which describes critical conditions for the bedload beginning and the functions which define critical conditions for the suspended transport , Bagnold (1966) e Van Rijn (1984)

1.1 Equilibrium condition for suspended transport

The suspended transport model traditionally employed in engineering is called *diffusive*. It is based on the hypothesis that the suspension is dilute enough, so interactions between solid particles can be ignored. Moreover the motion is thought as independent from the solid phase.
Let us define the volumetric sediment concentration $c(t)$:

$$c = \frac{\delta \mathcal{V}_s}{\delta V} \tag{1.9}$$

where $\delta \mathcal{V}_s$ is the volume contained in $\delta \mathcal{V}$, that is considered small enough to allow for a local definition. This type of model is strictly correct if we model suspended transport of very fine material. In any case this limit is largely ignored in engineering, extending to cases of one millimeter order diameter, and when volumetric concentration of solid phase reach values of the order of 10^{-1}.
Let us examine the simplest case: suspended transport in equilibrium condition. Equilibrium imposes turbulent flow in free surface to be uniform. In these conditions, the nature of solid load transported in suspension is so that particles averaging flow moved by the flow exactly balance the averaging depositional flux towards the bed.
In the following C will denote averaged concentration value over the turbulence.
Hence, it is purified by turbulence's fluctuations. In equilibrium conditions, we get:

$$\mathbf{V} = (U(z), 0, 0)\ , C = C(z) \tag{1.10}$$

where V is velocity vector averaged on the turbulence and z the coordinate perpendicular to bottom.

Let us examine the case of concentration distribution with uniform flow. We define the equilibrium concentration C_e as the concentration value which take place in equilibrium conditions, at the distance a from the bottom. If we known this value, we can obtain the vertical distribution of concentration C:

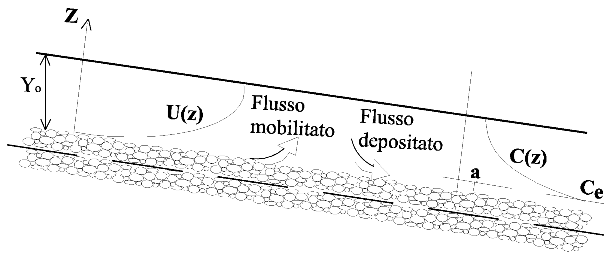

Figure 1.2: Sketch illustrating suspended transport in equilibrium condition

$$C = C_e \left[\frac{(1-\zeta)/\zeta}{(1-\zeta_a)/\zeta_a} \right]^Z \tag{1.11}$$

where:

$$\zeta = \frac{z}{Y} \qquad \zeta_a = \frac{a}{Y} \qquad Z = \frac{W_s}{k u_\tau} \tag{1.12a, b, c}$$

Note that k is Von Karman's constant, that assumes the value: $k = 0,41$.
A comparison between this distribution and several results available in literature, is reported in figure (1.3), (Vanoni, 1946).

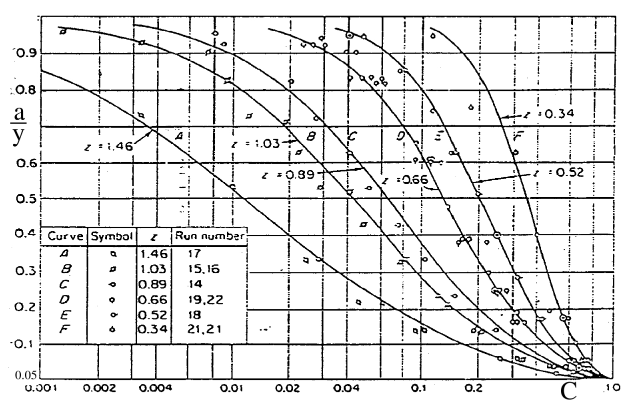

Figure 1.3: Sketch illustrating comparison between theoretic distribution of averaging concentration and experimental results (Vanoni, 1946).

According with the previous results, we can also find the vertically averaged concentration $\tilde{C}$, by integrating over the vertical to find:

$$\tilde{C} = \frac{C_e \zeta_a}{0,216} I_1 \tag{1.13}$$

$$\tilde{C} = \frac{1}{Y}\int_a^h C = \int_{\xi_a}^1 C \, d\xi =$$
$$= C_e \int_{\xi_a}^1 \left(\frac{1-\xi}{1-\xi_a}\frac{\xi_a}{\xi}\right)^Z d\xi = \frac{C_e \xi_a}{0.216} I_1 \tag{1.14}$$

with:

$$I_1 = \frac{0.216}{\xi_a}\int_{\xi_a}^1 \left(\frac{1-\xi}{1-\xi_a}\frac{\xi_a}{\xi}\right)^Z d\xi \tag{1.15}$$

where I_1 is the known integral, reported in figure (1.4).
Quantification of averaged concentration $\tilde{C}$ needs two evaluations, as follows:

i) evaluation of *equilibrium concentration* C_e, basic problem that has been object of several studies and investigations in the last few decades .

ii) evaluation of *conventional distance from the bottom* a where is necessary to impose the condition.

Let us report the most significant closures:
• Smith e Mc Lean (1977)

$$C_e = 0,65 \frac{\gamma_o T}{1 + \gamma_o T} \tag{1.16}$$

$$a = 26,3(\tau'_* - \tau_{*c})d + \epsilon \tag{1.17}$$

where:

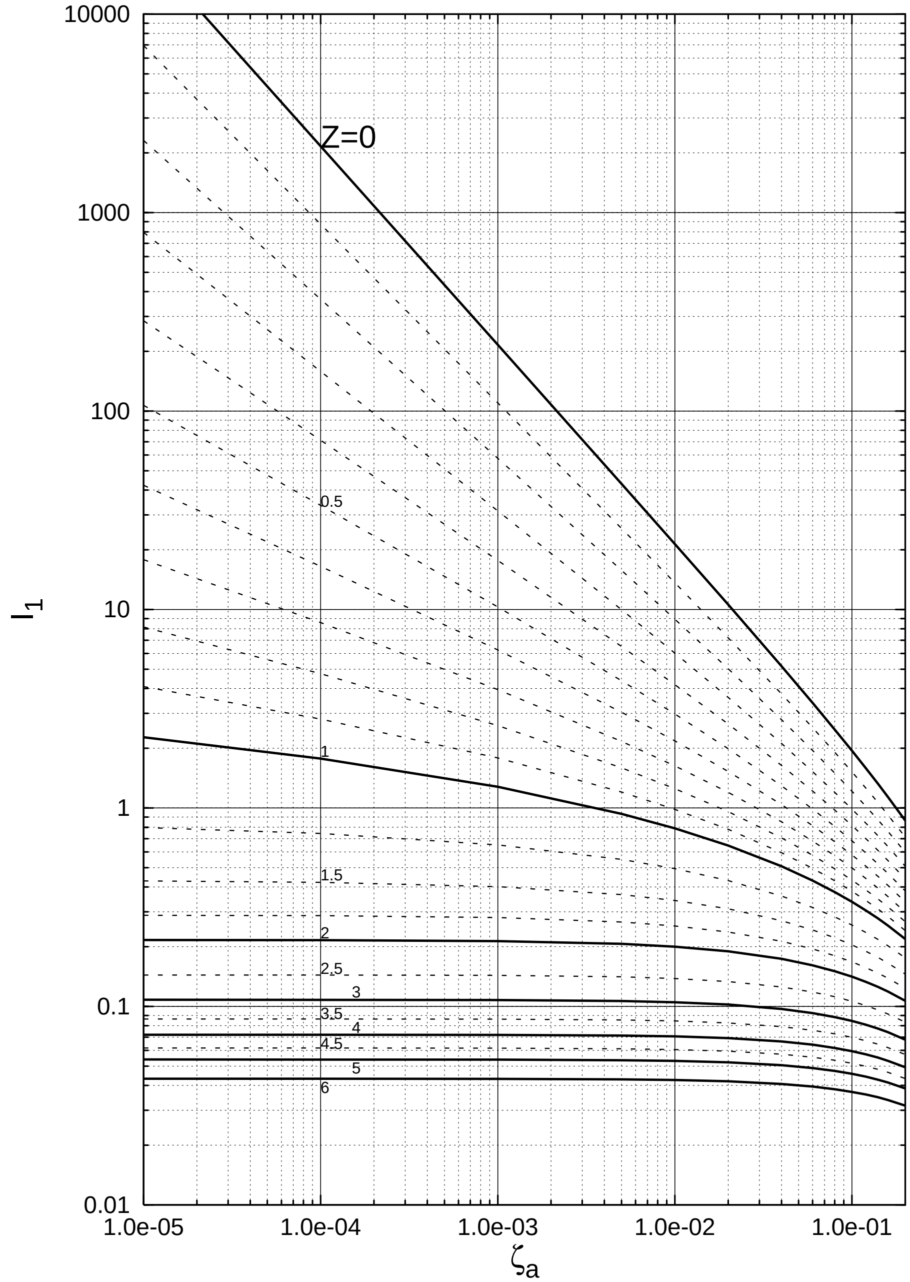

Figure 1.4: Tabulation of I_1 in function of ζ_a e Z parameters

$$\gamma_o = 0,0024 \ , \ T = \frac{\tau'_* - \tau_{*c}}{\tau_{*c}} \tag{1.18}$$

Note that apex denotes *friction component* of Shields's stress. This notion will be afterward explained in detail, when we will discuss the case of presence of bed forms. When equilibrium conditions are verified, friction component τ'_* of τ_* coincides with τ_* itself, being plane bed. Moreover,ϵ denotes absolute roughness of the bed, that we can evaluate as 2.5 d. Note that is necessary to calculate τ_{*c} as a function of particle Reynold's number R_p, using for example the formulation given by Brownlie.

• Van Rijn (1984)

$$C_e = 0,015 \frac{d}{a} \left(\frac{\tau'_*}{\tau_{*c}} - 1 \right)^{1,5} R_p^{-0,2} \tag{1.19}$$

$$a = \epsilon_e \qquad (\epsilon_e > 0,01Y) \tag{1.20a}$$
$$a = 0,01Y (\epsilon_e < 0,01Y) \tag{1.20b}$$

In this case, referring to the case of plane bed, we take $\epsilon_e = 3d_{90}$

• Garcia e Parker (1991)

$$C_e = \frac{A Z_u^5}{1 + \frac{A}{0,3} Z_u^5} \; , \; Z_u = \frac{u'_\tau}{W_s} R_p^{0,6} \; , \; A = 1,3 \times 10^{-7} \tag{1.21}$$

$$a = 0,05Y \tag{1.22}$$

Let us formulate how come to the solid discharge. Multiplying the concentration $C(z)$ for longitudinal velocity $U(z)$ and integrating along the vertical, we can obtain sediment volumetric flow in suspension for unit of width of the channel:

$$q_s = \frac{u_\tau C_e a}{k \times 0,216} \left[\ln \left(\frac{30Y}{k_s} \right) I_1 - I_2 \right] \tag{1.23}$$

where:

$$\frac{\zeta_a I_2}{0,216} = \int_{\zeta_a}^{1} \ln \zeta \left[\frac{(1-\zeta)/\zeta}{(1-\zeta_a)/\zeta_a} \right]^Z d\zeta \tag{1.24}$$

The function I_2 is tabulated in figure (1.5).

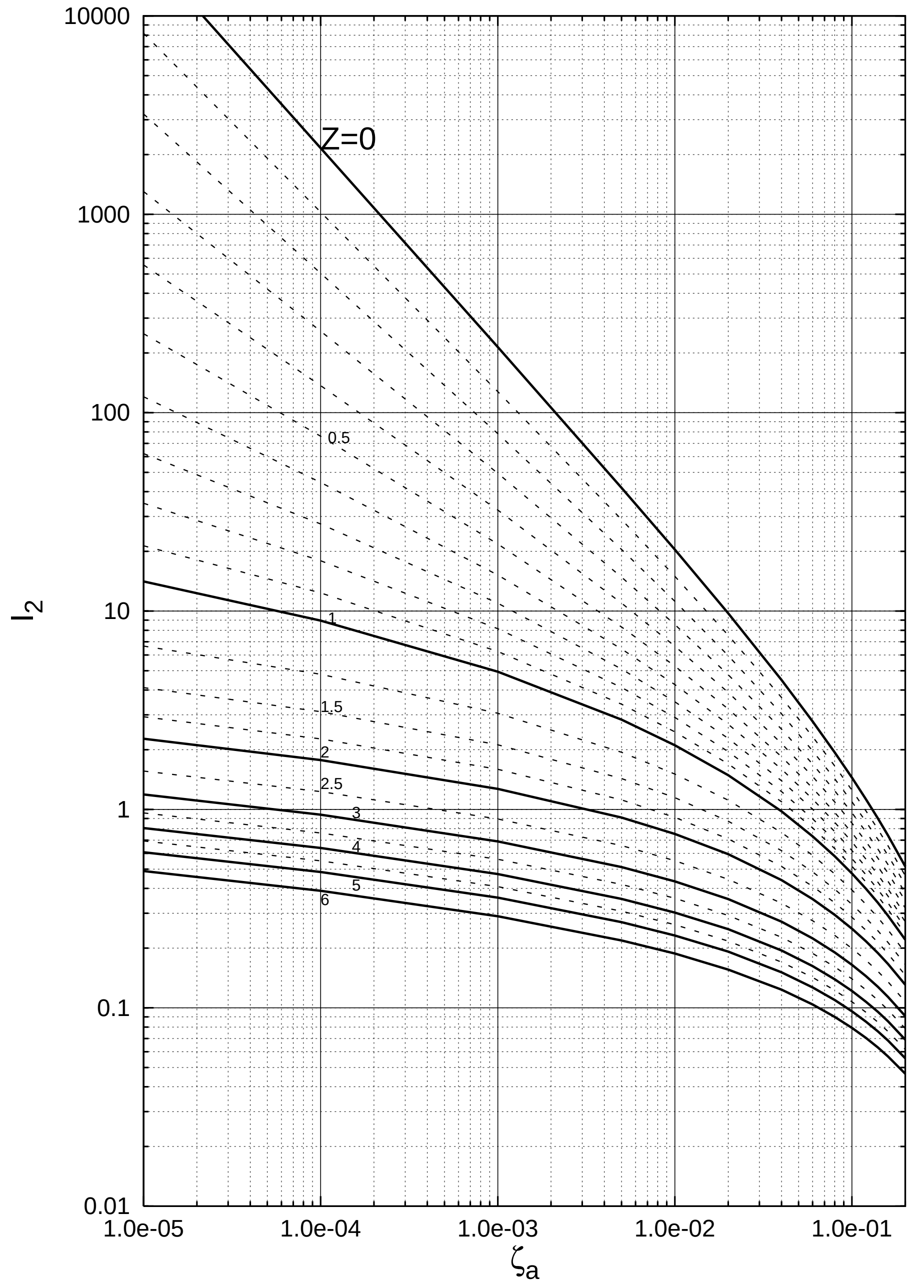

Figure 1.5: Tabulation of I_2 in function of ζ_a e Z parameters

Chapter 2

Non linear theory of slowly varying meanders: mathematical formulation

River morphodynamics deals with the turbulent free surface flow of a low concentration two phase mixture of water and sediment particles bounded by a granular medium consisting of sediment particles packed at their highest concentration: in river morphodynamics one ultimately wishes to determine the configuration of the bed interface. In other words, the mathematical problem of river morphodynamics is essentially a *free boundary problem*. In the following we will formulate it. In particular we intend to determine flow structure, suspended sediment concentration and bed topography of sinuous erodible channels. A longitudinal channel axis, is taken to describe a curve in space characterized by constant slope, and such that its projection onto an 'average bottom' plane is given by the so called 'sine-generated curve'.

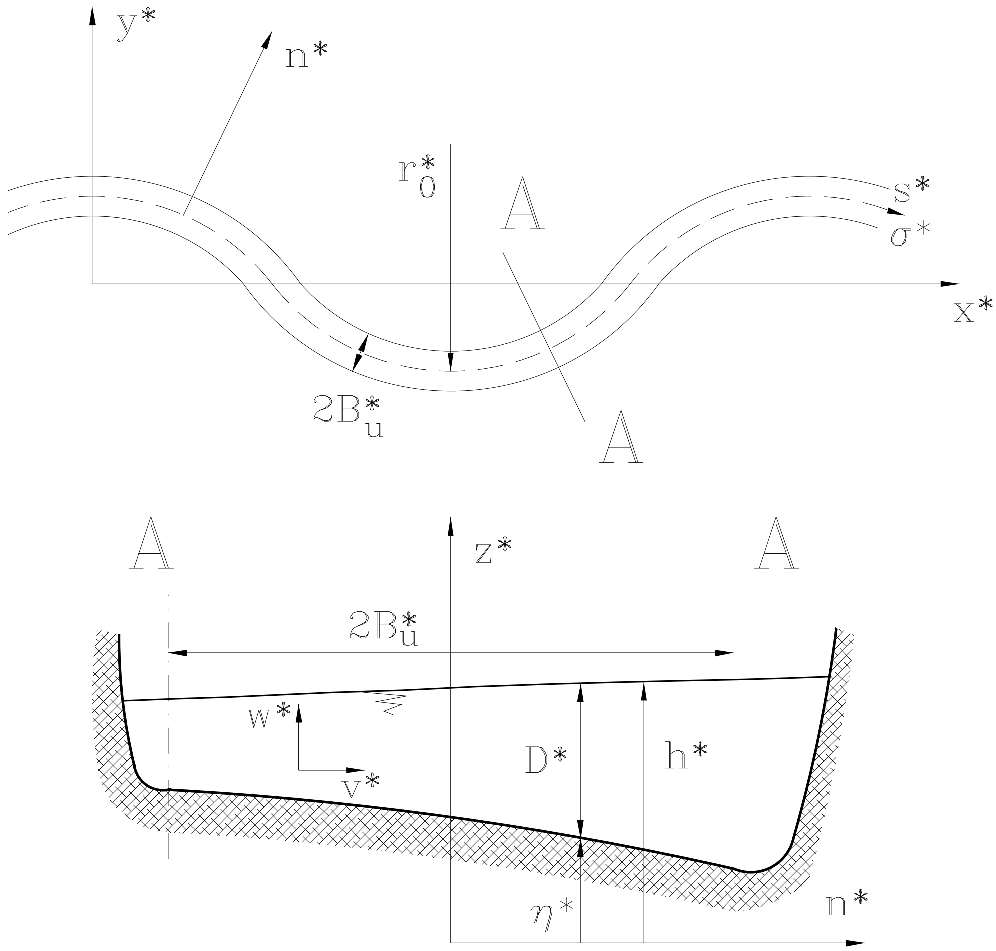

Figure 2.1: Sketch illustrating the meandering channel and notations

2.1 Formulation of the problem

Let us consider a sinuous alluvial channel with a non cohesive bed and refer it to the Cartesian coordinates sketched in figure 2.1:

Furthermore let us assume:

$$\cos\ \theta_s \simeq 1 \tag{2.1}$$

where θ_s is the angle the tangent to the axis forms with a horizontal plane. Let us then consider the flow of a constant discharge Q^* in the meandering reach. In the case of channels with constant width, say $2B_u^*$, the appropriate scaling for the intrinsic coordinates, the local mean velocity averaged over turbulence $\mathbf{u}^*{=}(u^*, v^*, w^*)^T$, the flow depth D^*, the free surface elevation h^*, the eddy viscosity ν_T^*, and the eddy diffusivity D_T.

$$(s^*, n^*) = B_u^*(s, n) \tag{2.2a}$$

$$(z^*, D^*, h^*) = D_u^*(z, D, F_u^2 h) \tag{2.2b}$$

$$\mathbf{u} = (u^*, v^*, w^*) = U_u^*(u, v, \frac{w}{\beta_u}) \tag{2.2c}$$

$$\mathcal{C}(s) = r_0^*\ \mathcal{C}^*(s) \tag{2.2d}$$

$$\nu_T^* = (\sqrt{C_{fu}} U_u^* D_u^*)\nu_T \tag{2.2b}$$

$$D_T^* = (U_u^* D_u^* \sqrt{C_{fu}}) D_T \tag{2.2c}$$

where a star denotes dimensional quantities. Note that Curvature is scaled with the minimum bending radius r_0.

According with the main aim of this paper, we consider the sediment flux per unit width in terms both of bedload and suspended transport. Consequently, we write the two components as follows:

$$q_b^* = \sqrt{(s_p - 1)gd^{*3}}(q_{b_s}, q_{b_n}) \tag{2.3a}$$

$$q_s^* = U_u^* D_u^* (q_{s_s}, q_{s_n}) \tag{2.3b}$$

$$q^* = q_b^* + q_s^* = \sqrt{(s_p - 1)gd^{*3}}\,[q_b + Q_0 q_s] \tag{2.3c}$$

where:

$$Q_0 = \frac{U_u^* D_u^*}{\sqrt{(s_p - 1)gd^{*3}}}$$

Moreover, s_p is the relative particle density ($= \rho_s/\rho$) with ρ and ρ_s water and particle density respectively, d^* is the particle diameter taken to be uniform. The index $_u$ refers to reference quantities consisting of the properties of uniform flow in a straight channel with the same flow discharge Q^* and the average channel slope S_u. In particular C_{fu} is the friction coefficient, β_u is the aspect ratio of the channel, F_u is the Froude number.

These parameters read:

$$\beta_u = \frac{B_u^*}{D_u^*} \tag{2.4a}$$

$$F_u^2 = \frac{U_u^{*2}}{g D_u^*} \tag{2.4b}$$

$$C_{fu} = \left[6 \;+\; 2.5 \;\ln\; \frac{D_u^*}{2.5\; d^*}\right]^{-2} \tag{2.5}$$

having estimated the absolute bottom roughness as $2.5d^*$.

We then take advantage of the hydrostatic approximation which applies when the spatial scale of the relevant hydrodynamic processes largely exceeds the flow depth. The steady turbulent flow of water in a channel characterized by a slowly varying spatial distribution of channel curvature $\mathcal{C}^*(s)$, is then governed by the longitudinal and lateral components of Reynolds equations, along with the continuity equations for the fluid and solid phases. Moreover, an additional continuity equation has to be satisfied which imposes a mass balance for the total load carried by the stream in suspension. In dimensional form, these equations read as follows.

Continuity equation for fluid phase:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \tag{2.6}$$

Reynold's equation:

$$\begin{aligned} &\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} + g \frac{\partial h^*}{\partial x^*} \\ &+ \frac{\partial}{\partial x^*}\left(\overline{u'^* u'^*}\right) + \frac{\partial}{\partial y^*}\left(\overline{u'^* v'^*}\right) + \frac{\partial}{\partial z^*}\left(\overline{u'^* w'^*}\right) = 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} + g \frac{\partial h^*}{\partial y^*} \\ &+ \frac{\partial}{\partial x^*}\left(\overline{u'^* v'^*}\right) + \frac{\partial}{\partial y^*}\left(\overline{v'^* v'^*}\right) + \frac{\partial}{\partial z^*}\left(\overline{v'^* w'^*}\right) = 0 \end{aligned} \tag{2.8}$$

$$\frac{\partial P^*}{\partial z^*} = \rho g \tag{2.9}$$

Continuity equation for solid phase (Exner's equation):

$$(1 - p)\,\frac{\partial \eta^*}{\partial t^*} + \vec{\nabla}\vec{q^*} = 0 \tag{2.10}$$

Convection - diffusion equation for sediment concentration:

$$\frac{\partial C}{\partial t^*} + \vec{u^*}\vec{\nabla} C - W_s^* \frac{\partial C}{\partial z^*} = \vec{\nabla}\left(D_T^* \vec{\nabla} C\right) \tag{2.11}$$

where:

$$\vec{u^*}\vec{\nabla} C = u^* \frac{\partial C}{\partial x^*} + v^* \frac{\partial C}{\partial y^*} + w^* \frac{\partial C}{\partial z^*} \tag{2.12}$$

In order to find steady configurations, we impose steady condition. We find:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} + g\frac{\partial h^*}{\partial x^*} + \frac{\partial}{\partial x^*}\left(\overline{u'^* u'^*}\right) + \frac{\partial}{\partial y^*}\left(\overline{u'^* v'^*}\right) + \frac{\partial}{\partial z^*}\left(\overline{u'^* w'^*}\right) = 0 \tag{2.13}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} + g\frac{\partial h^*}{\partial y^*} + \frac{\partial}{\partial x^*}\left(\overline{u'^* v'^*}\right) + \frac{\partial}{\partial y^*}\left(\overline{v'^* v'^*}\right) + \frac{\partial}{\partial z^*}\left(\overline{v'^* w'^*}\right) = 0 \tag{2.14}$$

$$\frac{\partial P^*}{\partial z^*} = \rho g \tag{2.15}$$

$$\vec{\nabla}\vec{q^*} = 0 \tag{2.16}$$

$$\vec{u^*}\vec{\nabla} C - W_s^* \frac{\partial C}{\partial z^*} = \vec{\nabla}\left(D_T^* \vec{\nabla} C\right) \tag{2.17}$$

If we refer the channel to the intrinsic coordinates (s^*, n^* and z^*), sketched in figure 2.1, then the steady turbulent flow of water in a channel characterized by a slowly varying spatial distribution of channel curvature $\mathcal{C}^*(s)$, is governed by the following equations, written in dimensionless form. Notice that s denotes a longitudinal coordinate defined along the channel axis, n is a transverse coordinate spanning the entire cross section and z is a nearly vertical coordinate orthogonal to the plane (s, n) and pointing upwards.

Continuity equation for fluid phase:

$$N\frac{\partial u}{\partial s} + \left[N\nu_0\mathcal{C} + \frac{\partial}{\partial n}\right] v + \frac{\partial w}{\partial z} = 0 \tag{2.18}$$

Reynold's equation:

$$\begin{aligned} N\frac{\partial u^2}{\partial s} + \frac{\partial uv}{\partial n} + \frac{\partial (uw)}{\partial z} + 2\nu_0 N\mathcal{C}uv \\ = \\ -N\frac{\partial h}{\partial s} + N\beta_u C_{fu} + \beta_u\sqrt{C_{fu}}\frac{\partial (\nu_T u_{,z})}{\partial z} \end{aligned} \tag{2.19}$$

$$\begin{aligned} N\frac{\partial}{\partial s}(uv) + \frac{\partial}{\partial n}\left(v^2\right) + \frac{\partial}{\partial z}(vw) + N\nu_0\mathcal{C}\left(v^2 - u^2\right) \\ = \\ -\frac{\partial h}{\partial n} + \beta_u\sqrt{C_{fu}}\frac{\partial}{\partial z}(\nu_T v_{,z}) \end{aligned} \tag{2.20}$$

$$\frac{\partial P}{\partial z} = -\frac{1}{F_u^2} \tag{2.21}$$

Finally an additional continuity equation is to be satisfied which imposes a mass balance for the total load carried by stream both at the bed and in suspension. This reads:

$$N\frac{\partial}{\partial s}\left(q_{b_s} + Q_0\ q_{s_s}\right) + \left(\frac{\partial}{\partial n} + \frac{\nu_0 N}{r_0}\right)\left(q_{b_n} + Q_0\ q_{s_n}\right) = 0 \tag{2.22}$$

where $\mathbf{q_s} = (q_{s_s}, q_{s_s})$ is the volume flux of suspended sediment (convective plus diffusive) defined in the form:

$$q_{s_s} = \int_{\eta+a}^{h} \left(UC - \frac{\sqrt{C_{fu}}}{\beta_u} \Psi(z) \frac{\partial C}{\partial s} \right) \, dz \tag{2.23}$$

$$q_{s_n} = \int_{\eta+a}^{h} \left(VC - \frac{\sqrt{C_{fu}}}{\beta_u} \Psi(z) \frac{\partial C}{\partial n} \right) \, dz \tag{2.24}$$

In (2.23) and (2.24), η and h denote non dimensional bed and free surface respectively. Moreover, $z = a$ is the conventional reference level at which the appropriate boundary condition for C is imposed under uniform condition.
Similarly the convection - diffusion equation for the sediment transported in suspension reads:

$$\begin{aligned} Nu\frac{\partial C}{\partial s} + v\frac{\partial C}{\partial n} + (w - W_s)\frac{\partial C}{\partial z} = \\ \beta_u \sqrt{C_{fu}} \frac{\partial}{\partial z}\left(D_T \frac{\partial C}{\partial z} \right) \end{aligned} \tag{2.25}$$

For the evaluation of the flux of suspended sediment we will neglect the role of diffusive terms, being $O\left(\frac{\sqrt{C_{fu}}}{\beta_u}\right) << 1$.
Moreover ν_0 is a curvature parameter, $\mathcal{C}(s)$ is dimensionless curvature and N is a metric coefficient of the orthogonal curvilinear coordinates:

$$\nu_0 = \frac{B_u^*}{r_0^*} \tag{2.26a}$$

$$\mathcal{C}(s) = r_0^* \mathcal{C}^*(s) \tag{2.26b}$$

$$N = \frac{1}{1 + \nu_0 n \mathcal{C}(s)} \tag{2.26c}$$

where r_0^* is some typical radius of curvature of the channel axis.
In the following we will assume the channel to be *wide* and *weakly curved*. Hence we write:

$$\beta_u \gg 1 \tag{2.27a}$$

$$\nu_0 \ll 1 \tag{2.27b}$$

Note that the assumption (2.27a) allows one to ignore the role of the side walls, concentrating the attention on the central region of the flow. The latter does not interact with the side wall boundary layers at least under natural conditions due to the relatively low slope of natural banks. The assumption (2.27b) implies that the flow field is slightly perturbed with respect to that in a straight channel.
Let us show in detail the boundary conditions needed to solve the concentration problem. At the free surface we impose no sediment flux through the interface, hence we write:

$$q_s^* \cdot \vec{n} = 0 \qquad (z^* = h^*) \tag{2.28}$$

where:

$$\vec{n} = \frac{\left(-N\frac{\partial h^*}{\partial s^*}, -\frac{\partial h^*}{\partial n^*}, 1\right)}{\sqrt{\left(N\frac{\partial h^*}{\partial s^*}\right)^2 + \left(\frac{\partial h^*}{\partial n^*}\right)^2 + 1}} \simeq \left(-N\frac{\partial h^*}{\partial s^*}, -\frac{\partial h^*}{\partial n^*}, 1\right) \tag{2.29}$$

and

$$q_s^* = \left(u^* C \ , \ v^* C \ , \ (w^* - W_s^*)C - D_T \frac{\partial C}{\partial z^*}\right) \tag{2.30}$$

It is important to note that, in this case, diffusive term is not negligible.
Now, substituting from (2.30) into (2.28), we find:

$$q_s^* \cdot \vec{n} = -u^* C N \frac{\partial h^*}{\partial s^*} - v^* C \frac{\partial h^*}{\partial n^*} + (w^* - W_s^*)C - D_T^* \frac{\partial C}{\partial z^*} = 0 \tag{2.31}$$

The condition takes the dimensionless form:

$$C\left[-uNF_u^2\frac{\partial h}{\partial s}-vF_u^2\frac{\partial h}{\partial n}+w\right]-W_sC-\beta_u\sqrt{C_{fu}}D_T\frac{\partial C}{\partial z}=0 \quad (2.32)$$

Hence, from the kinematic boundary condition on free surface, we get:

$$\beta_u\sqrt{C_{fu}}D_T\frac{\partial C}{\partial z}+W_sC=0 \qquad (z=F_u^2h) \qquad (2.33)$$

This constrain means that downward flux resulting from sedimentation process, must be balanced by upward diffusivity flux. By analogy with free surface condition it is easy to find the boundary condition which has to applied at the bed:

$$q_s^*\cdot\vec{n}=W_s^*(C_e-C) \qquad (z^*=z_R^*=\eta^*+a^*) \qquad (2.34)$$

that, in dimensionless form reads:

$$\beta_u\sqrt{C_{fu}}D_T\frac{\partial C}{\partial z}+W_sC_e=0 \qquad (z=z_R) \qquad (2.35)$$

Finally, we are able to write the full dimensionless system of boundary conditions in the form:

$$u=v=w=0 \qquad (z=z_0) \qquad (2.36)$$

$$P=\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w-\frac{u}{h_s}F_u^2\frac{\partial h}{\partial s}-vF_u^2\frac{\partial h}{\partial n}=0 \quad (z=F_u^2h) \qquad (2.37)$$

$$\int_{z_0}^{F_u^2h}v\ dz=q_{b_n}+Q_0q_{s_n}=0 \qquad (n=\pm 1) \qquad (2.38)$$

$$\beta_u\sqrt{C_{fu}}D_T\frac{\partial C}{\partial z}+W_sC=0 \qquad (z=F_u^2h) \qquad (2.39)$$

$$\beta_u \sqrt{C_{fu}} D_T \frac{\partial C}{\partial z} + W_s C_e = 0 \qquad (z = z_R) \tag{2.40}$$

The equations (2.36) impose no slip at the conventional reference level z_0; the equations (2.37) impose the conditions of no stress at the free surface and the requirement that the latter must be a material surface; finally, the condition (2.38) imposes the constraint that both the water and the sediment flux must vanish at the banks. Moreover, the equations (2.67) and (2.68) impose that the flux has to vanish through the free surface and the bottom respectively.
Closure relationships are then needed for the sediment flux per unit width $\mathbf{q}$ and for the eddy viscosity ν_T. We now take advantage of the *slowly varying* character of flow field and bed topography to assume that the turbulent structure is in *quasi* equilibrium with the local conditions, i.e. it is only slightly perturbed by weak curvature effects. Hence we write:

$$\nu_T = \left(\frac{|\boldsymbol{\tau}^*|}{\rho C_{fu} U_u^{*2}} \right)^{1/2} D(n,s) \mathcal{N}(\xi) \tag{2.41}$$

where $\boldsymbol{\tau}^*$ is the local value of the bottom stress, $D(n,s)$ is the local dimensionless value of the flow depth and $\mathcal{N}(\xi)$ is the vertical distribution of the eddy viscosity in a plane uniform free surface flow. Note that ξ is a normalized vertical coordinate which reads:

$$\xi = \frac{z - [F_u^2 h(n,s) - D(n,s)]}{D(n,s)} \tag{2.42}$$

Hence, ξ maps the actual cross section into the rectangle:

$$\xi_0 \leq \xi \leq 1 \qquad -1 \leq n \leq 1 \tag{2.43}$$

with ξ_0 normalized reference level, a weakly dependent function of the longitudinal and lateral coordinates, here assumed to be constant and equal to:

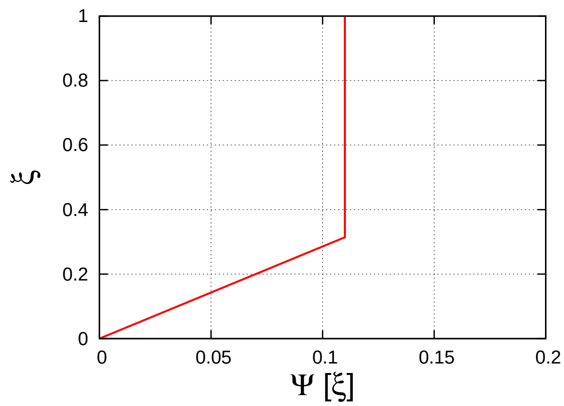

Figure 2.2: McTigue distrubution's function

$$\xi_0 \simeq \exp(-\frac{k}{\sqrt{C_{fu}}} - 0.777) \tag{2.44}$$

with $k = 0.41$ the Von Karman's constant. The distribution $\mathcal{N}(\xi)$ is taken to coincide with the classical parabolic distribution characteristic of uniform flows corrected by Dean's wake function (1974):

$$\mathcal{N}(\xi) = \frac{k\xi(1-\xi)}{1+2A\xi^2+3B\xi^3}, \qquad A = 1.84 \qquad B = -1.56 \tag{2.45}$$

For the eddy diffusivity $\Psi(\xi)$ we employ McTigue's distribution, reported in figure (2.2) which reads:

$$\Psi(\xi) = \begin{cases} 0.11 & (\xi_R \leq \xi < 0.314) \\ 0.35\xi & (0.314 \leq \xi \leq 1) \end{cases} \tag{2.46}$$

The closure for the bedload component of sediment flux per unit width $\mathbf{q_b}$ derives from a well established approach of semi empirical nature. In uniform open channel flow over a homogeneous non cohesive plane bed no significant sediment transport occurs below some *critical value* θ_c of a dimensionless form θ of the average shear stress $\boldsymbol{\tau}^*$ acting on the bed, depending on the *particle Reynolds number* R_p. With ν_f kinematic viscosity of the fluid, the Shields stress (Shields (1936)) and R_p read:

$$\theta = \frac{|\boldsymbol{\tau}^*|}{(\varrho_s - \varrho)\mathrm{g}d^*} \qquad (2.47a)$$

$$R_p = \frac{\sqrt{(s_p - 1)\mathrm{g}d^{*3}}}{\nu_f} \qquad (2.47b)$$

For values of θ exceeding θ_c but lower than a second threshold value θ_s, particles are transported as bedload with a distinct dynamics driven by, but different from, the dynamics of fluid particles. Moreover, flux direction deviates from the direction of bottom stress because of the effect of gravity. Under these conditions, on pure dimensional ground, the average bedload flux per unit width on a weakly sloping bottom may be given the general form:

$$\mathbf{q_b} = \Phi(\theta - \theta_c; R_p)\left(\frac{\boldsymbol{\tau}^*}{|\boldsymbol{\tau}^*|} + \mathbf{G}\cdot\nabla_h\eta\right) \qquad (2.48)$$

where η $(= F_u^2 h - D)$ is the dimensionless bed elevation.
Furthermore ∇_h is $(h_s^{-1}\partial/\partial s, \partial/\partial n)$, Φ is a monotonically increasing function of the excess Shields stress $(\theta - \theta_c)$ for given particle Reynolds number R_p, while $\mathbf{G}$ is a (2 x 2) matrix dependent on θ, θ_c and the angle of repose of the sediment. The function Φ can be estimated through well known empirical of semi empirical relationships: in the following we use the relation proposed by Parker (1990).

Moreover we only account for the lateral effect of gravity on the particle motion and write (Parker (1984)):

$$G_{ss} = G_{sn} = G_{ns} = 0 \qquad (2.49a)$$

$$G_{nn} = -R \tag{2.49b}$$

with R a typically small parameter which reads:

$$R = \frac{r_c}{\beta_u \sqrt{\theta}} \tag{2.50}$$

r_c being an empirical constant ranging about 0.56 (Talmon *et al.* (1995)). The reader should note that (2.48) fails close to sharp fronts (for the case of *arbitrarily sloping beds*, see Kovacs & Parker (1994) and Seminara *et al.* (2003)).

At last, the problem formulated above is subject to two integral constraints stipulating that flow and sediment supply must be constant at any cross section, hence:

$$\int_{-1}^{+1} D \int_{\xi_0}^{+1} u(\xi, n, s)\ d\xi\ dn = constant \tag{2.51}$$

$$\int_{-1}^{+1} \left(q_{b_s} + Q_0 q_{s_s}\right)\ dn = constant \tag{2.52}$$

2.2 Solution for channels with slowly varying distribution of curvature

Let us consider a sinuous channel characterized by a *slowly varying* distribution of curvature of the channel axis. Flow and bottom topography are then assumed to be *slowly varying* in both longitudinal and lateral directions. It's important to note that the above assumptions do not imply that perturbations of flow and bottom topography are necessarily small. It is then appropriate to rescale the longitudinal coordinate s introducing a *slowly varying coordinate* σ as follows:

$$\sigma = \frac{s^*}{r_0^*} = \nu_0 s \tag{2.53}$$

It is now useful to employ the new system of coordinates (σ, n, ξ), sketched in figure 2.1. Using the relations (2.53) and (2.42) the chain rule gives:

$$(s, n, z) \qquad \rightarrow \qquad (\sigma, n, \xi)$$

$$\begin{cases} \frac{\partial}{\partial s} & \rightarrow \quad \nu_0 \frac{\partial}{\partial \sigma} + \nu_0 \left[\frac{(1-\xi) D_{,\sigma} - F_u^2 h_{,\sigma}}{D} \right] \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial n} & \rightarrow \quad \frac{\partial}{\partial n} + \left[\frac{(1-\xi) D_{,n} - F_u^2 h_{,n}}{D} \right] \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial z} & \rightarrow \quad \frac{1}{D} \frac{\partial}{\partial \xi} \end{cases} \tag{2.54}$$

As a consequence of the change of coordinates system, the system (2.18 - 2.4) can be rewritten in the following form.

Continuity equation for fluid phase:

$$\begin{aligned} N\nu_0\frac{\partial u}{\partial \sigma} + N\nu_0\ \frac{1}{D}\left[(1-\xi)\frac{\partial D}{\partial \sigma} - F_u^2\frac{\partial h}{\partial \sigma}\right]\frac{\partial u}{\partial \xi} \\ +\frac{\partial v}{\partial n} + \frac{1}{D}\left[(1-\xi)\frac{\partial D}{\partial n} - F_u^2\frac{\partial h}{\partial n}\right]\frac{\partial v}{\partial \xi} \\ +\nu_0 N\mathcal{C}v \\ = \\ -\frac{1}{D}\frac{\partial w}{\partial \xi} \end{aligned} \tag{2.55}$$

Reynold's equations:

$$\begin{aligned} N\nu_0\frac{\partial u^2}{\partial \sigma} + \frac{N\nu_0}{D}\left[(1-\xi)\frac{\partial D}{\partial \sigma} - F_u^2\frac{\partial h}{\partial \sigma}\right]\frac{\partial u^2}{\partial \xi} + \frac{\partial uv}{\partial n} \\ +\frac{1}{D}\left[(1-\xi)\frac{\partial D}{\partial n} - F_u^2\frac{\partial h}{\partial n}\right]\frac{\partial uv}{\partial \xi} + \frac{1}{D}\frac{\partial uw}{\partial \xi} \\ +2\nu_0 N\mathcal{C}uv \\ = \\ -N\nu_0\frac{\partial h}{\partial \sigma} + N\beta_u C_{fu} + \frac{\beta_u\sqrt{C_{fu}}}{D^2}\frac{\partial}{\partial \xi}\left(\nu_T\frac{\partial u}{\partial \xi}\right) \end{aligned} \tag{2.56}$$

$$\begin{aligned} N\nu_0\frac{\partial uv}{\partial \sigma} + N\nu_0\frac{1}{D}\left[(1-\xi)\frac{\partial D}{\partial \sigma} - F_u^2\frac{\partial h}{\partial \sigma}\right]\frac{\partial uv}{\partial \xi} \\ +\frac{\partial v^2}{\partial n} + \frac{1}{D}\left[(1-\xi)\frac{\partial D}{\partial n} - F_u^2\frac{\partial h}{\partial n}\right]\frac{\partial v^2}{\partial \xi} \\ +\frac{1}{D}\frac{\partial uw}{\partial \xi} \\ +\nu_0 N\mathcal{C}(v^2-u^2) \\ = \\ -\frac{\partial h}{\partial n} + \frac{\beta_u\sqrt{C_{fu}}}{D^2}\frac{\partial}{\partial \xi}\left(\nu_T\frac{\partial v}{\partial \xi}\right) \end{aligned} \tag{2.57}$$

$$\frac{P_{,\xi}}{D} = -\frac{1}{F_u^2} \tag{2.58}$$

Continuity for solid phase (Exner's equation):

$$N\delta\frac{\partial}{\partial\sigma}\left[q_{b_\sigma} + Q_0\ q_{s_\sigma}\right] + \frac{\partial}{\partial n}\left[q_{b_n} + Q_0 q_{s_n}\right] + \delta N\mathcal{C}\nu_0\left[q_{b_n} + Q_0 q_{s_n}\right] = 0 \tag{2.59}$$

Convection - diffusion equation for sediment concentration:

$$Nu\frac{\partial C}{\partial s} + v\frac{\partial C}{\partial n} + (w - W_s)\frac{\partial C}{\partial z} = \beta_u\sqrt{C_{fu}}\frac{\partial}{\partial z}\left(D_T\frac{\partial C}{\partial z}\right) \tag{2.60}$$

To simplify further the system (2.55-2.59) we can use the continuity equation for the liquid phase (2.55) and perform simple algebra, to obtain:

$$\begin{aligned} N\nu_0 u u_{,\sigma} + N\nu_0\left[\frac{(1-\xi)D_{,\sigma} - F_u^2 h_{,\sigma}}{D}\right] u u_{,\xi} + v u_{,n} \\ + \left[\frac{(1-\xi)D_{,n} - F_u^2 h_{,n}}{D}\right] v u_{,\xi} + \frac{w u_{,\xi}}{D} \\ + N\nu_0\mathcal{C}uv \\ = \\ -N\nu_o h_{,\sigma} + N\beta_u C_{fu} + \frac{\beta_u\sqrt{C_{fu}}}{D^2}(\nu_T u_{,\xi})_{,\xi} \end{aligned} \tag{2.61}$$

$$\begin{aligned} N\nu_0 u v_{,\sigma} + N\nu_0\left[\frac{(1-\xi)D_{,\sigma} - F_u^2 h_{,\sigma}}{D}\right] u v_{,\xi} + v v_{,n} \\ + \left[\frac{(1-\xi)D_{,n} - F_u^2 h_{,n}}{D}\right] v v_{,\xi} + \frac{w v_{,\xi}}{D} \\ - N\nu_0\mathcal{C}u^2 \\ = \\ -h_{,n} + \frac{\beta_u\sqrt{C_{fu}}}{D^2}(\nu_T v_{,\xi})_{,\xi} \end{aligned} \tag{2.62}$$

The differential problem (2.61-2.62) will be solved with the following boundary conditions:

$$u = v = w = 0 \qquad (\xi = \xi_0) \tag{2.63}$$

$$P = u,_{\xi} = v,_{\xi} = 0 \quad (\xi = 1) \tag{2.64}$$

$$w - \nu_0 N u F_u^2 h,_{\sigma} - v F_u^2 h,_n = 0 \quad (\xi = 1) \tag{2.65}$$

$$\int_{\xi_0}^{1} v \; d\xi = q_n = q_{b_n} + Q_0 q_{s_n} = 0 \qquad (n = \pm 1) \tag{2.66}$$

$$\beta_u \sqrt{C_{fu}} D_T \frac{\partial C}{\partial z} + W_s C = 0 \qquad (z = F_u^2 h) \tag{2.67}$$

$$\beta_u \sqrt{C_{fu}} D_T \frac{\partial C}{\partial z} + W_s C_e = 0 \qquad (z = z_R) \tag{2.68}$$

Integrating the continuity equation of the liquid phase (2.55) in the vertical direction we get a relation for the vertical component of velocity in the form:

$$\begin{aligned} w = -N\nu_0 \frac{\partial}{\partial \sigma}\left[D \int_{\xi_0}^{\xi} u \; d\xi\right] - \frac{\partial}{\partial n}\left[D \int_{\xi_0}^{\xi} v \; d\xi\right] \\ -\nu_0 N \mathcal{C} D \left[\int_{\xi_0}^{\xi} v \; d\xi\right] - N\nu_0 U \left[(1-\xi) D,_{\sigma} - F_u^2 h,_{\sigma}\right] \\ -V\left[(1-\xi) D,_n - F_u^2 h,_n\right] \end{aligned} \tag{2.69}$$

Evaluating (2.69) at the free surface ($\xi = 1$) and using the kinematic boundary condition (2.65) we obtain the *depth-averaged form of the continuity equation for the liquid phase*:

$$N\nu_0 \frac{\partial}{\partial\sigma}\left[D\int_{\xi_0}^{1} u\; d\xi\right] + \frac{\partial}{\partial n}\left[D\int_{\xi_0}^{1} v\; d\xi\right] + \nu_0 N\mathcal{C}\left[D\int_{\xi_0}^{1} v\; d\xi\right] = 0 \tag{2.70}$$

Substituting from (2.69) and (2.55) into (2.56-2.57), dividing by $\beta_u\sqrt{C_{fu}}$ we finally derive an integro-differential system, which reads:

$$\begin{aligned} \delta N u u_{,\sigma} + \frac{1}{\beta_u\sqrt{C_{fu}}} v u_{,n} - \delta\frac{N u_{,\xi}}{D}\frac{\partial}{\partial\sigma}\left[D\int_{\xi_0}^{\xi} u\; d\xi\right] \\ -\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{u_{,\xi}}{D}\frac{\partial}{\partial n}\left[D\int_{\xi_0}^{\xi} v\; d\xi\right] - \delta\mathcal{C}N u_{,\xi}\int_{\xi_0}^{\xi} v\; d\xi \\ +\delta\mathcal{C}Nuv \\ = \\ -\delta N h_{,\sigma} + N\sqrt{C_{fu}} + \frac{1}{D^2}\left(\nu_T u_{,\xi}\right)_{,\xi} \end{aligned} \tag{2.72}$$

$$\begin{aligned} \delta N u v_{,\sigma} + \frac{1}{\beta_u\sqrt{C_{fu}}} v v_{,n} - \delta\frac{N v_{,\xi}}{D}\frac{\partial}{\partial\sigma}\left[D\int_{\xi_0}^{\xi} u\; d\xi\right] \\ -\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{v_{,\xi}}{D}\frac{\partial}{\partial n}\left[D\int_{\xi_0}^{\xi} v\; d\xi\right] - \delta\mathcal{C}N v_{,\xi}\int_{\xi_0}^{\xi} v\; d\xi \\ -\delta\mathcal{C}Nu^2 \\ = \\ -\frac{h_{,n}}{\beta_u\sqrt{C_{fu}}} + \frac{1}{D^2}\left(\nu_T v_{,\xi}\right)_{,\xi} \end{aligned} \tag{2.73}$$

$$\begin{aligned} \delta N\frac{\partial}{\partial\sigma}\left[D\int_{\xi_0}^{1} u\; d\xi\right] \\ +\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{\partial}{\partial n}\left[D\int_{\xi_0}^{1} v\; d\xi\right] + \delta N\mathcal{C}\left[D\int_{\xi_0}^{1} v\; d\xi\right] = 0 \end{aligned} \tag{2.74}$$

$$ \tag{2.75} $$

$$N\delta q_{\sigma},_{\sigma} + \frac{1}{\beta_u \sqrt{C_{fu}}} q_{n},_{n} + \delta N \mathcal{C} q_n = 0 \tag{2.76}$$

$$\delta N \frac{\partial}{\partial \sigma} \left[q_{b_\sigma} + Q_0 \; q_{s_\sigma} \right] + \frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n} \left[q_{b_n} + Q_0 \; q_{s_n} \right] + \delta N \mathcal{C}(s) \left[q_{b_n} + Q_0 \; q_{s_n} \right] = 0 \tag{2.77}$$

$$\begin{aligned} \frac{w(\xi)}{\beta_u \sqrt{C_{fu}}} = & -N\delta \frac{\partial}{\partial \sigma} \left(D \int_{\xi_0}^{\xi} u \; d\xi \right) \\ & -N\delta u \left[(1-\xi) \frac{\partial D}{\partial \xi} - F_r^2 \frac{\partial h}{\partial \sigma} \right] \\ & -\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n} \left(D \int_{\xi_0}^{\xi} v \; d\xi \right) \\ & -N\delta \mathcal{C}(s) \left(D \int_{\xi_0}^{\xi} v \; d\xi \right) \\ & -\frac{v}{\beta_u \sqrt{C_{fu}}} \left[(1-\xi) \frac{\partial D}{\partial n} - F_r^2 \frac{\partial h}{\partial n} \right] \end{aligned} \tag{2.78}$$

By using the expression found for w, the equation (2.78) can be simplified, as follows:

$$\begin{aligned} & N\delta u \frac{\partial C}{\partial \sigma} + \frac{1}{\beta_u \sqrt{C_{fu}}} v \frac{\partial C}{\partial n} - \frac{1}{\beta_u \sqrt{C_{fu}}} \frac{W_s}{D} \frac{\partial C}{\partial \xi} \\ & + \frac{1}{D} \frac{\partial C}{\partial \xi} \left[-N\delta \frac{\partial}{\partial \sigma} \left(D \int_{\xi_0}^{\xi} u \; d\xi \right) \right. \\ & \left. -\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n} \left(D \int_{\xi_0}^{\xi} v \; d\xi \right) - N\delta \mathcal{C}(s) \int_{\xi_0}^{\xi} v \; d\xi \right] = \\ & + \frac{1}{D^2} \frac{\partial}{\partial \xi} \left(D_T \frac{\partial C}{\partial \xi} \right) \end{aligned} \tag{2.79}$$

where δ is the small parameter:

$$\delta = \frac{\nu_0}{\beta_u \sqrt{C_{fu}}}$$

We may then expand the unknown functions in a neighborhood of the solution for uniform flow in a *straight channel with an unknown shape of the cross section and an unknown slope of the free surface*, the latter described by a slowly varying function $D_0(n, \sigma)$ of both the longitudinal and lateral coordinates and a slowly varying function $h_{00,\sigma}(\sigma)$ of the longitudinal coordinate only. Hence:

$$\begin{aligned}(u, v, w, D) = [u_0(\xi, n, \sigma), 0, 0, D_0(n, \sigma)] + \\ + \sum_{m=1}^{\infty} (u_m, v_m, w_m, D_m)(\delta)^m\end{aligned} \quad (2.80)$$

By analogy with (2.80), we expand concentration, as follows:

$$C = C_0(\xi, n, \sigma) + \sum_{m=1}^{\infty} (C_m)(\delta)^m \quad (2.81)$$

Note that, in order to account for the small variations of the longitudinal free surface slope associated with channel curvature, the free surface elevation must have distinct contributions, according to the following expansion:

$$h(\sigma, n) = \frac{h_{00}(\sigma)}{\delta} + \delta h_1(\sigma, n) + \sum_{m=1}^{\infty} \left[\left(\frac{h_{0m}}{\delta} + \delta h_{m+1}(\sigma, n) \right) \delta^m \right] \quad (2.82)$$

In chapter 2.3 we show in detail the expansion for solid discharge. We may now substitute from (2.80, 2.81) into (2.72 - 2.76) and equate likewise powers of δ, to obtain a sequence of differential problems, to be solved in terms of the unknown functions D and $h_{,\sigma}$.

2.3 Expansion for the solid discharge

Let us formulate the sediment continuity equation. Firstly, the closure relationships (2.48 - 2.50) for the bedload flux **q**, rewritten in terms of the rescaled coordinates, which allow us to write:

$$q_{b_\sigma} = \Phi \tag{2.83}$$

$$q_{b_n} = q_{b_\sigma}\left[\frac{\tau_n}{|\boldsymbol{\tau}|} - \frac{R\sqrt{\theta_u}}{\sqrt{\theta}}(F_u^2 h - D)_{,n}\right] \tag{2.84}$$

Similarity to (2.80- 2.82), we can expand bedload discharge components:

$$q_{b_n} = \delta q_{b_{n_1}} + \delta^2 q_{b_{n_2}} + O(\delta^3) \tag{2.85}$$

$$q_{b_\sigma} = q_{b_{\sigma_0}} + \delta q_{b_{\sigma_1}} + O(\delta^2) \tag{2.86}$$

Longitudinal component reads:

$$q_{b_\sigma} = \Phi_0 + \delta\Phi_1 \tag{2.87}$$

Expanding the equation (2.84) at $O(\delta)$ we find:

$$q_{b_{n_1}} = q_{\sigma_0}\left[\frac{v_{1,\xi}}{u_{0,\xi}}|_{\xi_0} + \frac{R' D_{0,n}}{\sqrt{D_0 R_0}}\right] \tag{2.88}$$

having set $\theta_0 = \theta_u D_o R_0$.

Expanding the quantity q_n (see the equation 2.109) and using (2.83) and (2.88), at $O(\delta^2)$ we obtain:

$$q_{b_{n_2}} = q_{b_{\sigma 0}}\left[\frac{v_{2,\xi}}{u_{0,\xi}}|_{\xi_0} - \frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\frac{q_{b_{n_1}}}{q_{b_{\sigma_0}}} - \frac{R'(D_{1,n} - F_u^2 h_{1,n})}{\sqrt{D_0 R_0}} - \frac{\Phi_1 q_{b_{n_1}}}{q^2_{b_{\sigma_0}}}\right] \tag{2.89}$$

having set $\frac{\theta_1}{\theta_0} = 2\frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}$.

Similarly we can expand the longitudinal component of suspended sediment transport in the form:

$$q_{s\sigma} = q_{s_{\sigma_0}} + \delta q_{s_{\sigma_1}} + O(\delta^2) \tag{2.90}$$

obtaining:

$$q_{s_{\sigma_0}} = D_0 \int_{\xi_R}^{1} u_0 C_0 \, d\xi \tag{2.91}$$

$$q_{s_{\sigma_1}} = \left[D_1 \int_{\xi_R}^{1} u_0 C_0 \, d\xi + D_0 \int_{\xi_R}^{1} u_1 C_0 \, d\xi + D_0 \int_{\xi_R}^{1} u_0 C_1 \, d\xi \right] \tag{2.92}$$

Similarly to (2.90), the transverse component of suspended sediment transport can be expanded in the form:

$$q_{s_n} = \delta q_{s_{n_1}} + \delta^2 q_{s_{n_2}} + O(\delta^3) \tag{2.93}$$

to obtain:

$$q_{s_{n_1}} = \left[D_0 \int_{\xi_R}^{1} v_1 C_0 \, d\xi \right] \tag{2.94}$$

$$q_{s_{n_2}} = \left[D_1 \int_{\xi_R}^{1} v_1 C_0 \, d\xi + D_0 \int_{\xi_R}^{1} v_2 C_0 \, d\xi + D_0 \int_{\xi_R}^{1} v_1 C_1 \, d\xi \right] \tag{2.95}$$

2.4 Expansion for equilibrium concentration

In the following we will employ closure relation for equilibrium concentration defined by Van Rijn:

$$C_e = 0.015\ \frac{d_s^*}{D^*\ \xi_R}\left(\frac{\theta' - \theta_{cr}}{\theta_{cr}}\right)\ R_p^{-0.2} \tag{2.96}$$

with:ξ_R is defined as follows:

$$\xi_R = \frac{a^*}{D^*} = \begin{cases} 0.01 & (\epsilon_e^* < 0.01\ D^*) \\ \frac{\epsilon_e^*}{D^*} & (\epsilon_e^* < 0.01\ D^*) \end{cases} \tag{2.97}$$

By analogy with the expansion (2.81), we write:

$$C_e = C_{e0} + \delta\ C_{e1} + O(\delta^2)\left(\frac{\theta' - \theta_{cr}}{\theta_{cr}}\right)\ R_p^{-0.2} \tag{2.98}$$

Hence, at minimum order we can easily get: Since equilibrium concentration is a function of θ and the flow depth D^*, we get:

$$C_e = C_{e0}(\theta_0, D_0) + \delta\left[C_{e_T}\frac{\theta_1}{\theta_0} + C_{e_D}\frac{D_1}{D_0}\right] + O(\delta^2) \tag{2.99}$$

where:

$$C_{e_T} = \frac{\theta_0}{C_e 0}\frac{\partial C_e}{\partial \theta}|_D = \frac{3}{2}\left(\frac{\theta_0}{\theta_0' - \theta_{cr}}\right)\frac{\partial \theta_0'}{\partial \theta_0} - \xi_{R_T} \tag{2.100}$$

$$C_{e_D} = \frac{1}{C_e 0}\frac{\partial C_e}{\partial D}|_\theta = -\frac{1}{D_0} - \xi_{R_D} \tag{2.101}$$

If $\epsilon_e^* < 0.01\ D^*$ then we find:

$$\xi_{R_T} = \frac{\theta_0}{\xi_{R_0}}\frac{\partial \xi_R}{\partial \theta}|_D = \frac{\theta_0}{\xi_{R_0}}\frac{1}{D^*}\frac{\partial \epsilon_e^*}{\theta}|_D = \frac{1}{5\sqrt{C_{fu}}}C_T \tag{2.102}$$

$$\xi_{R_D} = \frac{1}{\xi_{R_0}}\frac{\partial \xi_R}{\partial D}|_\theta = \left(-\frac{\epsilon_e^*}{D_0^*}\frac{1}{D^2} + \frac{1}{D} + \frac{\partial \epsilon_e^*}{\partial D}|_\theta \right)\frac{1}{\xi_{R_0}} = \tag{2.103}$$

$$\left[-\frac{\epsilon_e^*}{D_0^2} + \frac{1}{D}\left(\frac{\epsilon_e^*}{D} + C_D\ \frac{\epsilon_e^*}{5\sqrt{C_{fu}}} \right)\right]\frac{1}{\xi_{R_0}} = C_D\frac{1}{5\sqrt{C_{fu}}} \tag{2.104}$$

Finally we can find the first order expression for the concentration:

$$\begin{aligned} \frac{C_{e1}}{C_{e0}} &= C_{e_T}\frac{\theta_1}{\theta_0} + C_{e_D}\frac{D_1}{D_0} = \\ \left[\frac{3}{2}\left(\frac{\theta_0}{\theta_0' - \theta_{cr}}\right)\frac{\partial \theta_0'}{\partial \theta_0} - \xi_{R_T}\right]\frac{\theta_1}{\theta_0} &+ \left[-\frac{1}{D_0} - \xi_{R_D}\right]\frac{D_1}{D_0} = \\ \left[\frac{3}{2}\left(\frac{\theta_0}{\theta_0' - \theta_{cr}}\right)\frac{\partial \theta_0'}{\partial \theta_0} - \left(\frac{1}{5\sqrt{C_{fu}}}C_T\right)\right]&\frac{\theta_1}{\theta_0} \\ + \left[-\frac{1}{D_0} - \left(C_D\frac{1}{5\sqrt{C_{fu}}}\right)\right]&\frac{D_1}{D_0} \end{aligned} \tag{2.105}$$

2.5 Leading order solution: Exner equation

By substituting expansions (2.84-2.95) into the Exner equation, we find at leading order $O(\delta)$:

$$\frac{\partial}{\partial \sigma}\left[q_{b\sigma_0} + Q_0 q_{s\sigma_0}\right] + \frac{1}{\beta_u \sqrt{C_{fu}}}\frac{\partial}{\partial n}\left[q_{bn_1} + Q_0 q_{sn_1}\right] = 0 \qquad (2.106)$$

Integrating over the cross section, between -1 and n, we get:

$$\frac{\partial}{\partial \sigma}\int_{-1}^{n}\left[q_{b\sigma_0} + Q_0 q_{s\sigma_0}\right] dn + \frac{1}{\beta_u \sqrt{C_{fu}}}\frac{\partial}{\partial n}\left[q_{bn_1} + Q_0 q_{sn_1}\right]_{-1}^{n} = 0 \qquad (2.107)$$

By using the boundary condition, which imposes that the net sediment flux trough the sidewalls has to vanish, we find:

$$\frac{\partial}{\partial \sigma}\int_{-1}^{n}\left[q_{b\sigma_0} + Q_0 q_{s\sigma_0}\right] dn + \frac{1}{\beta_u \sqrt{C_{fu}}}\frac{\partial}{\partial n}\left[q_{bn_1} + Q_0 q_{sn_1}\right] = 0 \qquad (2.108)$$

The lateral component of the bedload flux q_n can also be written in the form:

$$q_{b_n} = q_\sigma \left[\frac{v_{,\xi}}{u_{,\xi}}|_{\xi_0} - \delta \frac{R'\sqrt{\theta_u}}{\sqrt{\theta}}(F_u^2 h - D)_{,n}\right] \qquad (2.109)$$

Here, we have introduced the $O(1)$ parameter $R' = \frac{R}{\delta}$: in fact, the slowly varying character of the lateral distribution of flow depth is due to the quantity R being $O(\delta)$.

Now, performing simple algebra, we find:

$$\frac{\beta_u\sqrt{C_{fu}}}{q_{\sigma_0}}\frac{\sqrt{D_0R_0}}{R'}\frac{\partial}{\partial\sigma}\left[\frac{v_{1,\xi}}{u_{0,\xi}}|_{\xi_0}+\frac{R'D_{0,n}}{\sqrt{D_0R_0}}\right]$$
$$+\frac{\sqrt{D_0R_0}}{R'}\frac{v_{1,\xi}}{u_{0,\xi}}|_{\xi_0}+\frac{\partial D_0}{\partial n}$$
$$+\frac{Q_0}{q_{\sigma_0}}\frac{\sqrt{D_0R_0}}{R'}D_0\int_{\xi_R}^{1}v_1C_0d\xi=0 \qquad (2.110)$$

Finally, we easily find a non linear partial integro-differential equation for the unknown function $D_0(\sigma,n)$:

$$\frac{\partial D_0}{\partial n}=-\frac{\sqrt{D_0R_0}}{R'}\left[\frac{v_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\right.$$
$$+\frac{\beta_u\sqrt{C_{fu}}}{q_{\sigma_0}}\frac{\partial}{\partial\sigma}\int_{-1}^{n}(q_{b_{\sigma_0}}+Q_0q_{s_{\sigma_0}})dn$$
$$\left.+\frac{Q_0}{q_{\sigma_0}}D_0\int_{\xi_R}^{1}v_1C_0d\xi\right] \qquad (2.111)$$

This equation has to be solved subject to the integral constraints (2.51) and (2.52) evaluated at the order $O(\delta^0)$:

$$\int_{-1}^{+1}D_0^{3/2}I_{F_0}\;dn=2(1-\xi_u) \qquad (2.112)$$

$$\int_{-1}^{+1}q_{b_{\sigma_0}}+Q_0q_{s_{\sigma_0}}\;dn=2\Phi_u+Q_0C_{e_u}\int\;F_0C_0\;d\xi \qquad (2.113)$$

These constraints can be reinforced by choosing appropriately the quantity $h_{00,\sigma}$ and the boundary condition for $D_0(n,\sigma)$ at one of the banks. For a complete description of the numerical procedure employed to solve the latter equation the reader is referred to (Nobile (2008)).

2.6 Leading order solution for longitudinal flow velocity

At the leading order of approximation $O(\delta^0)$, the longitudinal component of the Reynolds equations (2.72) reduces to a uniform balance between gravity and friction in a channel with unknown distribution of flow depth $D_0(n,\sigma)$ and free surface slope -$h_{00},_{\sigma}(\sigma)$ with relative boundary condition (2.63 - 2.64):

$$\begin{cases} \frac{1}{D_0^2}\left[\nu_{T0} u_0,_{\xi}\right],_{\xi} & = \quad h_{00},_{\sigma} - \sqrt{C_{fu}} \\ u_0,_{\xi}\mid_1 = 0 \\ u_0 \mid_{\xi_0} = 0 \end{cases} \tag{2.114}$$

After setting:

$$u_0 = D_0^{1/2}(n,\sigma) R_0^{1/2}(\sigma) F_0(\xi,n,\sigma) \tag{2.115}$$

$$\nu_{T0} = D_0^{3/2}(n,\sigma) R_0^{1/2}(\sigma) \Psi(\xi) \tag{2.116}$$

with $R_0 = 1 - h_{00,\sigma}/\sqrt{C_{fu}}$, one finds:

$$\begin{cases} \left[\mathcal{N}(\xi) F_{0,\xi}\right],_{\xi} & = \quad -\sqrt{C_{fu}} \\ F_0,_{\xi}\mid_1 = 0 \\ F_0 \mid_{\xi_0} = 0 \end{cases} \tag{2.117}$$

The solution for F_0 is the classical logarithmic distribution corrected by a wake function:

$$F_0(\xi) = \frac{\sqrt{C_{fu}}}{k}\left[\ln\frac{\xi}{\xi_0} + A(\xi^2 - \xi_0^2) + B(\xi^3 - \xi_0^3)\right] \tag{2.118}$$

where ξ_0 is the normalized conventional reference level, here assumed to be constant. Also note that the function F_0 satisfies the following integral constraint:

$$\int_{\xi_0}^{1} F_0 \, d\xi = 1 - \xi_0 \tag{2.119}$$

2.7 Leading order solution for sediment concentration

At the leading order of approximation $O(\delta^0)$, the convection - diffusion equation for sediment concentration can be rewritten as:

$$-\frac{1}{\beta_u\ \sqrt{C_{fu}}}\frac{W_s}{D_0}\frac{\partial C_0}{\partial \xi} = +\frac{1}{D_0^2}\frac{\partial}{\partial \xi}\left(D_{T_0}\frac{\partial C_0}{\partial \xi}\right) \tag{2.120}$$

That can be solved by making use of the following boundary condition:

$$\begin{cases} \beta_u\sqrt{C_{fu}}D_T\frac{\partial C}{\partial z} + W_s C = 0 \qquad (\xi = 1) \\ \beta_u\sqrt{C_{fu}}D_T\frac{\partial C}{\partial z} + W_s C_e = 0 \qquad (\xi = \xi_R) \end{cases} \tag{2.121}$$

Here we employ Reynold's analogy, therefore the closure for eddy diffusivity reads:

$$D_{T_0} = \nu_{T_0} = D_0^{3/2} R_0^{1/2} \Psi(\xi) \tag{2.122}$$

Integrating along the vertical coordinate ξ the equation (2.117), we get:

$$\frac{D_{T_O}}{D_0^2}\frac{\partial C_0}{\partial \xi} + \frac{1}{\beta_u\ \sqrt{C_{fu}}D_0} W_s C_0 = constant \tag{2.123}$$

Where, using the boundary condition on the free surface (2.121), we can find:

$$constant = 0$$

Now by separating the variables we integrate (2.123) once, in order to find:

$$\ln\left(C_0\right) = -\frac{W_s}{\beta_u \sqrt{C_{fu}}} D_0(n) \int_{\xi_R}^{\xi} \frac{1}{D_{T_0}} d\xi + c_2 \tag{2.124}$$

Substituting from (2.122) into the latter expression, we can write:

$$C_0 = \exp\left[-\frac{W_s}{\beta_u \sqrt{C_{fu}}} \frac{1}{(D_0 R_0)^1/2} \int_{\xi_R}^{\xi} \frac{1}{\Psi(\xi)} d\xi + c_2\right] \tag{2.125}$$

Moreover, by using the boundary condition ate the bottom, imposing:

$$C_0 = C_{e0} \qquad (\xi = \xi_R)$$

and performing simple algebra, we finally obtain the leading order solution for sediment concentration:

$$C_0 = C_{e0} \exp\left[-k Z_0 \int_{\xi_R}^{\xi} \frac{1}{\Psi(\xi)}\ d\xi\right] \tag{2.126}$$

With the Rouse number Z_0 is defined as follows:

$$Z_0 = \frac{W_s}{k \beta_u \sqrt{Cfu} (D_0 R_0)^{1/2}} \tag{2.127}$$

It will also be useful to write the sediment concentration in the form:

$$C_0 = C_{e0}\ f \tag{2.128}$$

with:

$$f = \exp\left[-k Z_0 \int_{\xi_R}^{\xi} \frac{1}{\Psi\left(\xi\right)} d\xi\right] \tag{2.129}$$

2.8 First order: secondary flow induced by curvature and longitudinal variations

At first order $O(\delta^1)$, the transverse component of the Reynolds equations (2.73) reduces to a balance between lateral component of gravity, centripetal inertia and lateral friction in a channel with unknown, yet slowly varying, distributions of flow depth $D_0(n,\sigma)$ and free surface slope (-$h_{00},_\sigma$) as well as given slowly varying distribution of channel curvature $C(\sigma)$. We find:

$$\begin{cases} \frac{1}{D_0^2}\left[\nu_{T0}v_{1,\xi}\right],_\xi &= \frac{h_{1,n}}{\beta_u\sqrt{C_{fu}}} - u_0^2\mathcal{C} \\ v_1|_{\xi_0} = 0 \\ v_{1,\xi}|_1 = 0 \end{cases} \tag{2.130}$$

We then set:

$$v_1 = D_0^{3/2}(n,\sigma)R_0^{1/2}(\sigma)G_1(\xi,n,\sigma)\mathcal{C}(\sigma) \tag{2.131}$$

$$\frac{\partial h_1}{\partial n} = \beta_u\sqrt{C_{fu}}D_0(n,\sigma)R_0(\sigma)\mathcal{C}(\sigma)a_1(n,\sigma) \tag{2.132}$$

where G_1 is the solution of the following ordinary differential problem:

$$\begin{cases} \left[\mathcal{N}(\xi)G_{1,\xi}\right],_\xi &= a_1(n,\sigma) - F_0^2(\xi) \\ G_1|_{\xi_0} = 0 \\ G_{1,\xi}|_1 = 0 \end{cases} \tag{2.133}$$

Let us write the solution for G_1 in the form:

$$G_1 = a_1(n,\sigma)G_{11}(\xi) + G_{12}(\xi) \tag{2.134}$$

where:

$$G_{1j} = g_j(\xi) - \frac{g'_j|_{\xi=1}}{g'_0|_{\xi=1}} g_0(\xi) \qquad (j = 1,2) \tag{2.135}$$

and g_j $(j = 0,1,2)$ are solutions of ordinary differential systems:

$$\begin{cases} [\mathcal{N}(\xi) g_{j,\ \xi}]_{,\xi} = \delta_i \\ g_j|_{\xi_0} = 0 \\ g_{j,\ \xi}|_{\xi_0} = 1 \end{cases} \tag{2.136}$$

where:

$$\delta_0 = 0 \qquad \delta_1 = 1 \qquad \delta_2 = -F_0^2(\xi,\sigma,n) \tag{2.137}$$

The solutions for the functions g_j are obtained in an analytical form, (Nobile (2008)) and depend only on the normalized conventional reference level ξ_0. Also note that a relation between the functions F_0 (2.117) and G_1 (2.133) can be easily found in the form:

$$F_0 = -\sqrt{C_{fu}}G_{11} \tag{2.138}$$

We may then proceed to determine the function $a_1(n,\sigma)$ firstly expressing the *depth-averaged form of the continuity equation for the liquid phase* (2.70) at $O(\delta)$:

$$\frac{\partial}{\partial\sigma}\left[D_0\int_{\xi_0}^{1}u_0\ d\xi\right]+\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{\partial}{\partial n}\left[D_0\int_{\xi_0}^{1}v_1\ d\xi\right]=0 \qquad (2.139)$$

and secondly integrating the latter over the cross section, with the use of the boundary condition (2.66) written at $O(\delta)$:

$$\int_{\xi_0}^{1}v_1\ d\xi=0 \qquad (n=\pm 1) \qquad (2.140)$$

to obtain:

$$\frac{\partial}{\partial\sigma}\int_{1}^{n}\left[D_0\int_{\xi_0}^{1}u_0\ d\xi\right]dn+\frac{1}{\beta_u\sqrt{C_{fu}}}\left[D_0\int_{\xi_0}^{1}v_1\ d\xi\right]=0 \qquad (2.141)$$

Substituting (2.115) and (2.131) into the latter equation we find:

$$a_1(n,\sigma)=a_{10}-\frac{a_{11}}{\mathcal{C}D_0^{5/2}R_0^{1/2}}\frac{\partial}{\partial\sigma}\left(\int_{-1}^{n}D_0^{3/2}R_0^{1/2}I_{F_0}dn\right) \qquad (2.142)$$

where:

$$a_{10}=-\frac{I_{G_{12}}}{I_{G_{11}}} \qquad a_{11}=\frac{\beta_u\sqrt{C_{fu}}}{I_{G_{11}}} \qquad (2.143)$$

and I_f is the integral $(\int_{\xi_0}^{1}f d\xi)$.
It's important to note that the coefficients (2.143) are analytical and depend on the normalized conventional reference level ξ_0 only.

Furthermore we observe that the secondary flow expressed in (2.131) can be split in two different contributions: the former v_{10} is correlated to the curvature of the channel axis and has a vanishing depth average, the latter v_{11} is due to the longitudinal variations of the flow field. This can be easily shown rearranging the equation (2.131) using the relation (2.142):

$$v_1 = v_{10} + v_{11} \tag{2.144}$$

$$v_{10} = D_0^{3/2} R_0^{1/2} \mathcal{C} \left(a_{10} G_{11} + G_{12} \right) \tag{2.145}$$

$$v_{11} = -\frac{\beta_u \sqrt{C_{fu}}}{D_0} F_0 \frac{\partial}{\partial \sigma} \left(\int_{-1}^{n} D_0 U_0 dn \right) \tag{2.146}$$

where:

$$U_0 = \frac{1}{1-\xi_0} \int_{\xi_0}^{1} u_0 \; d\xi = \frac{1}{1-\xi_0} I_{u_0} \tag{2.147}$$

is the depth average of the longitudinal velocity.
Let us then calculate the depth average of the transverse velocity:

$$V_1 = \frac{1}{1-\xi_0} \int_{\xi_0}^{1} v_1 d\xi = \overline{v_{10}} + \overline{v_{11}} \tag{2.148}$$

Using the equations (2.119), (2.143) and (2.148) we can readily show that:

$$\overline{v_{10}} = 0 \tag{2.149}$$

$$\overline{v_{11}} = -\frac{\beta_u \sqrt{C_{fu}}}{D_0} \frac{\partial}{\partial \sigma} \left(\int_{-1}^{n} D_0 U_0 dn \right) = V_1 \tag{2.150}$$

Hence, comparing (2.150) with (2.146) and using (2.144) we can write:

$$v_1(\sigma, n, \xi) = v_{10}(\sigma, n, \xi) + F_0(\xi) V_1(\sigma, n) \tag{2.151}$$

2.9 First order: correction of longitudinal motion due to convective effects

Our leading-order solution for the longitudinal velocity was simply a uniform flow slowly varying in the longitudinal and lateral directions and characterized by a free surface slope corrected with respect to the average intrinsic slope. However, because of the effects of curvature and flow variations, longitudinal momentum is transported outward close to the free surface and inward close to the bed. As a result, a perturbation of longitudinal velocity is then produced. In fact, by perturbing (2.41) we obtain:

$$\nu_T = \nu_{T0}\left[1 + \delta\left(\frac{D_1}{D_0} + \frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\right) + O(\delta^2)\right] \tag{2.152}$$

We refer to (Nobile (2008)) for a complete derivation of (2.152).
At first order $O(\delta^1)$, in the Reynolds equation (2.72) contributions due to lateral, vertical and longitudinal transport of momentum appear. Further effects are due to metric transverse variation of curvature, topographic effects, perturbation of flow depth and longitudinal free surface slope. After setting:

$$\frac{\nu_{T1}}{\nu_{T0}} = \frac{D_1}{D_0} + \frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0} \tag{2.153}$$

and using the longitudinal differential problem at previous order (2.114) we find:

$$\begin{cases} \frac{1}{D_0^2}\left[\nu_{T0}u_{1,\xi}\right]_{,\xi} & = & -\sqrt{C_{fu}}R_0\left(\frac{D_1}{D_0}-\frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\right)+n\mathcal{C}R_0\beta_u C_{fu}+h_{01,\sigma} \\ & & +u_0 u_{0,\sigma} \\ & & \frac{1}{\beta_u\sqrt{C_{fu}}}v_1 u_{0,n} \\ & & -\frac{u_{0,\xi}}{D_0}\frac{\partial}{\partial\sigma}\left[D_0\int_{-1}^{n}u_0\,dn\right] \\ & & -\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{u_{0,\xi}}{D_0}\frac{\partial}{\partial n}\left[D_0\int_{-1}^{n}v_1\,dn\right] \\ u_1|_{\xi_0}=0 \\ u_{1,\xi}|_1=0 \end{cases} \tag{2.154}$$

Hence, defining:

$$u_1 = D_0^{1/2}(n,\sigma)R_0^{1/2}(\sigma)F_1(\xi,n,\sigma) \tag{2.155}$$

and using equations (2.115), (2.116), (2.131), some algebra allows us to derive the problem for F_1, which reads:

$$\begin{cases} \left[\mathcal{N}(\xi)F_{1,\xi}\right]_{,\xi} & = & R_1 \\ & & +\frac{1}{2}R_2\ \left[F_0^2\right] \\ & & +\frac{1}{2}R_3\ \left[F_0\mathcal{C}G_1\right] \\ & & -\frac{3}{2}R_{2b}\ \left[F_{0,\xi}\int_{\xi_0}^{\xi}F_0\,d\xi\right] \\ & & -\frac{5}{2}R_3\ \left[F_{0,\xi}\int_{\xi_0}^{\xi}\mathcal{C}G_1\,d\xi\right] \\ F_1|_{\xi_0}=0 \\ F_{1,\xi}|_1=0 \end{cases} \tag{2.156}$$

where:

$$\begin{array}{rcl} R_1 & = & -\sqrt{C_{fu}} R_0 \left(\frac{D_1}{D_0} - \frac{u_{1,\xi}}{u_{0,\xi}} |_{\xi_0} \right) + n\mathcal{C} R_0 \beta_u C_{fu} + \frac{h_{01,\sigma}}{R_0} \\ R_2 & = & D_{0,\sigma} + \frac{D_0}{R_0} R_{0,\sigma} \\ R_{2b} & = & D_{0,\sigma} + \frac{D_0}{3R_0} R_{0,\sigma} \\ R_3 & = & \frac{1}{\beta_u \sqrt{C_{fu}}} D_0 D_{0,n} \end{array} \tag{2.157}$$

The solution for F_1 can be given the form:

$$F_1 = R_1 F_{11} - \frac{1}{2} R_2 F_{12} - \frac{1}{2} R_3 F_{13} - \frac{3}{2} R_{2b} F_{14} - \frac{5}{2} R_3 F_{15} \tag{2.158}$$

where:

$$F_{1j} = f_j(\xi) - \frac{f'_j|_{\xi=1}}{f'_0|_{\xi=1}} f_0(\xi) \qquad (j = 1, 5) \tag{2.159}$$

and f_j $(j = 0, 1, 2, 3, 4, 5)$ are solutions of the following ordinary differential systems:

$$\left\{ \begin{array}{lcl} [\mathcal{N}(\xi) f_{j,\ \xi}]_{,\xi} & = & \delta_i \\ f_j|_{\xi_0} = 0 & & \\ f_{j,\ \xi}|_{\xi_0} = 1 & & \end{array} \right. \tag{2.160}$$

where:

$$\begin{array}{lcl} \delta_0 & = & 0 \\ \delta_1 & = & 1 \\ \delta_2 & = & -F_0^2 \\ \delta_3 & = & -F_0 \mathcal{C} G_1 \\ \delta_4 & = & F_{0,\xi} \int_{\xi_0}^{\xi} F_0 \; d\xi \\ \delta_5 & = & F_{0,\xi} \int_{\xi_0}^{\xi} \mathcal{C} G_1 \; d\xi \end{array} \tag{2.161}$$

We note from (2.136) that $f_0 = g_0$, $f_1 = g_1$, $f_2 = g_2$.
It can be easily shown that previous forcing terms come from the following contributions due to convective effects and coordinate transformation (refer to 2.72 and 2.73). In particular:

$$\begin{array}{lcl} -F_0^2 & \rightarrow & N u u_{,\sigma} \\ -F_0 \mathcal{C} G_1 & \rightarrow & v u_{,n} \\ F_{0,\xi} \int_{\xi_0}^{\xi} F_0 \; d\xi & \rightarrow & \frac{N u_{,\xi}}{D} \frac{\partial}{\partial \sigma} \left[D \int_{\xi_0}^{\xi} u \; d\xi \right] \\ F_{0,\xi} \int_{\xi_0}^{\xi} \mathcal{C} G_1 \; d\xi & \rightarrow & \frac{u_{,\xi}}{D} \frac{\partial}{\partial n} \left[D \int_{\xi_0}^{\xi} v \; d\xi \right] \end{array} \tag{2.162}$$

Functions F_{1j} are expressed as combinations of simpler contributions depending only on the normalized conventional reference level ξ_0, (Nobile (2008)). To evaluate the coefficient R_1 a closure relationship for $\frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}$ is needed. The latter can be readily obtained starting from (2.115) and (2.155):

$$\frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0} = \frac{F_{1,\xi}}{F_{0,\xi}}|_{\xi_0} \tag{2.163}$$

Then, using the relation (2.158) and solving (2.163) for $\frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}$ we find:

$$\frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0} = \frac{1}{1-\sqrt{C_{fu}}\frac{F_{11,\xi}}{F_{0,\xi}}|_{\xi_0}}\Bigg[\left(-\sqrt{C_{fu}}\frac{D_1}{D_0}+n\mathcal{C}\beta_u C_{fu}+\frac{h_{01,\sigma}}{R_0}\right)\frac{F_{11,\xi}}{F_{0,\xi}}|_{\xi_0} \\ -\frac{1}{2}R_2\frac{F_{12,\xi}}{F_{0,\xi}}|_{\xi_0}-\frac{1}{2}R_3\frac{F_{13,\xi}}{F_{0,\xi}}|_{\xi_0} \\ -\frac{3}{2}R_{2b}\frac{F_{14,\xi}}{F_{0,\xi}}|_{\xi_0}-\frac{5}{2}R_3\frac{F_{15,\xi}}{F_{0,\xi}}|_{\xi_0}\Bigg] \tag{2.164}$$

2.10 First order solution for suspended sediment concentration

Convection - diffusion equation for sediment concentration at first order takes the form:

$$\frac{1}{D_0^2}\frac{\partial}{\partial\xi}\left(D_{T_0}\frac{\partial C_1}{\partial\xi}\right)+\frac{1}{D_0^2}\frac{\partial}{\partial\xi}\left(D_{T_1}\frac{\partial C_0}{\partial\xi}\right)-2\frac{D_1}{D_0^3}\frac{\partial}{\partial\xi}\left(D_{T_0}\frac{\partial C_0}{\partial\xi}\right)=$$
$$+u_0\frac{\partial C_0}{\partial\sigma}+\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{W_s}{D_0}\frac{\partial C_1}{\partial\xi}+\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{D_1}{D_0^2}W_s\frac{\partial C_0}{\partial\xi}$$
$$+\frac{1}{D_0}\frac{\partial C_0}{\partial\xi}\left[-\frac{\partial}{\partial\sigma}\left(D_0\int_{\xi_0}^{\xi}u_0\ d\xi\right)\right.$$
$$-\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{\partial}{\partial n}\left(D_0\int_{\xi_0}^{\xi}v_1\ d\xi\right)$$
$$\left.-\mathcal{C}(s)\left(D_0\int_{\xi_0}^{\xi}v_1\ d\xi\right)\right] \tag{2.165}$$

It is now useful to substitute the problem for concentration found at minimum order, and obtain:

$$\frac{1}{kZ_0}\frac{\partial}{\partial\xi}\left[\Psi(\xi)\frac{\partial C_1}{\partial\xi}\right]+\frac{\partial C_1}{\partial\xi}=$$
$$+\frac{1}{kZ_0}D_0F_0C_0\left(\frac{1}{C_{e0}}\frac{\partial C_{e0}}{\partial\sigma}+\frac{1}{f}\frac{\partial f}{\partial\sigma}\right)$$
$$-\frac{C_{e0}f}{\Psi(\xi)\left(D_0R_0\right)^{1/2}}\left[\frac{\partial}{\partial\sigma}\left(D_0^{3/2}R_0^{1/2}\int_{\xi_0}^{\xi}F_0\ d\xi\right)+\frac{1}{\beta_u\sqrt{C_{fu}}}\right] \tag{2.166}$$

Hence, C_1 can assume the general form:

$$\begin{aligned} C_1 = & \\ & +\mu_1 C_{101} \\ & +\mu_2 C_{102} \\ & +\frac{1}{kZ_0} D_0 \frac{\partial C_{e0}}{\partial \sigma} C_{110}(\xi, n, \sigma) \\ & +\frac{1}{kZ_0} D_0 C_{e0} C_{120}(\xi, n, \sigma) \\ & +\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{1}{kZ_0} D_0^2 \frac{\partial C_{e0}}{\partial n} C_{130}(\xi, n, \sigma) \\ & +\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{1}{kZ_0} D_0^2 C_{e0} C_{140}(\xi, n, \sigma) \\ & +\frac{C_{e0}}{(D_0 R_0)^2} \frac{\partial}{\partial \sigma} \left(D_0^{3/2} R_0^{1/2} \right) C_{150}(\xi, n, \sigma) \\ & +\frac{C_{e0}}{(D_0 R_0)^2} \frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n} \left(D_0^{5/2} R_0^{1/2} \right) C_{160}(\xi, n, \sigma) \\ & +\frac{C_{e0}}{(D_0 R_0)^2} C D_0^{5/2} R_0^{1/2} C_{170}(\xi, n, \sigma) \\ & -kZ_0 C_{e0} \left[\frac{D_1}{D_0} + \frac{u_{1,\xi}}{u_{0,\xi}} |_{\xi_0} \right] C_{180}(\xi, n, \sigma) \end{aligned} \tag{2.167}$$

where μ_1 e μ_2 are two coefficients to be determined by making use of the boundary conditions at the free surface and at the bottom. Moreover, the functions C_{1j} are the solutions of the ordinary problem:

$$\begin{cases} \frac{1}{kZ_0} \frac{\partial}{\partial \xi} \left[\Psi(\xi) \frac{\partial C_{1j}}{\partial \xi} \right] + \frac{\partial C_{1j}}{\partial \xi} = a_j \\ \frac{\partial C_{1j}}{\partial \xi} = b_j \quad (\xi = \xi_R) \\ C_{1j} = 1 \quad (\xi = \xi_R) \end{cases} \tag{2.168}$$

with:

$$a_{101} = 0 \qquad b_{101} = 1 \tag{2.169}$$

$$a_{102} = 0 \qquad b_{101} = 0 \tag{2.170}$$

$$a_{110} = fF_0 \qquad b_{101} = 0 \tag{2.171}$$

$$a_{120} = F_0 \frac{\partial f}{\partial \sigma} \qquad b_{101} = 0 \tag{2.172}$$

$$a_{130} = \mathcal{C}(s) G_1 f \qquad b_{101} = 0 \tag{2.173}$$

$$a_{140} = \mathcal{C}(s) G_1 \frac{\partial f}{\partial n} \qquad b_{101} = 0 \tag{2.174}$$

$$a_{150} = \frac{f}{N(\xi)} \int_{\xi_0}^{\xi} \frac{1}{F_0} d\xi \qquad b_{101} = 0 \tag{2.175}$$

$$a_{160} = \frac{f}{N(\xi)} \int_{\xi_0}^{\xi} \mathcal{C}(s) G_1 d\xi \qquad b_{101} = 0 \tag{2.176}$$

$$a_{170} = \frac{f}{N(\xi)} \int_{\xi_0}^{\xi} \mathcal{C}(s) G_1 d\xi \qquad b_{101} = 0 \tag{2.177}$$

$$a_{180} = \frac{f}{N(\xi)} \qquad b_{101} = 0 \tag{2.178}$$

In order to find the solution for C_1 we make use of the boundary condition at the free surface, and define the operator:

$$Bf = \frac{1}{kZ_0} \Psi(\xi) \frac{\partial f}{\partial \xi} + f$$

Substituting the operator in (2.167), we can write:

$$
\begin{aligned}
&\mu_1 BC_{101}|_1 \\
&+\mu_2 BC_{102}|_1 \\
&+\frac{1}{kZ_0} D_0 \frac{\partial C_{e0}}{\partial \sigma} BC_{120}|_1 \\
&+\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{1}{kZ_0} D_0^2 \frac{\partial C_{e0}}{\partial n} BC_{130}|_1 \\
&+\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{1}{kZ_0} D_0^2 C_{e0} BC_{140}|_1 \\
&+\frac{C_{e0}}{(D_0 R_0)^{1/2}} \frac{\partial}{\partial \sigma} \left(D_0^{3/2} R_0^{1/2} \right) BC_{150}|_1 \\
&+\frac{C_{e0}}{(D_0 R_0)^{1/2}} \frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n} \left(D_0^{5/2} R_0^{1/2} \right) BC_{160}|_1 \\
&+\frac{C_{e0}}{(D_0 R_0)^{1/2}} \mathcal{C}(s) D_0^{5/2} R_0^{1/2} BC_{170}|_1 \\
&+kZ_0 C_{e0} \left(\frac{D_1}{D_0} + \frac{u_{1,\xi}}{u_{0,\xi}} |\xi_0 \right) BC_{180}|_1 \\
&= \\
&C_0 \frac{u_{1,\xi}}{u_{0,\xi}} |\xi_0 \qquad (\xi = 1)
\end{aligned}
\tag{2.179}
$$

Hence:

$$\mu_1 = \frac{kZ_0}{N(\xi)}\left[\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0 - C_{e1}\right] \tag{2.180}$$

$$\begin{aligned} &\mu_2 = \\ &\frac{1}{BC_{102}|_1}\Bigg[C_0\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0 - \frac{kZ_0}{N(\xi)}\left(\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0 - C_{e1}\right)BC_{101}|_1 \\ &-\frac{1}{kZ_0}D_0\frac{\partial C_{e0}}{\partial\sigma}BC_{110}|_1 - \frac{1}{kZ_0}D_0C_{e0}BC_{120}|_1 \\ &-\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{1}{kZ_0}D_0^2\frac{\partial C_{e0}}{\partial n}BC_{130}|_1 \\ &-\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{1}{kZ_0}D_0^2C_{e0}BC_{140}|_1 \\ &-\frac{C_{e0}}{(D_0R_0)^{1/2}}\frac{\partial}{\partial\sigma}\left(D_0^{3/2}R_0^{1/2}\right)BC_{150}|_1 \\ &+\frac{C_{e0}}{(D_0R_0)^{1/2}}\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{\partial}{\partial n}\left(D_0^{5/2}R_0^{1/2}\right)BC_{160}|_1 \\ &-\frac{C_{e0}}{(D_0R_0)^{1/2}}\mathcal{C}(s)D_0^{5/2}R_0^{1/2}BC_{170}|_1 \\ &-KZ_0C_{e0}\left(\frac{D_1}{D_0}\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0\right)BC_{180}|_1\Bigg] \end{aligned} \tag{2.181}$$

Finally, substituting from (2.180) and (2.181) into (2.167), we can find the solution for sediment flux concentration:

$$
\begin{aligned}
C_1 = \\
\frac{C_{102}}{BC_{102}|_1}\left[C_0\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0\right] \\
+\frac{kZ_0}{N(\xi)}\left[\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0 - C_{e1}\right]\left[1 - C_{102}\frac{BC_{101}|_1}{BC_{102}|_1}\right] \\
+\frac{1}{kZ_0}D_0\frac{\partial C_{e0}}{\partial\sigma}\left[C_{110} - C_{102}\frac{BC_{110}|_1}{BC_{102}|_1}\right] \\
+\frac{1}{kZ_0}D_0C_{e0}\left[C_{120} - C_{102}\frac{BC_{120}|_1}{BC_{102}|_1}\right] \\
+\frac{1}{kZ_0}\frac{1}{\beta_u\sqrt{C_{fu}}}D_0^2\frac{\partial C_{e0}}{\partial n}\left[C_{130} - C_{102}\frac{BC_{130}|_1}{BC_{102}|_1}\right] \\
+\frac{1}{kZ_0}\frac{1}{\beta_u\sqrt{C_{fu}}}D_0^2C_{e0}\left[C_{140} - C_{102}\frac{BC_{140}|_1}{BC_{102}|_1}\right] \\
+\frac{C_{e0}}{(D_0R_0)^{1/2}}\frac{\partial}{\partial\sigma}\left(D_0^{3/2}R_0^{1/2}\right)\left[C_{150} - C_{102}\frac{BC_{150}|_1}{BC_{102}|_1}\right] \\
+\frac{C_{e0}}{(D_0R_0)^{1/2}}\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{\partial}{\partial n}\left(D_0^{5/2}R_0^{1/2}\right)\left[C_{160} - C_{102}\frac{BC_{160}|_1}{BC_{102}|_1}\right] \\
+\frac{C_{e0}}{(D_0R_0)^{1/2}}\mathcal{C}(s)D_0^{5/2}R_0^{1/2}\left[C_{170} - C_{102}\frac{BC_{170}|_1}{BC_{102}|_1}\right] \\
-kZ_0C_{e0}\left(\frac{D_1}{D_0}\frac{u_{1,\xi}}{u_{0,\xi}}|\xi_0\right)\left[C_{180} - C_{102}\frac{BC_{180}|_1}{BC_{102}|_1}\right]
\end{aligned}
\tag{2.182}
$$

Solution for $\xi < 0.314$

At first order of approximation O (δ^1), concentration consist of the solution of a second order differential problem. McTigue's distribution nature suggests to find two separate solutions for C_{1j}, one for each of the defined ranges. For value of ξ between ξ_R and 0.314, concentration can be obtained by solving the following differential system (2.168): Integrating the equation in the vertical direction we get:

$$\frac{\partial C_{1j}}{\partial \xi} = -\frac{kZ_0}{\Psi(\xi)} C_{1j} + \frac{kZ_0}{\Psi(\xi)} \int_{\xi_R}^{\xi} a_j d\xi + \frac{kZ_0}{\Psi(\xi)} \left[\frac{1}{kZ_0} \Psi(\xi_R)\, b_j + 1 \right] \tag{2.183}$$

Employing Lagrange's method of variation of parameters, Abramovitz (1964), Boyce & DiPrima (1965), one finds:

$$C_{1j}(\xi > 0.314) = \exp \int_{\xi_R}^{\xi} a(x)dx$$
$$\left[constant + \int_{\xi_R}^{\xi} b(x) \exp\left(- \int_{\xi_R}^{x} a(s)ds \right) dx \right] \tag{2.184}$$

where:

$$constant = C_{1j}|_{\xi_R} = 1 \tag{2.185}$$

$$a(x) = -\frac{kZ_0}{\Psi(\xi)} \tag{2.186}$$

$$b(x) = \frac{kZ_0}{\Psi(\xi)} \int_{\xi_R}^{\xi} a_j d\xi + \frac{\Psi(\xi_R)}{\Psi(\xi)}\, b_j + \frac{kZ_0}{\Psi(\xi)} \tag{2.187}$$

It is useful to note that is possible to write:

$$\exp \int_{\xi_R}^{\xi} a(x)\, dx = \left(\frac{\xi_R}{\xi} \right)^{k_1} \tag{2.188}$$

where:

$$k_1 = \frac{kZ_0}{0.35} \tag{2.189}$$

Moreover, the function f defined at (2.129), assumes the following simple form:

$$f = \left(\frac{\xi_R}{\xi}\right)^{k_1} \tag{2.190}$$

We may then proceed to determine each of the function C_{1j}:

Solution for C_{101}, valid in ($\xi_R \leq \xi < 0.314$):

$$C_{101} = 1 + \frac{\xi_R}{k_1}(1 - f) \tag{2.191}$$

Solution for C_{102}, valid in ($\xi_R \leq \xi < 0.314$):

$$C_{102} = 1 \tag{2.192}$$

Solution for C_{110}, valid in ($\xi_R \leq \xi < 0.314$):

$$C_{110} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{0.314\sqrt{C_{fu}}Z_0}{0.11} \sum_{i=1}^{5} (IBi - IBi|_{\xi_R})\right] + 1 \tag{2.193}$$

Solution for C_{120}, valid in ($\xi_R \leq \xi < 0.314$):

$$C_{120} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{k_1\sqrt{C_{fu}}}{0.35} \sum_{i=9}^{17} (IOi - IOi|_{\xi_R})\right] + 1 \tag{2.194}$$

Solution for C_{130}, valid in $(\xi_R \leq \xi < 0.314)$:

$$C_{130} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{-\mathcal{C}(s)a_1k_1}{k}\sum_{i=1}^{5}(IBi - IBi|_{\xi_R}) + \mathcal{C}(s)k_1II05\right] + 1 \quad (2.195)$$

Solution for C_{140}, valid in $(\xi_R \leq \xi < 0.314)$:

$$C_{140} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[-\mathcal{C}(s)a_1\sum_{i=9}^{17}(IOi - IOi|_{\xi_R}) + \mathcal{C}(s)kII14\right] + 1 \quad (2.196)$$

Solution for C_{150}, valid in $(\xi_R \leq \xi < 0.314)$:

$$C_{150} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{k_1\sqrt{C_{fu}}}{0.35k}\sum_{i=6}^{11}(ICi - ICi|_{\xi_R})\right] + 1 \quad (2.197)$$

Solution for C_{160}, valid in $(\xi_R \leq \xi < 0.314)$:

$$C_{160} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{-\mathcal{C}(s)a_1k_1}{0.35k}\sqrt{C_{fu}}0.35\sum_{i=6}^{11}(ICi - ICi|_{\xi_R}) + \frac{\mathcal{C}(s)k_1}{0.35}III08\right] + 1 \quad (2.198)$$

Solution for C_{170}, valid in $(\xi_R \leq \xi < 0.314)$:

$$C_{170} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{-\mathcal{C}(s) a_1 k_1}{0.35k}\sqrt{C_{fu}} 0.35 \sum_{i=6}^{11} (ICi - ICi|_{\xi_R}) + \frac{\mathcal{C}(s) k_1}{0.35} III08\right] + 1 \quad (2.199)$$

Solution for C_{180}, valid in $(\xi_R \leq \xi < 0.314)$:

$$C_{180} = \left(\frac{\xi_R}{\xi}\right)^{k_1} \left[\frac{\log \xi_R - \log \xi}{0.35} - \frac{1}{kZ_0}\right] + \frac{1}{kZ_0} + 1 \quad (2.200)$$

The solution for the integral coefficients IC_i, IB_i, IO_i, $II05$, $II14$, $III08$, appearing in the expression for C_{1j} are reported in Appendix A.

Solution for $\xi > 0.314$

For value of ξ exceeding 0.314, the vertical distribution of sediment concentration can be obtained by solving the following ordinary problem:

$$\begin{cases} \frac{1}{kZ_0}\frac{\partial}{\partial\xi}\left[\Psi\left(\xi\right)\frac{\partial C_{1j}}{\partial\xi}\right]+\frac{\partial C_{1j}}{\partial\xi}=a_j \\ \frac{\partial C_{1j}}{\partial\xi}=\widehat{b}_j \qquad (\xi=0.314) \\ C_{1j}=C_{1j}|_{0.314} \qquad (\xi=0.314) \end{cases} \tag{2.201}$$

Integrating the equation in the vertical direction we get:

$$\frac{\partial C_{1j}}{\partial\xi}=-\frac{kZ_0}{\Psi\left(\xi\right)}C_{1j}+\frac{kZ_0}{\Psi\left(\xi\right)}\int_{0.314}^{\xi}a_j d\xi + \frac{kZ_0}{\Psi\left(\xi\right)}\left[\frac{1}{kZ_0}\Psi\left(0.314\right)\widehat{b}_j+C_{1j}|_{0.314}\right] \tag{2.202}$$

Employing Lagrange's method of variation of parameters, one finds:

$$C_{1j}(\xi\leq 0.314)=\exp\int_{0.314}^{\xi}a(x)dx \tag{2.203}$$

$$\left[constant+\int_{0.314}^{\xi}b(x)\exp\left(-\int_{0.314}^{x}a(s)ds\right)dx\right] \tag{2.204}$$

where:

$$constant=C_{1j}|_{0.314} \tag{2.205}$$

$$a\left(x\right)=-\frac{kZ_0}{\Psi\left(\xi\right)} \tag{2.206}$$

$$b\left(x\right)=k_2\int_{\xi_R}^{0.314}a_j d\xi+b_j+k_2C_{1j}|_{0.314} \tag{2.207}$$

$$k_2 = \frac{kZ_0}{0.11} \tag{2.208}$$

$$\widehat{b}_j = k_2 \left[\int_{0.314}^{\xi} a_j d\xi - C_{1j}|_{0.314} + \frac{\xi_R}{k_1} \widehat{b}_j + 1 \right] \tag{2.209}$$

It is useful to note that is possible to write:

$$\exp \int_{\xi_R}^{\xi} a\left(x\right) dx = \exp\left[-k_2\left(\xi - 0.314\right)\right] \tag{2.210}$$

Moreover, the function f defined at (2.129), can be written as follows:

$$f = \left(\frac{0.314}{\xi}\right)^{k_1} \exp[-k_2\left(\xi - 0.314\right)] \tag{2.211}$$

In order to simplify the following expression, we define:

$$f_{01} = \exp\left[\int_{0.314}^{\xi} -s d\xi\right] = \exp\left[-s(\xi - 0.314)\right] \tag{2.212}$$

$$f_{02} = \exp\left[\int_{0.314}^{\xi} s d\xi\right] = \exp\left[s(\xi - 0.314)\right] \tag{2.213}$$

$$f_{03} = \exp\left[-s(0.314)\right] \tag{2.214}$$

$$I2aj = \int_{\xi_R}^{0.314} a_j \; d\xi \tag{2.215}$$

These parameters satisfy the following property:

$$f_{01} f_{02} = 1$$

We may then proceed to determine each of the function C_{1j}:

Solution for C_{101}, valid in $(0.314 \leq \xi < 1)$:

$$C_{101} = f_{01} \left[C_{101}|_{0.314} + \left(\frac{\xi_R}{k_1} + 1 \right) (f_{02} - 1) \right] \tag{2.216}$$

Solution for C_{102}, valid in $(0.314 \leq \xi < 1)$:

$$C_{102} = 1 \tag{2.217}$$

Solution for C_{110}, valid in $(0.314 \leq \xi < 1)$:

$$C_{110} = f_{01} \Bigg[C_{110}|_{0.314} + tt_{001} \exp(-0.314 k_2) k_2 \sum_{i=5}^{9} (ITi - ITi|_{0.314}) + (I2aj + 1)(f_{02} - 1) \Bigg] \tag{2.218}$$

Solution for C_{120}, valid in $(0.314 \leq \xi < 1)$:

$$C_{120} = f_{01} \Bigg[C_{120}|_{0.314} + tt_{007} \exp(-0.314 k_2) k_2 \sum_{i=9}^{17} (IRi - IRi|_{0.314}) + (I2aj + 1)(f_{02} - 1) \Bigg] \tag{2.219}$$

Solution for C_{130}, valid in $(0.314 \leq \xi < 1)$:

$$C_{130} = f_{01}\Bigg[C_{130}|_{0.314}$$

$$+tt_{010}\exp(-0.314k_2)k_2\Bigg(-\frac{\mathcal{C}(s)a_1}{k}\sum_{i=5}^{9}(ITi - ITi|_{0.314}) + \mathcal{C}(s)II02\Bigg)$$

$$+(I2aj+1)(f_{02}-1)\Bigg] \quad (2.220)$$

Solution for C_{140}, valid in $(0.314 \leq \xi < 1)$:

$$C_{140} = f_{01}\Bigg[C_{140}|_{0.314} +$$

$$+tt_{017}\exp(-0.314k_2)k_2\Bigg(-\mathcal{C}(s)a_1\sum_{i=9}^{17}(IRi - IRi|_{0.314}) + \mathcal{C}(s)kII16\Bigg)$$

$$+(I2aj+1)(f_{02}-1)\Bigg] \quad (2.221)$$

Solution for C_{150}, valid in $(0.314 \leq \xi < 1)$:

$$C_{150} = f_{01}\Bigg[C_{150}|_{0.314} +$$

$$tt_{011}\exp(-0.314k_2)k_2\sum_{i=6}^{11}(ILi - ILi|_{0.314})$$

$$+(I2aj+1)(f_{02}-1)\Bigg] \quad (2.222)$$

Solution for C_{160}, valid in $(0.314 \leq \xi < 1)$:

$$C_{160} = f_{01} \Bigg[C_{160}|_{0.314}$$
$$+tt_{015} \exp(-0.314 k_2) k_2 \Bigg(-\frac{\mathcal{C}(s) a_1}{k} \sum_{i=6}^{11} (ILi - ILi|_{0.314}) + \mathcal{C}(s) III12 \Bigg)$$
$$+(I2aj + 1)(f_{02} - 1) \Bigg] \quad (2.223)$$

Solution for C_{170}, valid in $(0.314 \leq \xi < 1)$:

$$C_{170} = f_{01} \Bigg[C_{170}|_{0.314}$$
$$+tt_{015} \exp(-0.314 k_2) k_2 \Bigg(-\frac{\mathcal{C}(s) a_1}{k} \sum_{i=6}^{11} (ILi - ILi|_{0.314}) + \mathcal{C}(s) III12 \Bigg)$$
$$+(I2aj + 1)(f_{02} - 1) \Bigg] \quad (2.224)$$

Solution for C_{180}, valid in $(0.314 \leq \xi < 1)$:

$$C_{180} = f_{01} \Bigg[C_{180}|_{0.314} +$$
$$\left(\frac{1}{kZ_0} \left(\frac{\xi_R}{0.314} \right)^{k_1} + I2aj + 1 \right) (f_{02} - 1) \quad (2.225)$$
$$+ \frac{0.314 - \xi}{0.11} \left(\frac{\xi_R}{0.314} \right)^{k_1} \Bigg] \quad (2.226)$$

The solution for the integral coefficients IT_i, IRi, ILi, $II05$, $II14$, $III08$, appearing in the expression for C_{1j} are reported in Appendix A.

2.11 Second order: correction of secondary flow due to convective effects

Many second-order effects arise in the equation governing the secondary flow. Indeed the $O(\delta^1)$ correction of longitudinal velocity affects the centrifugal term and further contributions are due to the lateral variations of the lateral component of momentum, to topographic effects and perturbations of the eddy viscosity forced by perturbations of flow depth and longitudinal velocity. The lateral component of Reynolds' equation at order $O(\delta^2)$, using the differential problem at previous order (2.130) and the perturbation of the eddy viscosity (2.153), is readily written in the form:

$$\left\{\begin{array}{lcl}
\frac{1}{D_0^2}\left[\nu_{T0} v_{2,\xi}\right],_\xi & = & \left(\frac{h_{1,n}}{\beta_u\sqrt{C_{fu}}} - \mathcal{C}u_0^2\right)\left(\frac{D_1}{D_0} - \frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\right) \\
 & & +\beta_u\sqrt{C_{fu}}n\mathcal{C}^2u_0^2 + \frac{h_{2,n}}{\beta_u\sqrt{C_{fu}}} \\
 & & +u_0 v_{1,\sigma} \\
 & & \frac{1}{\beta_u\sqrt{C_{fu}}} v_1 v_{1,n} \\
 & & -\frac{v_{1,\xi}}{D_0}\frac{\partial}{\partial\sigma}\left[D_0\int_{-1}^n u_0\ dn\right] \\
 & & -\frac{1}{\beta_u\sqrt{C_{fu}}}\frac{v_{1,\xi}}{D_0}\frac{\partial}{\partial n}\left[D_0\int_{-1}^n v_1\ dn\right] \\
 & & -2\mathcal{C}u_0u_1 \\
v_2|_{\xi_0} = 0 & & \\
v_{2,\xi}|_1 = 0 & &
\end{array}\right. \tag{2.227}$$

Hence, defining:

$$v_2 = D_0^{3/2}(n,\sigma)R_0^{1/2}(\sigma)G_2(\xi,n,\sigma)\mathcal{C}(\sigma) \tag{2.228}$$

$$\frac{\partial h_2}{\partial n} = \beta_u \sqrt{C_{fu}} D_0(n,\sigma) R_0(\sigma) \mathcal{C}(\sigma) a_2(n,\sigma) \tag{2.229}$$

and using equations (2.115), (2.116), (2.131), (2.132), (2.153), (2.155), some algebra allows us to derive the differential system for G_2:

$$\begin{cases} \left[\mathcal{N}(\xi) G_{2,\xi}\right]_{,\xi} = & a_2 + R_5\ a_1 \\ & +R_{4b}\ \left[F_0^2\right] \\ & \frac{R_{6b}}{\mathcal{C}}\ \left[F_0 \mathcal{C} G_1\right] \\ & +\frac{3}{2}\frac{R_3}{\mathcal{C}}\ \left[(\mathcal{C}G_1)^2\right] \\ & -2\ \left[F_0 F_1\right] \\ & -\frac{3}{2}\frac{R_{2b}}{\mathcal{C}}\ \left[\mathcal{C}G_{1,\xi} \int_{\xi_0}^{\xi} F_0\ d\xi\right] \\ & -\frac{5}{2}\frac{R_3}{\mathcal{C}}\ \left[\mathcal{C}G_{1,\xi} \int_{\xi_0}^{\xi} \mathcal{C}G_1\ d\xi\right] \\ & +\frac{R_7}{\mathcal{C}}\ \left[F_0 \frac{\partial}{\partial\sigma}\left(\mathcal{C}G_1\right)\right] \\ G_2|_{\xi_0} = 0 & \\ G_{2,\xi}|_1 = 0 & \end{cases} \tag{2.230}$$

where:

$$\begin{aligned} R_{4b} &= -\left(\frac{D_1}{D_0} - \frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\right) + n\mathcal{C}\beta_u\sqrt{C_{fu}} \\ R_5 &= \left(\frac{D_1}{D_0} - \frac{u_{1,\xi}}{u_{0,\xi}}|_{\xi_0}\right) \\ R_{6b} &= \frac{3}{2}R_{2b} \\ R_7 &= D_0 \end{aligned} \tag{2.231}$$

and the other coefficients are the same as shown in equations (2.157). In the previous analysis terms deriving from $G_{1,n}$ were neglected taking advantage of the slowly varying assumption.

The solution for G_2 can be given the form:

$$G_2 = a_2\ G_{21} + \tilde{G}_{22} \tag{2.232}$$

with:

$$\begin{aligned}\tilde{G}_2 = a_1 R_5 G_{21} - R_{4b} G_{22} - 2G_{23} - \frac{R_{6b}}{\mathcal{C}} G_{24} - \frac{3}{2}\frac{R_3}{\mathcal{C}} G_{25} \\ -\frac{3}{2}\frac{R_{2b}}{\mathcal{C}} G_{26} - \frac{5}{2}\frac{R_3}{\mathcal{C}} G_{27} - \frac{R_7}{\mathcal{C}} G_{28}\end{aligned} \tag{2.233}$$

where:

$$G_{2j} = g_j(\xi) - \frac{g'_j|_{\xi=1}}{g'_0|_{\xi=1}} g_0(\xi) \qquad (j = 1, 8) \tag{2.234}$$

and g_j ($j = 0..8$) are solutions of the following ordinary differential systems:

$$\left\{\begin{array}{lcl} [\mathcal{N}(\xi) g_{j,\ \xi}]_{,\xi} & = & \delta_i \\ g_j|_{\xi_0} = 0 & & \\ g_{j,\ \xi}|_{\xi_0} = 1 & & \end{array}\right. \tag{2.235}$$

where:

$$
\begin{array}{lcl}
\delta_0 & = & 0 \\
\delta_1 & = & 1 \\
\delta_2 & = & -F_0^2 \\
\delta_3 & = & F_0 F_1 \\
\delta_4 & = & -F_0 \mathcal{C} G_1 \\
\delta_5 & = & -(\mathcal{C} G_1)^2 \\
\delta_6 & = & \mathcal{C} G_{1,\xi} \int_{\xi_0}^{\xi} F_0 \ d\xi \\
\delta_7 & = & \mathcal{C} G_{1,\xi} \int_{\xi_0}^{\xi} \mathcal{C} G_1 \ d\xi \\
\delta_8 & = & -F_0 \frac{\partial}{\partial \sigma} (\mathcal{C} G_1)
\end{array}
\tag{2.236}
$$

Obviously g_0, g_1 and g_2 are the same solutions found in (2.136), hence:

$$
G_{21} = G_{11} \qquad\qquad G_{22} = G_{12} \tag{2.237}
$$

Furthermore, we note that g_3 is the solution forced by perturbations of the centrifugal term driven by the longitudinal flow; the remaining terms represent the contributions of vertical and longitudinal variations of the secondary flow at order $O(\delta)$. Similarly to (2.162) it can be easily shown that terms come from following contributions:

$$
\begin{array}{lcl}
-F_0^2 & \longrightarrow & \mathcal{C} N u^2 \\
F_0 F_1 & \longrightarrow & \mathcal{C} N u^2 \\
-F_0 \mathcal{C} G_1 & \longrightarrow & N u v_{,\sigma} \\
-(\mathcal{C} G_1)^2 & \longrightarrow & v v_{,n} \\
\mathcal{C} G_{1,\xi} \int_{\xi_0}^{\xi} F_0 \ d\xi & \longrightarrow & \frac{N v_{,\xi}}{D} \frac{\partial}{\partial \sigma} \left[D \int_{\xi_0}^{\xi} u \ d\xi \right] \\
\mathcal{C} G_{1,\xi} \int_{\xi_0}^{\xi} \mathcal{C} G_1 & \longrightarrow & \frac{v_{,\xi}}{D} \frac{\partial}{\partial n} \left[D \int_{\xi_0}^{\xi} v \ d\xi \right] \\
-F_0 \frac{\partial}{\partial \sigma} (\mathcal{C} G_1) & \longrightarrow & N u v_{,\sigma}
\end{array}
\tag{2.238}
$$

We refer (Nobile (2008)) for a complete derivation of functions G_{2j}.

We proceed to determine the function $a_2(n, \sigma)$ in a quite similar way as done for (2.141). The *depth-averaged form of the continuity equation for the liquid phase* (2.70) at $O(\delta^2)$, after some algebra, takes the form:

$$\frac{\partial}{\partial \sigma}\left[D_1 \int_{\xi_0}^{1} u_0 \, d\xi\right] + \frac{\partial}{\partial \sigma}\left[D_0 \int_{\xi_0}^{1} u_1 \, d\xi\right] + \frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n}\left[D_1 \int_{\xi_0}^{1} v_1 \, d\xi\right]$$
$$\frac{1}{\beta_u \sqrt{C_{fu}}} \frac{\partial}{\partial n}\left[D_0 \int_{\xi_0}^{1} v_2 \, d\xi\right] + \mathcal{C} D_0 \int_{\xi_0}^{1} v_1 \, d\xi$$
$$-n \mathcal{C} \beta_u \sqrt{C_{fu}} \frac{\partial}{\partial \sigma}\left[D_0 \int_{\xi_0}^{1} u_0 \, d\xi\right] = 0 \tag{2.239}$$

We then integrate the latter equation over the cross section. With the use of the boundary condition (2.66) written at $O(\delta)$ (2.140) and at $O(\delta^2)$, namely:

$$\int_{\xi_0}^{1} v_2 \, d\xi = 0 \qquad (n = \pm 1) \tag{2.240}$$

we find:

$$+\frac{1}{\beta_u \sqrt{C_{fu}}} D_1 \int_{\xi_0}^{1} v_1 \, d\xi + \mathcal{C} \int_{-1}^{n} \left[D_0 \int_{\xi_0}^{1} v_1 \, d\xi\right] dn$$
$$+\frac{\partial}{\partial \sigma}\left[\int_{-1}^{n} \left(D_1 \int_{\xi_0}^{1} u_0 \, d\xi + D_0 \int_{\xi_0}^{1} u_1 \, d\xi\right) dn\right]$$
$$-\mathcal{C} \beta_u \sqrt{C_{fu}} \int_{-1}^{n} n \frac{\partial}{\partial \sigma}\left[D_0 \int_{\xi_0}^{1} u_0 \, d\xi\right] dn$$
$$+\frac{1}{\beta_u \sqrt{C_{fu}}} D_0 \int_{\xi_0}^{1} v_2 \, d\xi = 0 \tag{2.241}$$

Now a useful relationship is obtained multiplying by $n\mathcal{C}\beta_u\sqrt{C_{fu}}$ the depth-averaged continuity equation (2.141) at $O(\delta)$ and integrating the latter in the lateral direction using (2.140):

$$-\mathcal{C}\beta_u\sqrt{C_{fu}}\int_{-1}^{n} n\frac{\partial}{\partial\sigma}\left[D_0\int_{\xi_0}^{1} u_0\ d\xi\right]\ dn =$$
$$\mathcal{C}\left\{nD_0\int_{\xi_0}^{1} v_1\ d\xi - \int_{-1}^{n}\left(D_0\int_{\xi_0}^{1} v_1\ d\xi\right)\ dn\right\} \qquad (2.242)$$

Substituting from (2.242) into (2.241) we find:

$$+\frac{1}{\beta_u\sqrt{C_{fu}}}\left[D_1\int_{\xi_0}^{1} v_1\ d\xi + D_0\int_{\xi_0}^{1} v_2\ d\xi\right]$$
$$+\frac{\partial}{\partial\sigma}\left[\int_{-1}^{n}\left(D_1\int_{\xi_0}^{1} u_0\ d\xi + D_0\int_{\xi_0}^{1} u_1\ d\xi\right)\ dn\right]$$
$$+\mathcal{C}nD_0\int_{\xi_0}^{1} v_1 d\xi = 0 \qquad (2.243)$$

Finally, using (2.228) and (2.232), we end up with an expression for a_2:

$$a_2(n,\sigma) = a_{20} - \frac{a_{21}}{\mathcal{C}D_0^{5/2}R_0^{1/2}}\left\{+\frac{1}{\beta_u\sqrt{C_{fu}}}D_1\int_{\xi_0}^{1} v_1\ d\xi\right.$$
$$+\frac{\partial}{\partial\sigma}\left[\int_{-1}^{n}\left(D_1\int_{\xi_0}^{1} u_0\ d\xi + D_0\int_{\xi_0}^{1} u_1\ d\xi\right)\ dn\right]$$
$$\left.+\mathcal{C}nD_0\int_{\xi_0}^{1} v_1 d\xi\right\} \qquad (2.244)$$

where:

$$a_{20} = -\frac{I_{\tilde{G}_{22}}}{I_{G_{21}}} \qquad a_{21} = \frac{\beta_u \sqrt{C_{fu}}}{I_{G_{21}}} \tag{2.245}$$

The second order correction of secondary flow can be split in a way similar to (2.151), with a first contribution characterized by vanishing vertical average and a second contribution providing the non zero part of the vertical average of the lateral velocity. This can be easily shown rearranging the equation (2.228) using the relations (2.119) and (2.237):

$$v_2 = v_{20} + v_{21} \tag{2.246}$$

$$v_{20} = D_0^{3/2} R_0^{1/2} \mathcal{C} \left(a_{20} G_{11} + \tilde{G}_{22} \right) \tag{2.247}$$

$$\begin{aligned} v_{21} = -\frac{\beta_u \sqrt{C_{fu}}}{D_0(1-\xi_0)} F_0 \Bigg\{ \frac{1}{\beta_u \sqrt{C_{fu}}} D_1 \int_{\xi_0}^1 v_1 \, d\xi \\ + \frac{\partial}{\partial \sigma} \left[\int_{-1}^{n} \left(D_1 \int_{\xi_0}^1 u_0 \, d\xi + D_0 \int_{\xi_0}^1 u_1 \, d\xi \right) dn \right] \\ + \mathcal{C} n D_0 \int_{\xi_0}^1 v_1 d\xi \Bigg\} \end{aligned} \tag{2.248}$$

Let us then calculate the vertical average of the lateral velocity:

$$V_2 = \frac{1}{1-\xi_0} \int_{\xi_0}^1 v_2 d\xi = \overline{v_{20}} + \overline{v_{21}} \tag{2.249}$$

Using the equations (2.119), (2.237), (2.245) and (2.249) we can readily show that:

$$\overline{v_{20}} = 0 \tag{2.250}$$

$$\overline{v_{21}} = V_2 \tag{2.251}$$

Hence, comparing (2.251) with (2.248) and using (2.246), we can finally express v_2 in the form:

$$v_2(\sigma, n, \xi) = v_{20}(\sigma, n, \xi) + F_0(\xi)V_2(\sigma, n) \tag{2.252}$$

We may finally come to the sediment continuity equation (2.76). Integrating the latter over the cross section, with the use of the boundary condition (2.66) at $O(\delta^2)$, it takes the form:

$$\begin{aligned} &\beta_u\sqrt{C_{fu}}\frac{\partial}{\partial\sigma}\left[q_{b_{\sigma_1}} + Q_0\ q_{s_{\sigma_1}}\right] \\ -(\beta_u\sqrt{C_{fu}})^2\ n\ &\mathcal{C}\frac{\partial}{\partial\sigma}\left[q_{b_{\sigma_0}} + Q_0\ q_{s_{\sigma_0}}\right] \\ &+\frac{\partial}{\partial n}\left[q_{b_{n_2}} + Q_0\ q_{s_{n_2}}\right] \\ &+\beta_u\sqrt{C_{fu}}\mathcal{C}\left[q_{b_{n_1}} + Q_0\ q_{s_{n_1}}\right] = 0 \end{aligned} \tag{2.253}$$

Integrating between -1 and n, over the cross section, and using the boundary condition on the side walls, we get:

$$\begin{aligned} q_{b_{n_2}} + Q_0\ q_{s_{n_2}} &= \\ -\beta_u\sqrt{C_{fu}}\frac{\partial}{\partial\sigma}&\left[\int_{-1}^{n} q_{b_{\sigma_1}} + Q_0\ q_{s_{\sigma_1}}\ dn\right] \\ +(\beta_u\sqrt{C_{fu}})^2\int_{-1}^{n}&\left[n\ \mathcal{C}\frac{\partial}{\partial\sigma}\left(q_{b_{\sigma_0}} + Q_0\ q_{s_{\sigma_0}}\right)\ dn\right] \end{aligned}$$

$$-\beta_u \sqrt{C_{fu}} \left[\int_{-1}^{n} \mathcal{C} \left(q_{b_{n_1}} + Q_0 \ q_{s_{n_1}} \right) \ dn \right] \tag{2.254}$$

After some algebra, substituting in the first order Exner expression, we can finally reduced the system to a non linear partial integro-differential equation for the unknown functions $D_1(n, \sigma)$ and $h_{01},_{\sigma}(\sigma)$. We find:

$$\frac{\partial D_1}{\partial n} = \frac{\sqrt{D_0 R_0}}{R'}$$

$$\left[- \frac{\beta_u \sqrt{C_{fu}}}{\Phi_0} \frac{\partial}{\partial \sigma} \int_{-1}^{n} (\ q_{b_{\sigma_1}} + Q_0 \ q_{s_{\sigma_1}}) \ dn \right.$$

$$- \frac{\beta_u \sqrt{C_{fu}}}{\Phi_0} n \mathcal{C} (q_{b_{n_1}} + Q_0 \ q_{s_{n_1}}) - \frac{v_2,_{\xi}}{u_0,_{\xi}}|_{\xi_0} \tag{2.255}$$

$$\left. + \frac{q_{b_{n_1}}}{\Phi_0} \left(\frac{u_1,_{\xi}}{u_0,_{\xi}}|_{\xi_0} - \frac{\Phi_1}{\Phi_0} \right) - \frac{Q_0 q_{s_{n_2}}}{\Phi_0} \right]$$

$$+ F_u^2 \frac{\partial h_1}{\partial n} \tag{2.256}$$

Clearly this equation has to be solved subject to the integral constraints (2.51) and (2.52) evaluated at the order $O(\delta^1)$:

$$\int_{-1}^{+1} \left(D_0^{3/2} I_{F_1} + D_0^{1/2} D_1 I_{F_0} \right) \ dn = 0 \tag{2.257}$$

$$\int_{-1}^{+1} \ q_{b_{\sigma 1}} + Q_0 q_{s_{\sigma 1}} \ dn = 0 \tag{2.258}$$

Again we refer to (Nobile (2008)) for a complete description of the numerical procedure employed to solve.

Chapter 3

Suspended transport effect: results

In this chapter we report some results on flow and bed topography in curved channels where both bedload and suspended load are accounted for. In particular we report some solutions for the vertical distribution of suspended sediment concentration and evaluate its sensitivity to varying the Rouse Number Z_0 and the reference level ξ_R.
Comparisons between the present solution and results obtained in the case of dominant bedload are also reported and discussed.

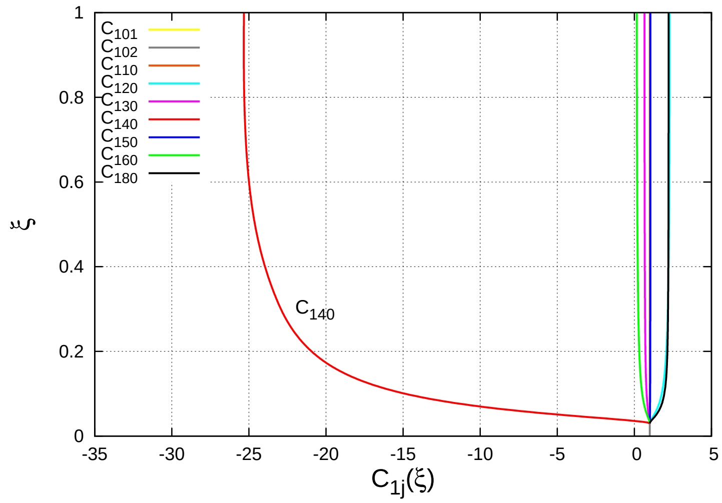

Figure 3.1: Sketch illustrating a comparison between the components of C_1, for the following parameter: $Z_0 = 2.0$, $d_{50} = 10^{-3}$ and $\xi_R = 0.03$

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3.1 First order solution for suspended sediment concentration

In this paragraph we report sediment concentration distributions along the vertical axis ξ. We will separately analyze each of the contributions that form the first order solution C_1. Note that the function C_{140}, reported in figures (3.1) is the most relevant term, being one order of magnitude greater than the others.
Let us now consider the sensitivity of each functions C_{1j} to varying Rouse number Z_0, the dimensionless reference value ξ_R and the relative roughness d_{50}. Let us consider figure (3.2), where we can see the relation between C_{101} and Rouse number. When vertical coordinate ξ increases, concentration value rises, and this variation is greater for higher value of Rouse number. The concentration function is very similar to a vertical straight line, if Rouse number tends to the unity, and approximates to a parabolic distribution for high value of Z_0.We can also observe how concentration increases if dimensionless diameter decreases and ξ_R rises. In any case the variations of the first order concentration driven by ξ_R and d_{50} are smaller than those induced by the Rouse number. Note also that C_{101} and C_{180} functions do not show any dependency from the relative roughness.

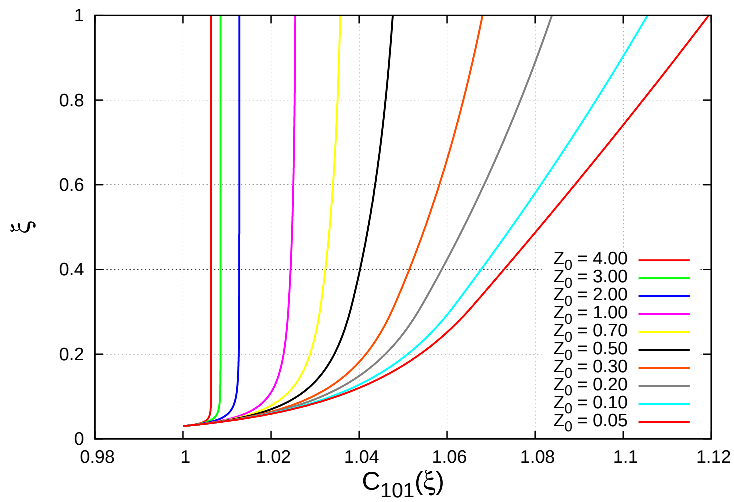

Figure 3.2: Sketch illustrating C_{101} as a function of Rouse number

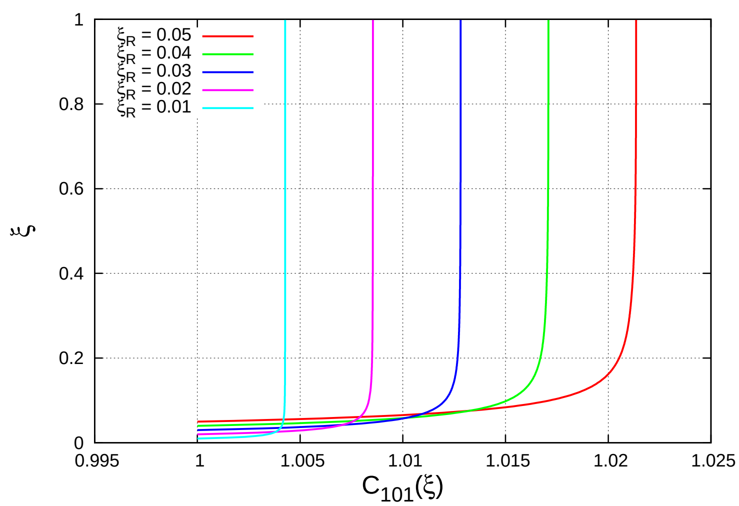

Figure 3.3: Sketch illustrating C_{101} as a function of bottom level reference ξ_R

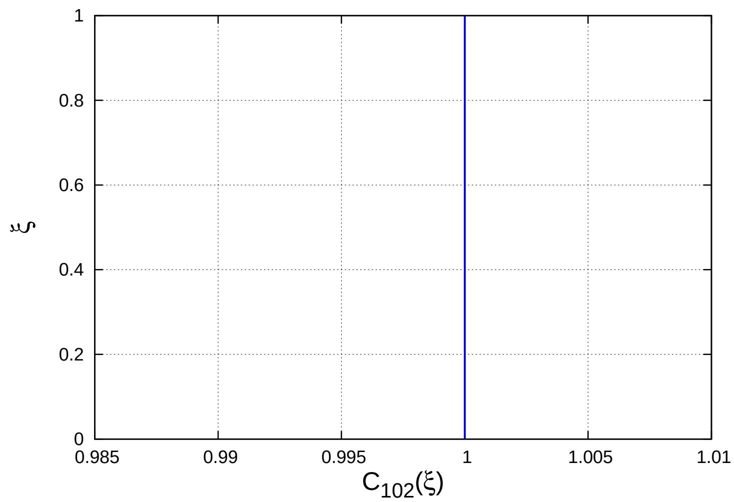

Figure 3.4: Sketch illustrating C_{102} as a function of Rouse number

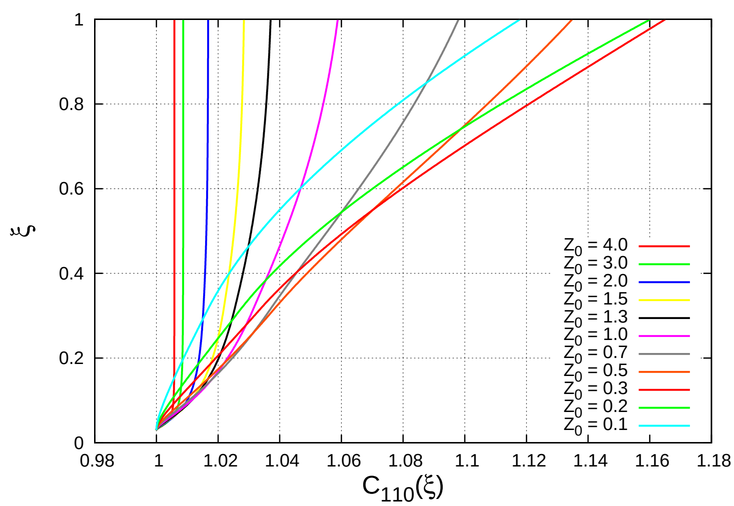

Figure 3.5: Sketch illustrating C_{110} as a function of Rouse number

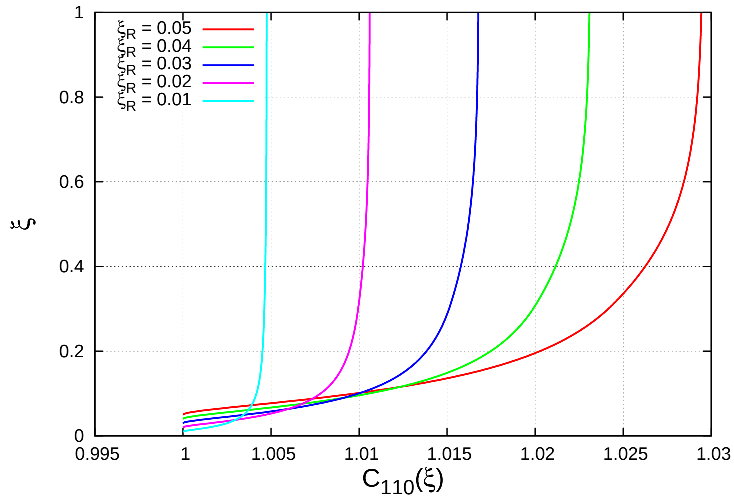

Figure 3.6: Sketch illustrating C_{110} as a function of bottom level reference ξ_R

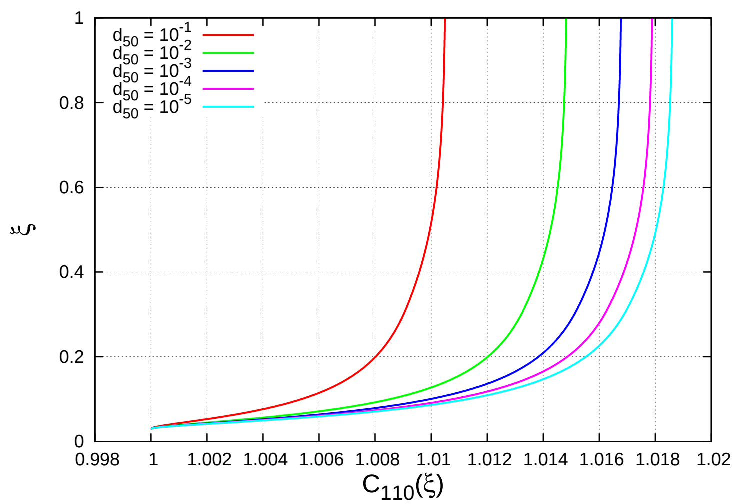

Figure 3.7: Sketch illustrating C_{110} as a function of dimensionless sediment diameter d_{50}

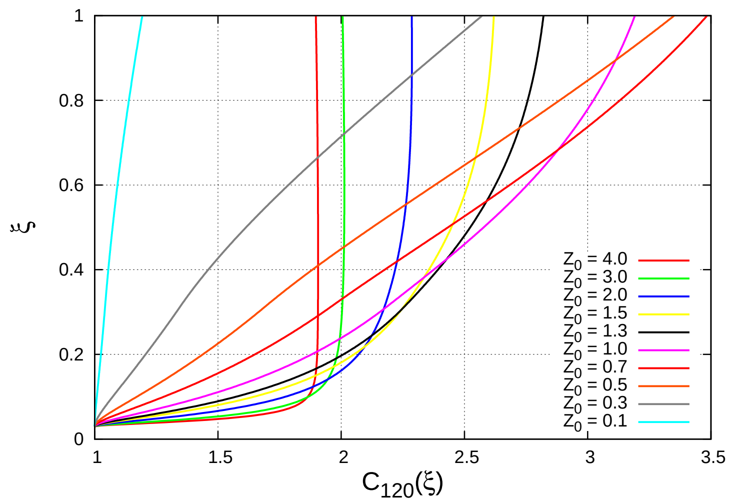

Figure 3.8: Sketch illustrating C_{120} as a function of Rouse number

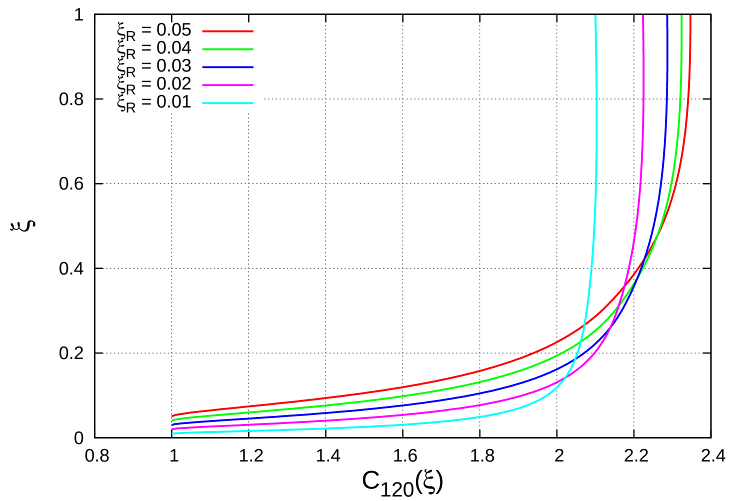

Figure 3.9: Sketch illustrating C_{120} as a function of bottom level reference ξ_R

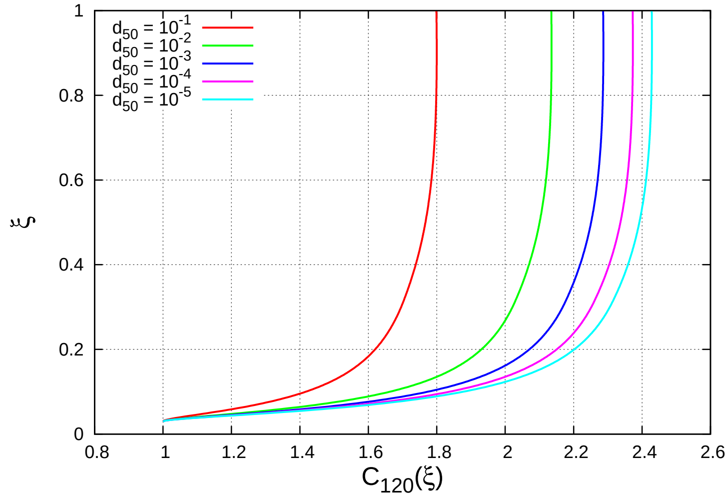

Figure 3.10: Sketch illustrating C_{120} as a function of dimensionless sediment diameter d_{50}

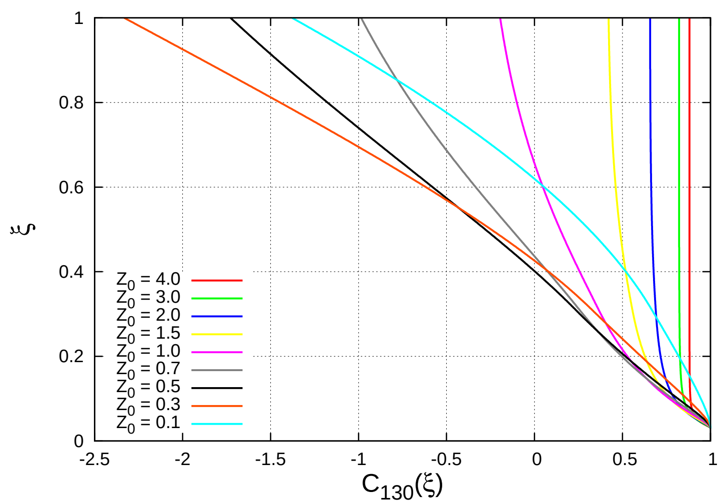

Figure 3.11: Sketch illustrating C_{130} as a function of Rouse number

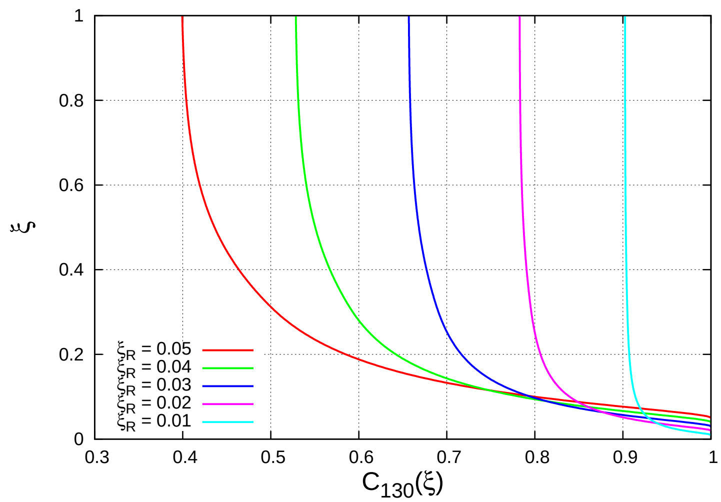

Figure 3.12: Sketch illustrating C_{130} as a function of bottom level reference ξ_R

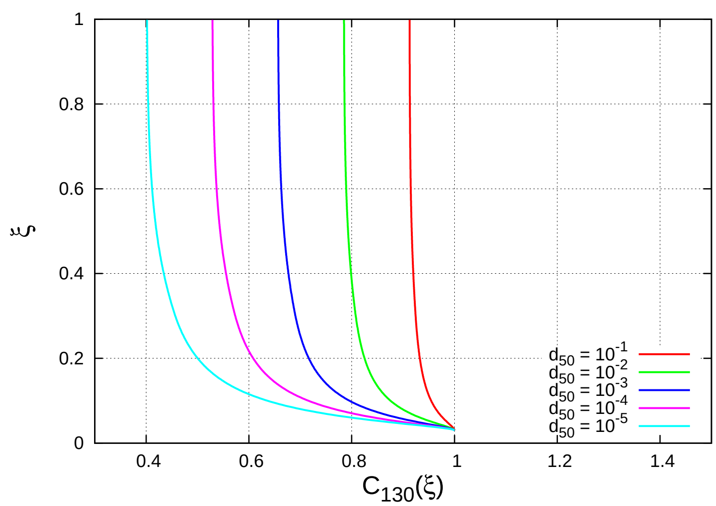

Figure 3.13: Sketch illustrating C_{130} as a function of dimensionless sediment diameter d_{50}

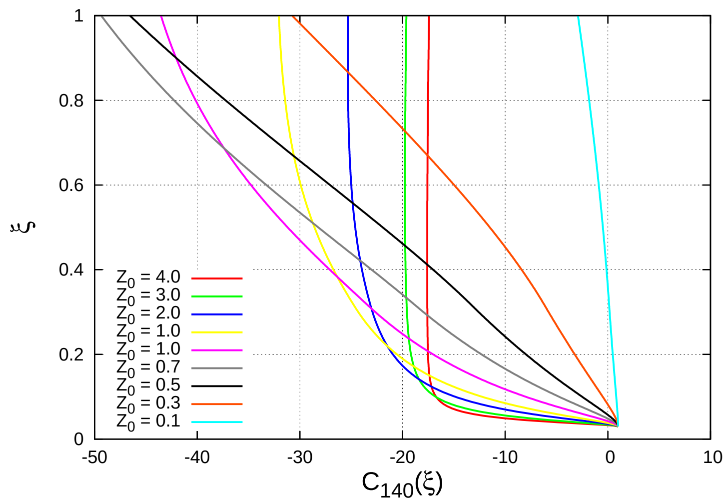

Figure 3.14: Sketch illustrating C_{140} as a function of Rouse number

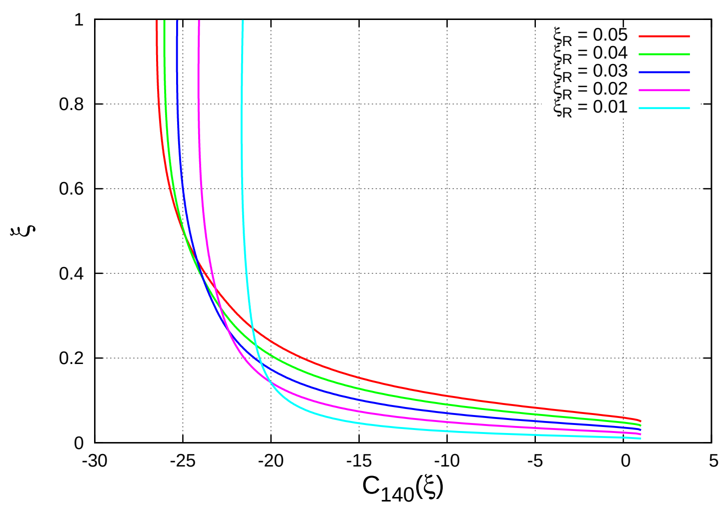

Figure 3.15: Sketch illustrating C_{140} as a function of bottom level reference ξ_R

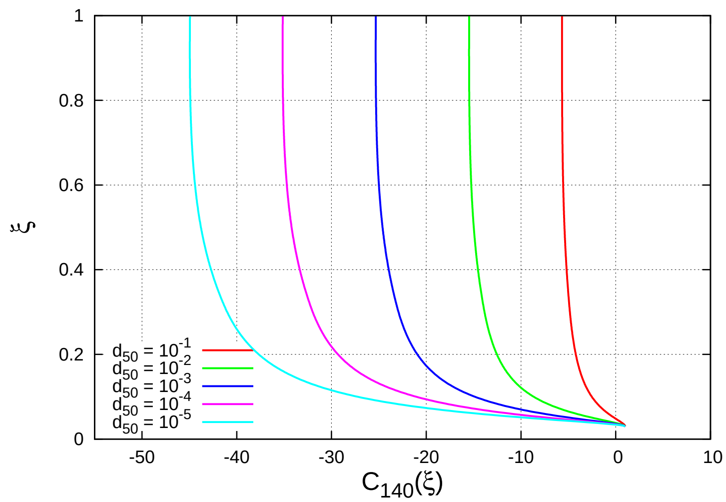

Figure 3.16: Sketch illustrating C_{140} as a function of dimensionless sediment diameter d_{50}

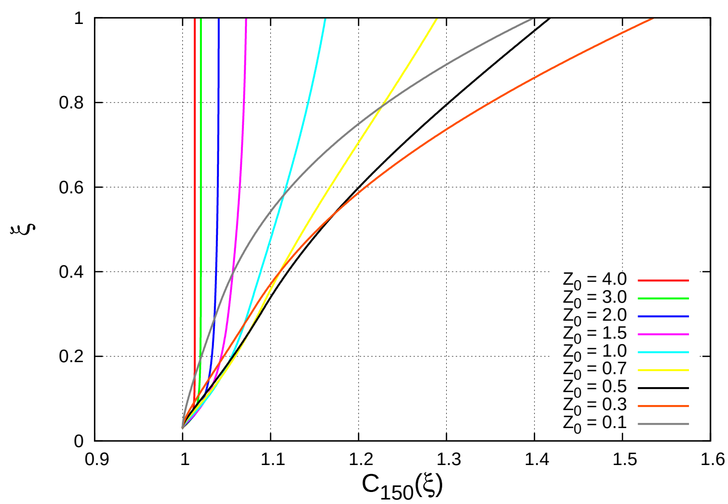

Figure 3.17: Sketch illustrating C_{150} as a function of Rouse number

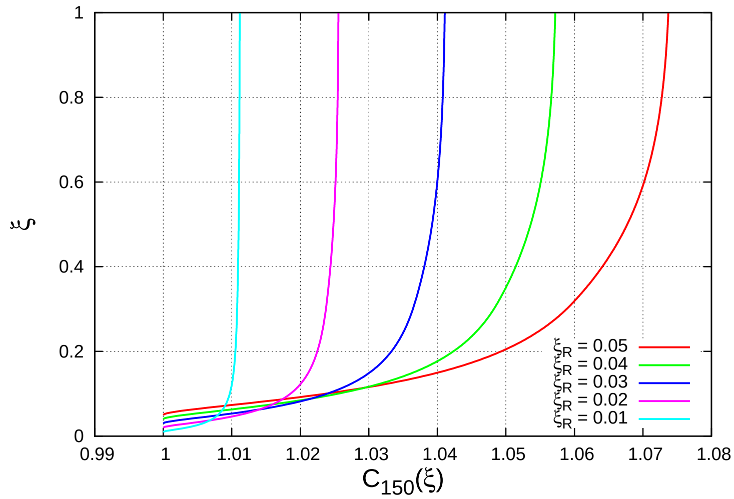

Figure 3.18: Sketch illustrating C_{150} as a function of bottom level reference ξ_R

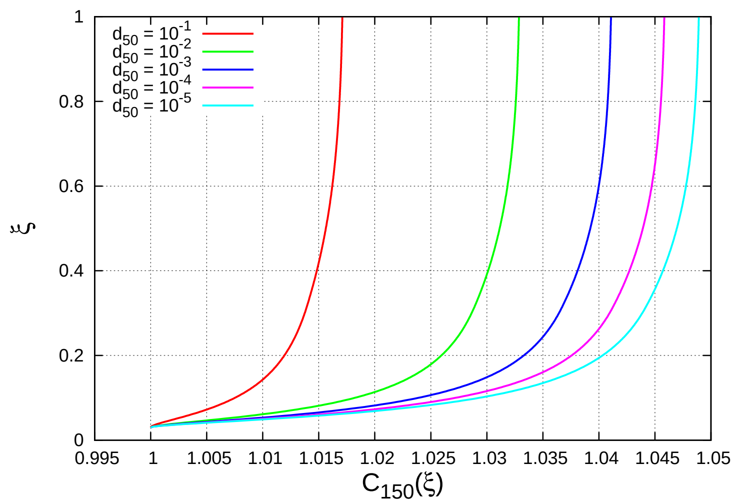

Figure 3.19: Sketch illustrating C_{150} as a function of dimensionless sediment diameter d_{50}

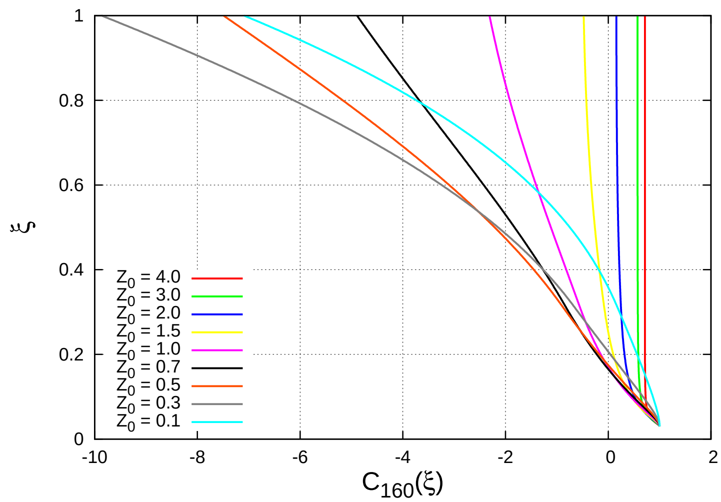

Figure 3.20: Sketch illustrating C_{160} as a function of Rouse number

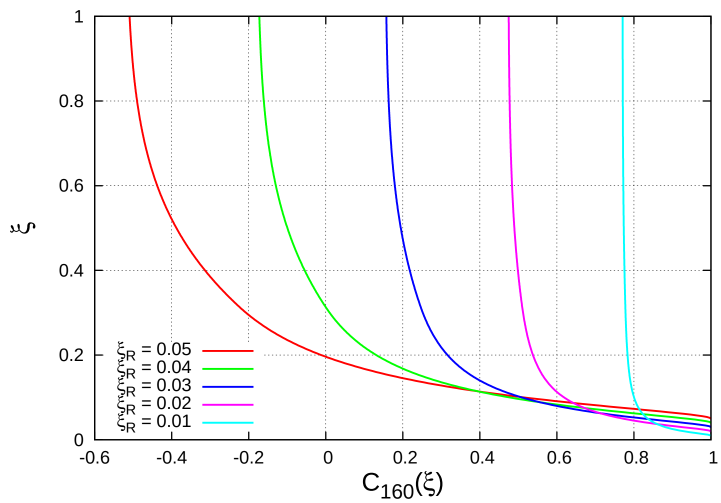

Figure 3.21: Sketch illustrating C_{160} as a function of bottom level reference ξ_R

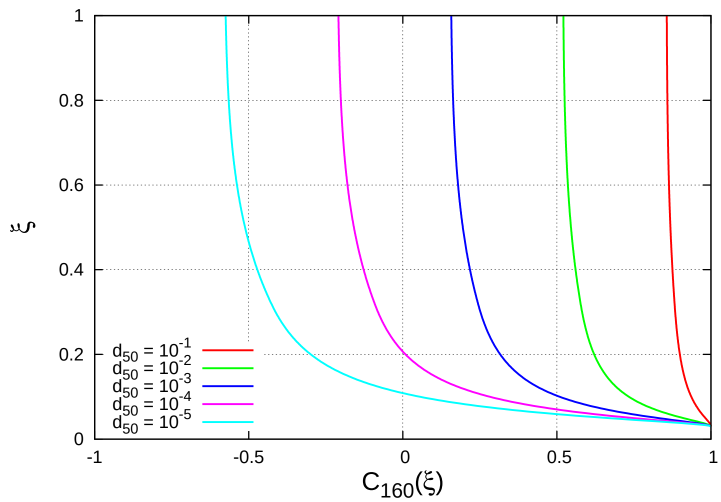

Figure 3.22: Sketch illustrating C_{160} as a function of dimensionless sediment diameter d_{50}

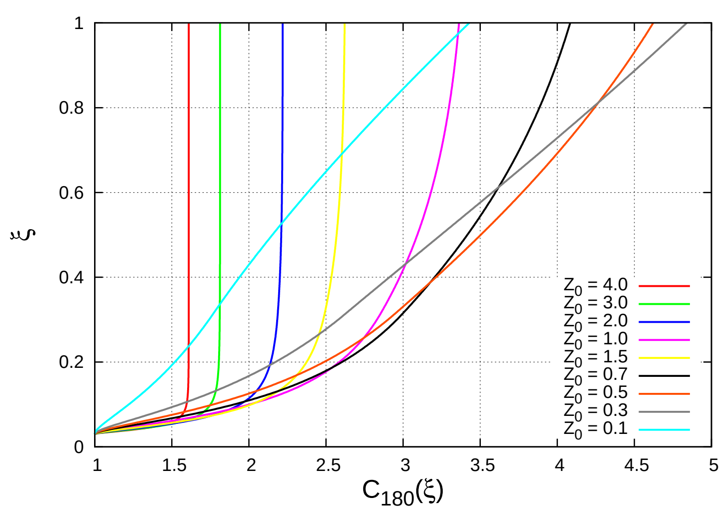

Figure 3.23: Sketch illustrating C_{180} as a function of Rouse number

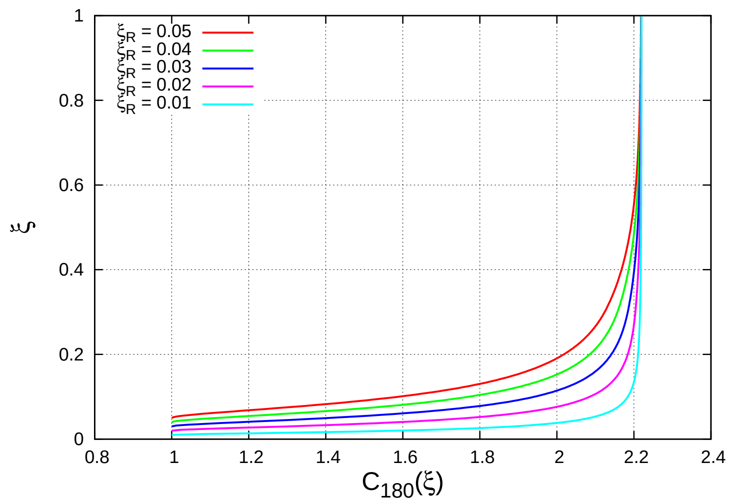

Figure 3.24: Sketch illustrating C_{180} as a function of bottom level reference ξ_R

Let us consider the secondary flow as fully developed, with vanishing average. We are going to explain the role of the most important term that contributes to form the first order solution of the concentration. We analyze in detail following term, taken from the C_1 expression reported at (2.167):

$$\frac{\beta_u \sqrt{C_{fu}}}{C_{e0}} \frac{D_0^2}{K Z_0} \left[\frac{1}{C_{e0}} \frac{\partial C_{e0}}{\partial n} C_{130} + C_{140} \right]$$

The contributions of C_{130} C_{140} are both negative, and in every case it turn out that C_{140} is one order of magnitude greater then the others.
In order to analyze in detail the effect of this them, and in particular the variation along the n axis, we distinguish between layer close to the bottom and one close to the free surface..
Let us firstly consider the layer close to the bed. If we refer to leading order solution for sediment concentration, it is clear that in this part the function has its maximum values. Here the secondary flow leads the fluid, and therefore the sediments, towards places where they have a smaller distance from the bed. Hence, the secondary flow has the effect to reduce the concentration at the bed.
Now we consider a generic coordinate near the free surface. Even though the secondary flow drive leads to rise the concentration, the

effect is less important, since the concentration presents low value. Altogether, the secondary flow decreases the concentration, seeing that his effect is more intense at the bed.

In this paragraph we want to underline the importance of the correction to apply if we perform the analysis at the first order of approximation As we have found at Chapter 2, solid discharge at second order of approximation depend on C_1 by means of the following term:

$$\chi_1 = \int_{\xi_R}^{1} u_0 \; C_1 \; d\xi \tag{3.1}$$

According with second order expression of sediment concentration we can find these terms:

$$\chi_{1j} = \int_{\xi_R}^{1} u_0 \; C_{1j} \; d\xi \tag{3.2}$$

Reminding single contribute forms of C_1 we basically know that $\chi_{1,j}$ are related to Rouse number, bottom reference level ξ_R and dimensionless diameter of the particles d_{50}, that is a measure of roughness of the bed. In figure (3.27-3.27) is represented this dependence. Again, it is clear that 140 term is largely the most important one, and variation related to ξ_R and d_{50} are smaller in respect of how much χ change with Rouse number.

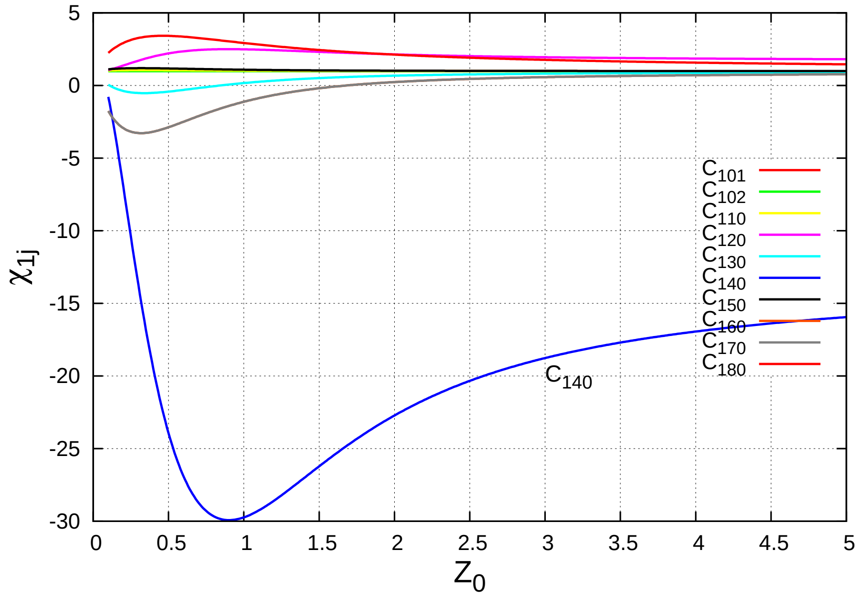

Figure 3.25: Sketch illustrating $\chi_{1,j}$ as a function of Rouse number

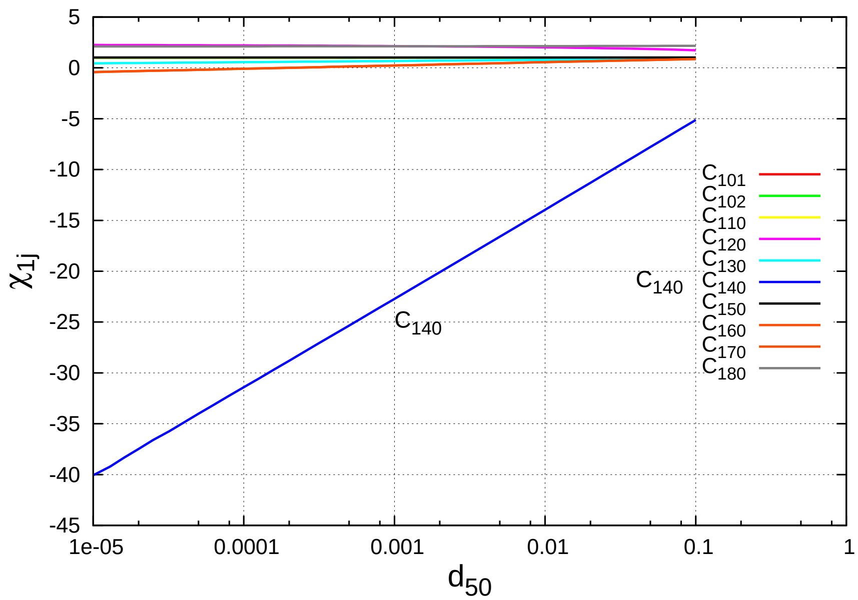

Figure 3.26: Sketch illustrating $\chi_{1,j}$ as a function of d_{50}

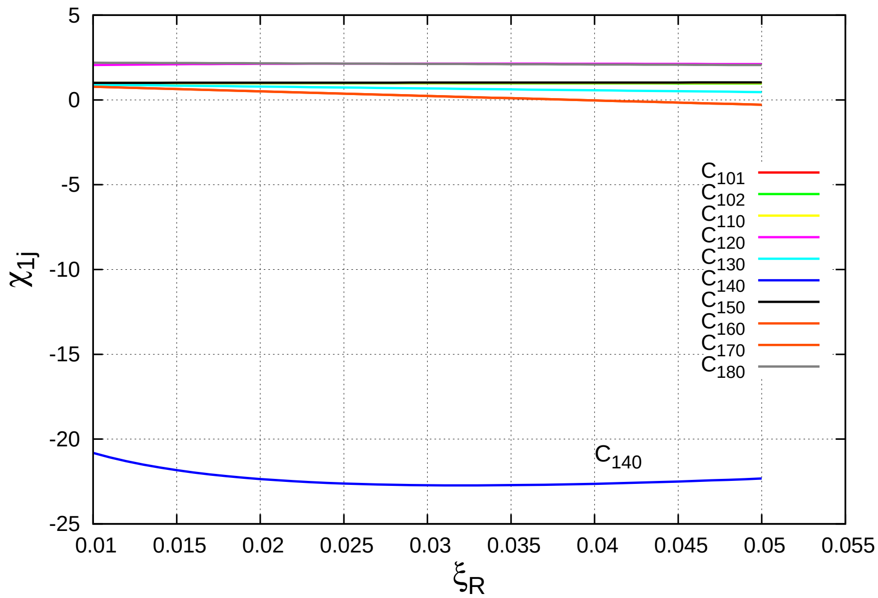

Figure 3.27: Sketch illustrating $\chi_{1,j}$ as a function of ξ_R

3.2 Prediction of maximum scour depth in channels with constant curvature

In figure (3.28) we reported the maximum scour at the banks for different value of particle Reynods number R_p. First of all we notice that there will be a greater scour where the curvature of the axis will be higher. In fact, $\bar{d_0}$ parameter in defined as follows:

$$\bar{d_0} = \frac{\nu_0\sqrt{\theta_u}}{C_{fu}r} \tag{3.3}$$

It turns out how the scour tends to rise when dimensional diameter of the particle decreases, i.e. for decreasing values of R_p. In the same figure is also reported the maximum value of the scour predicted by the theory in the case of bedload transport only. It turns out that if we only take into account the bedload transport, the value of scour will be lower. Indeed, the contribution driven by suspended sediment transport cannot be neglected.
In figure (3.41) maximum scour at second order of approximation is reported. In this case second order correction does not seem to be so important because the bed is relatively tough ($d_{50} = 10^{-3}$). The

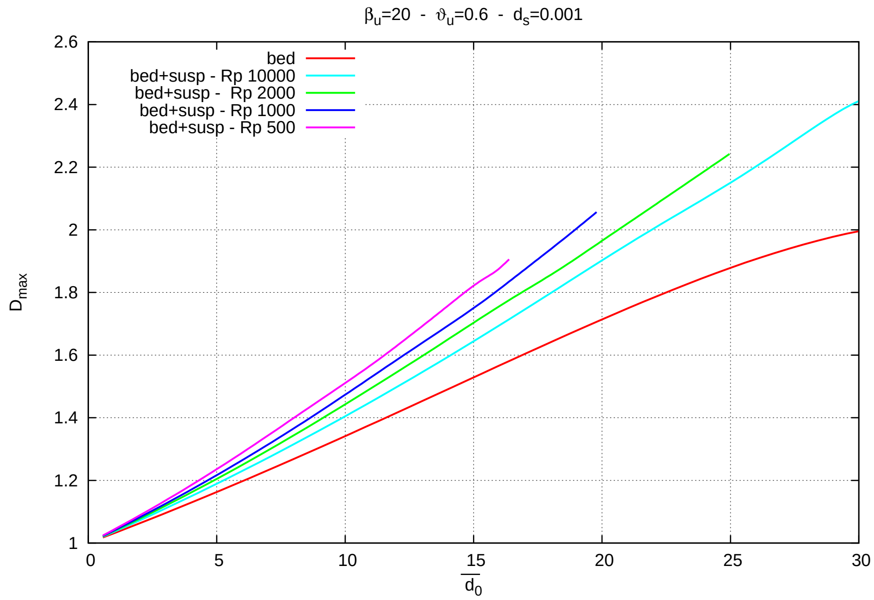

Figure 3.28: Sketch illustrating maximum scour at leading order

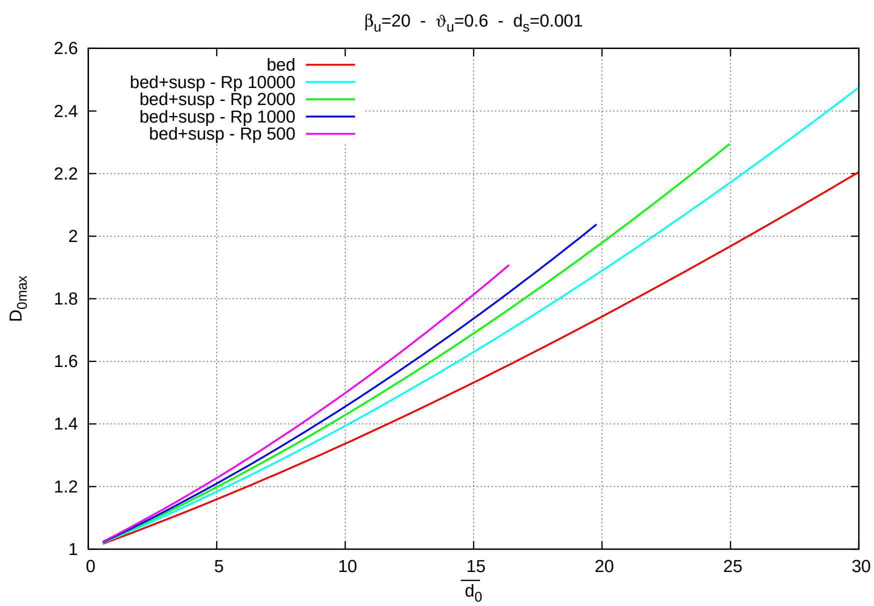

Figure 3.29: Sketch illustrating maximum scour at first order

second order correction increases exponentially for decreasing values of the relative roughness d_{50}.

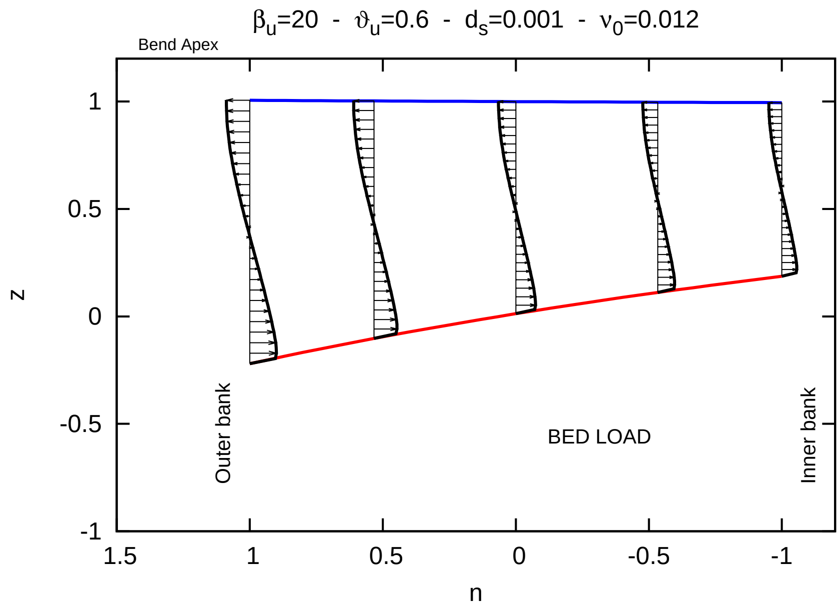

Figure 3.30: Section at bend apex, with $\nu_0 = 0.012$

Let us now show the effect of secondary flow on the transverse slope, and consequently on the flow field in the cross section of the channel in a case reference: ($\beta_u = 20$; $R_p = 1000$, $d^* = 4mm$, $\theta_u = 0.6$, and $d_{50} = 10^{-3}$). Let us consider initially the case of bedload transport only. We report some sections for different value of curvature parameter ν_0. We can notice how the scour rises at the outer bank, when the curvature increases.

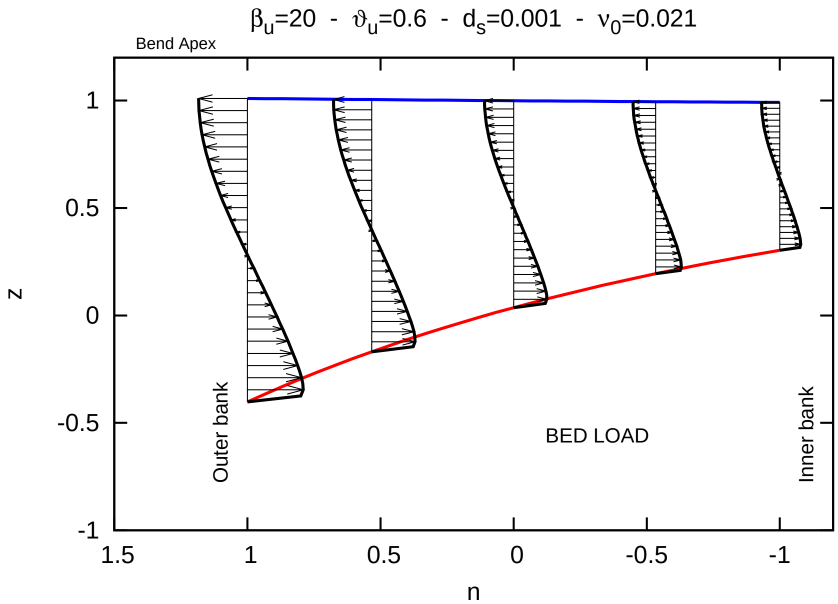

Figure 3.31: Section at bend apex, with $\nu_0 = 0.021$

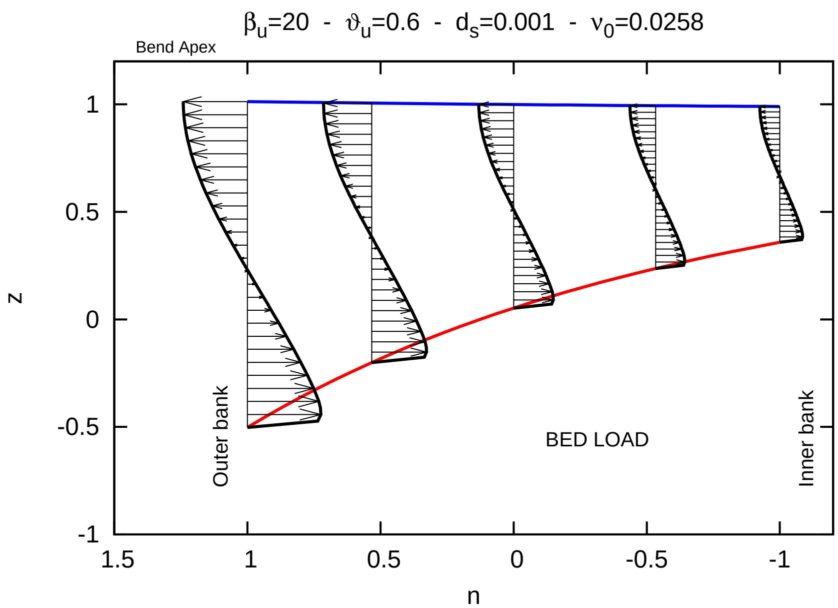

Figure 3.32: Section at bend apex, with $\nu_0 = 0.0258$

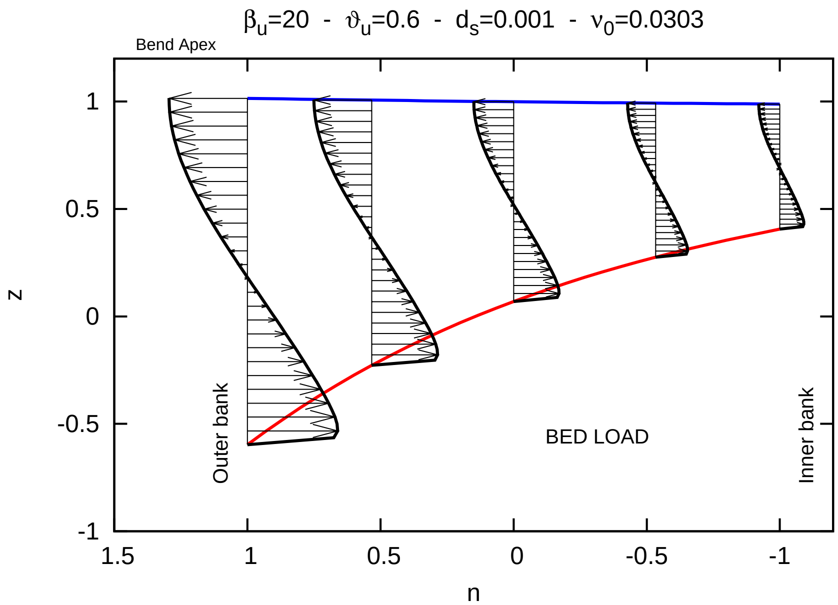

Figure 3.33: Section at bend apex, with $\nu_0 = 0.0303$

If we now consider the case of both bedload and suspended sediment transport, we get the following picture. Again, we report sections for different value of curvature parameter ν_0. The trend is basically the same of the previous analysis. In other words, the scour always rises at the outer bank, when the curvature increases.

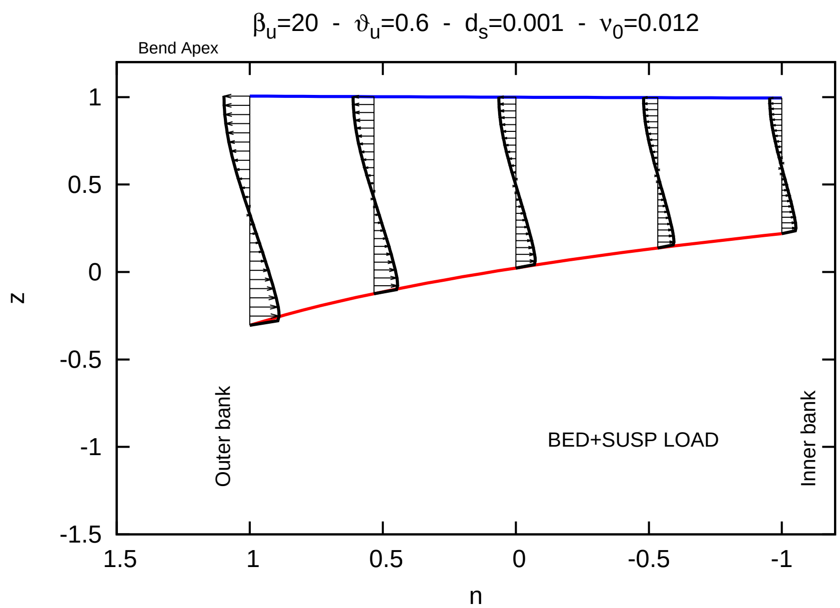

Figure 3.34: Section at bend apex, with $\nu_0 = 0.012$

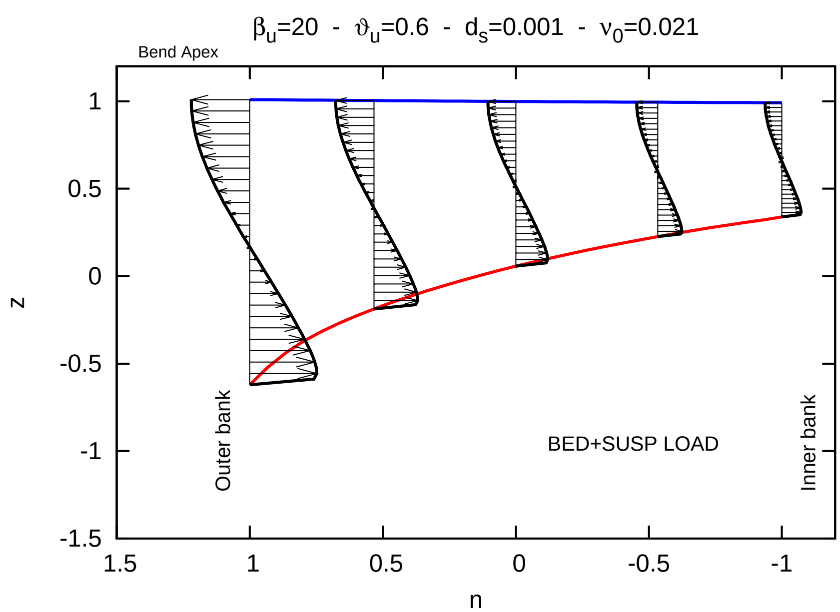

Figure 3.35: Section at bend apex, with $\nu_0 = 0.021$

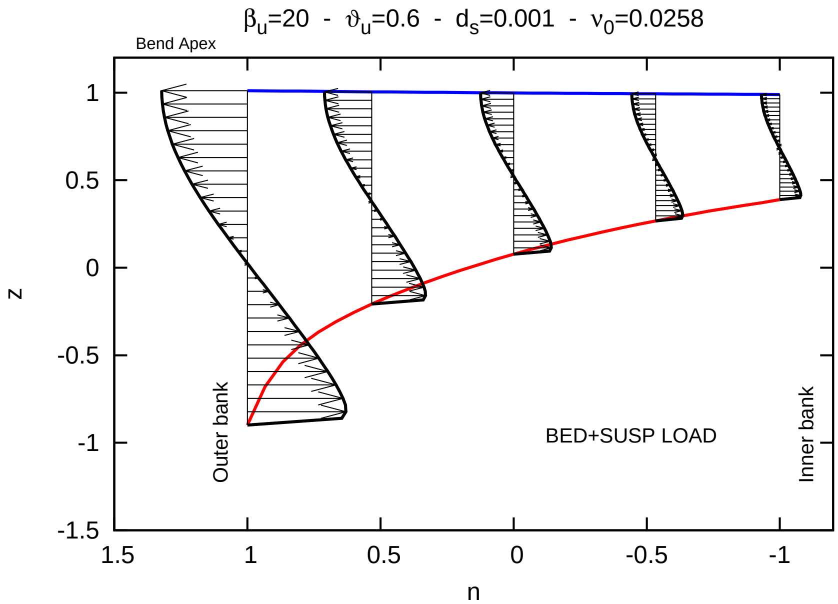

Figure 3.36: Section at bend apex, with $\nu_0 = 0.0258$

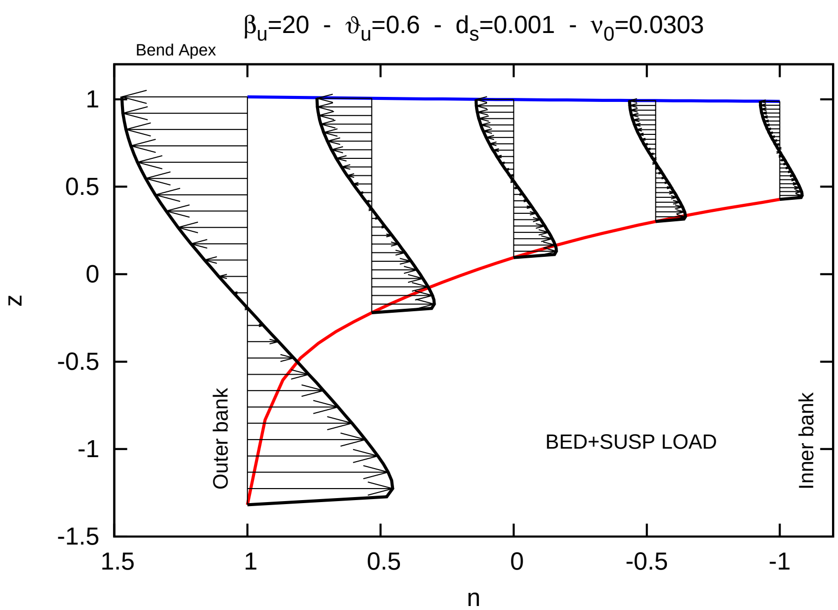

Figure 3.37: Section at bend apex, with $\nu_0 = 0.0303$

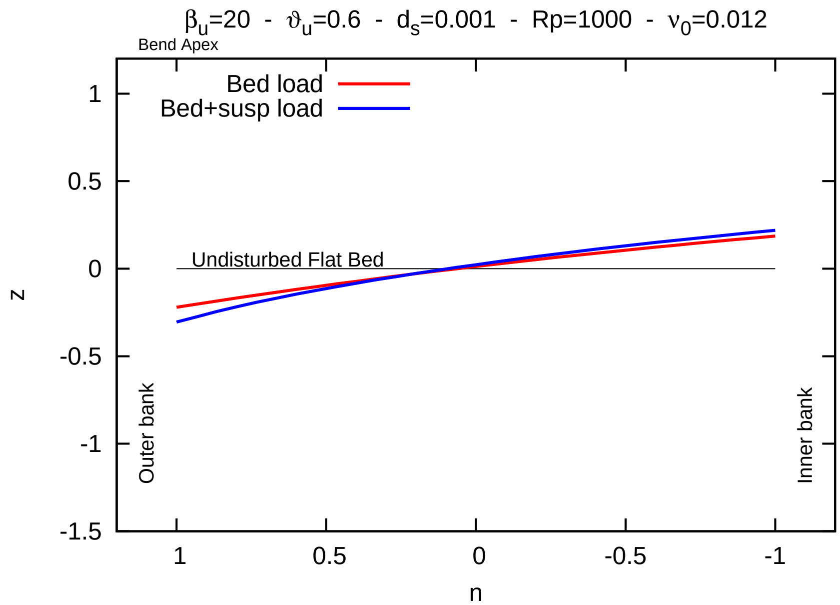

Figure 3.38: Section at bend apex, with $\nu_0 = 0.012$

Most important results that we want to underline is the rise of the scour when we consider suspended transport, compared to the scour predicted with only bedload transport. This effect is evident if we compare the two cases, only plotting the bed elevation in the cross section. This is a consequence of the secondary flow, that at the outer bank is stronger, in order to balance the gravity effect on the transverse inclination of the bed.

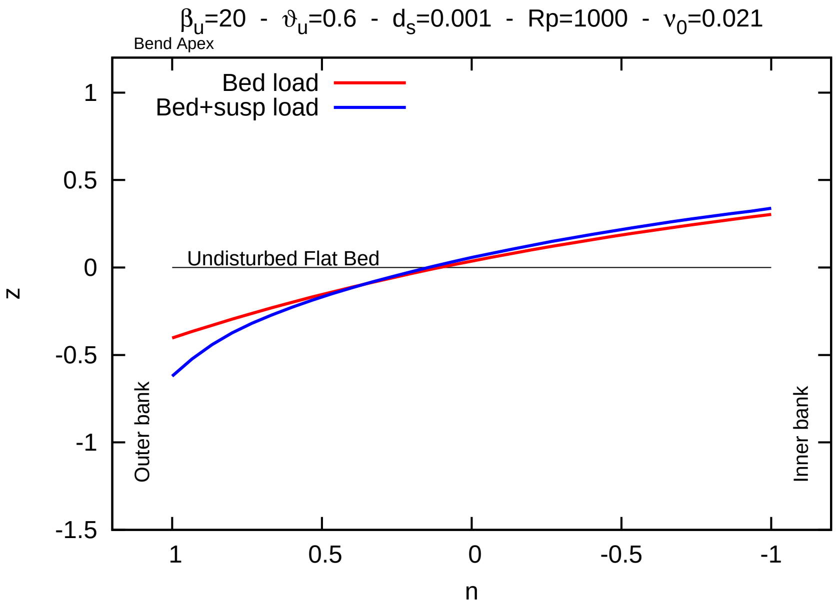

Figure 3.39: Section at bend apex, with $\nu_0 = 0.021$

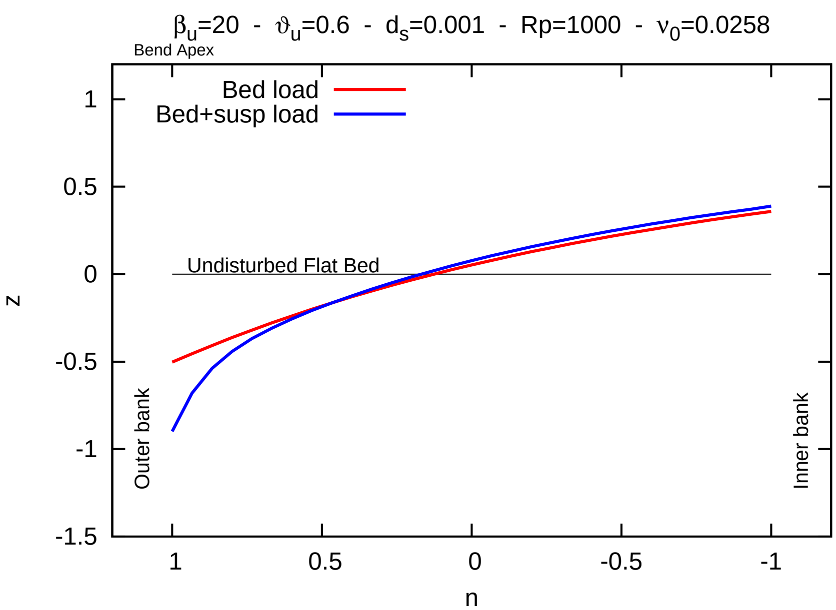

Figure 3.40: Section at bend apex, with $\nu_0 = 0.0258$

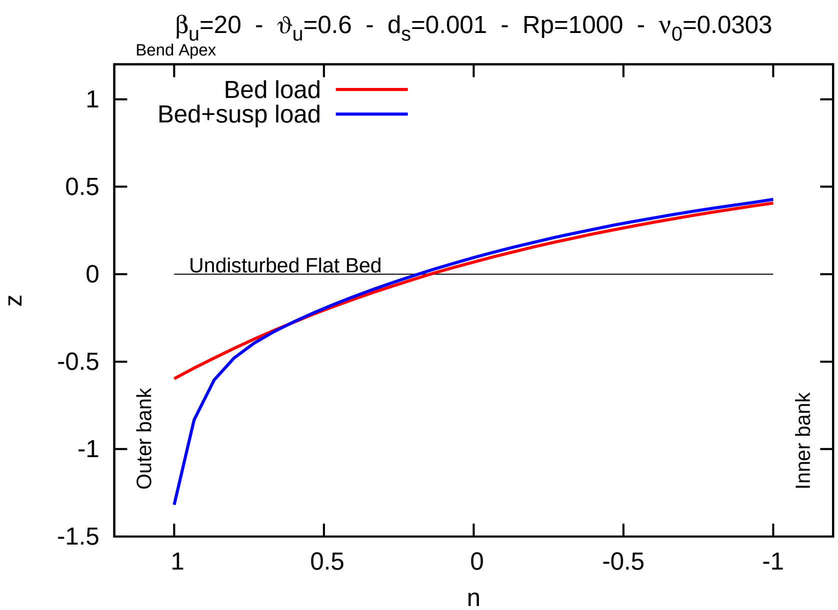

Figure 3.41: Section at bend apex, with $\nu_0 = 0.0303$

Conclusions

A non linear asymptotic theory of flow and bed topography in meandering channels able to describe finite amplitude perturbations of bottom topography and account for arbitrary, yet slow, variations of channel curvature has been developed. Both bedload and suspended load have been accounted for. This model appears to be a potentially useful and powerful tool for many purposes. Firstly, we have shown the fundamental role of suspended sediment transport component, in order to predict maximum scour at the bank in curved channels. In particular, we have found that:

- Correction of distribution of suspended sediment concentration with respect to the leading order solution (classical Rouse distribution) turns out to lead to a net decrease of sediment in suspension. We have shown that this dilution in strictly related to the advection of sediment in suspension by leading order solutions for secondary flow and concentration distribution.

- The effect of suspended load on maximum scour in curved channels increases as curvature increases and sediment diameter decreases.

- Evaluation of maximum scour at the banks, taking account for both bedload and suspended load, suggests that suspended load contribution can not be neglected. In fact it leads to higher transverse slope of the bed and consequently, to greater values of maximum scour. This contribution is a consequence of the increase in transversal sediment transport induced by the suspended load component.

- The maximum scour depth evaluated in presence of suspended load, shows an increase if curvature rises. This result is in agreement with previous results obtained in the case of bedload only, (Seminara & Solari (1998)).

Moreover, it is clear that this model can be useful, in order to predict several features of the actual meandering process.

In conclusion, an accurate prediction meandering process can not ignore the role of the suspended transport. In particular, the evaluation of this contribute is absolutely needed if it is necessary to predict maximum scour at the banks.
The availability of this model suggests a number of interesting future developments.

Firstly, the present model can be readily extended to allow also for slowly varying variations of channel width: such an extension may allow insight on the mechanism controlling the development of width oscillations correlated with channel curvature observed in nature. Moreover, the stability of the solution for flow and bed topography thus obtained may shed some light on the observed occurrence of island formation in river bends, along with the tendency of the stream to bifurcate eventually leading to chute cutoff.

A second extension of the present model can be obtained by coupling the flow field with a bank erosion law, leading to prediction of the plan form evolution of the meandering pattern accounting not only for geometric non linearity but also for flow nonlinearity, the present approach requiring a computational effort much smaller than that needed by numerical solutions of the full 3-D governing equations or their shallow water version (e.g. Mosselman (1991), Shimizu (2002) among others).

Finally, a third line of development can be readily obtained by allowing for slow temporal variations of flow and sediment supply: the morphological response of the channel to a sequence of flood events can then be investigated. Such an investigation would possibly provide a rational interpretation of the as yet loosely defined notion of 'formative discharge of an alluvial river'.

Appendix A

Analytic integrals

Below we rewrite the solutions of the ordinary differential problems (2.136, 2.160, 2.235), in a reordered and simplified way to make their numerical evaluation simpler and faster. In particular, we decompose the functions isolating constant terms which depend only on the following parameter: Rouse number, friction coefficient,Von Karman coefficient and bottom reference lever ξ_R.
Firstly we have to redefine the Dean's wake function $\mathcal{N}(\xi)$:

$$\mathcal{N}(\xi) = \frac{k\xi(1-\xi)}{1+2A\xi^2+3B\xi^3} = \frac{k\xi}{1+\xi-3B\xi^2} = k\hat{\mathcal{N}}(\xi) \tag{A.1}$$

where

$$\hat{\mathcal{N}}(\xi) = \frac{\xi}{1+\xi-3B\xi^2} \tag{A.2}$$

In a similar way we redefine function F_0:

$$F_0(\xi) = \frac{\sqrt{C_{fu}}}{k}\hat{F}_0 \tag{A.3}$$

where

$$\hat{F}_0(\xi) = \left[\ln\frac{\xi}{\xi_0} + A(\xi^2 - \xi_0^2) + B(\xi^3 - \xi_0^3)\right] \tag{A.4}$$

Hence, integrating once the system (2.136) for the forcing function δ_0 and using (A.1), we obtain:

$$g_{0,\xi} = \frac{\hat{N}_0}{\hat{N}} = \hat{g}_{0,\xi} \tag{A.5}$$

where $\hat{\mathcal{N}}_0 = \hat{\mathcal{N}}(\xi)|_{\xi_0}$ and $\hat{\mathcal{N}} = \hat{\mathcal{N}}(\xi)$. Furthermore integrating again (A.5) we obtain:

$$g_0 = \int_{\xi_0}^{\xi} \frac{\hat{N}_0}{\hat{N}}\, d\xi = \hat{g}_0 \tag{A.6}$$

Proceeding in a similar way with the other forcing functions δ_j $(j = 1, 2)$ of the system (2.136), after some algebra and using (A.1), (A.3), (A.5) we find:

$\boxed{g_1}$

$$\begin{aligned} g_{1,\xi} &= \tfrac{1}{k}\hat{g}_{1,\xi} + g_{0,\xi} \\ g_1 &= \tfrac{1}{k}\hat{g}_1 + g_0 \\ \hat{g}_{1,\xi} &= \tfrac{\xi-\xi_0}{\hat{N}} \\ \hat{g}_1 &= \int \tfrac{\xi-\xi_0}{\hat{N}} \end{aligned} \tag{A.7}$$

$\boxed{g_2}$

$$\begin{aligned} g_{2,\xi} &= \tfrac{C_{fu}}{k^3}\hat{g}_{2,\xi} + g_{0,\xi} \\ g_2 &= \tfrac{C_{fu}}{k^3}\hat{g}_2 + g_0 \\ \hat{g}_{2,\xi} &= -\tfrac{1}{\hat{N}}\int \hat{F}_0^2 \\ \hat{g}_2 &= -\int\left(\tfrac{1}{\hat{N}}\int \hat{F}_0^2\right) \end{aligned} \qquad (A.8)$$

For the sake of clarity we will abbreviate the quantity $\int_{\xi_0}^{\xi} f\, d\xi$ through the notation $\int f$.

A.1 Solution for sediment concentration between ξ_R and 0.314

Analytical integrals used to find concentration value between conventional reference level ξ_R and 0.314, are here reported. For sake of clarity we will abbreviate the quantity $\int_{\xi_R}^{\xi}$ through the notation $\int$

$\boxed{IBi}$

$$\begin{aligned}
IB1 &= \int \xi^{k_1-1}\ IC1\ d\xi = \left[-\frac{\xi[(k_1-1)\log\xi-k_1+2]}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IB2 &= 3\int \xi^{k_1-1}\ IC2\ d\xi = \left[-\frac{A\xi^3}{3(k_1-3)}\right]_{\xi_R}^{\xi} \\
IB3 &= 4\int \xi^{k_1-1}\ IC3\ d\xi = \left[-\frac{B\xi^4}{4(k_1-4)}\right]_{\xi_R}^{\xi} \\
IB4 &= \int \xi^{k_1-1}\ IC4\ d\xi = \left[\frac{(1-tt5)\xi}{k_1-1}\right]_{\xi_R}^{\xi} \\
IB5 &= \int \xi^{-1}\ tt8\ d\xi = -tt8\left[\frac{\xi^{k_1}}{k_1}\right]_{\xi_R}^{\xi}
\end{aligned} \qquad (A.9)$$

$\boxed{IOi}$

$$
\begin{aligned}
IO1 &= -\tfrac{\partial Z_0}{\partial\sigma}\int\ \xi^{-k_1}\ (\log\xi)^2\ d\xi = \\
&+\tfrac{\partial Z_0}{\partial\sigma}\left[\left[(k_1-1)^2\ (\log\xi)^2+2(k_1-1)\log\xi+2\right]\tfrac{\xi^{1-k_1}}{(k_1-1)^3}\right]_{\xi_R}^{\xi} \\
IO2 &= tt20\int\ \xi^{-k_1}\log\xi d\xi = \\
&-tt20\left[\left[(k_1-1)\log\xi+1\right]\tfrac{\xi^{1-k_1}}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IO3 &= -A\tfrac{\partial Z_0}{\partial\sigma}\int\ \xi^{2-k_1}\log\xi\ d\xi = \\
&+A\tfrac{\partial Z_0}{\partial\sigma}\left[\left[(k_1-3)\log\xi+1\right]\tfrac{\xi^{3-k_1}}{(k_1-3)^2}\right]_{\xi_R}^{\xi} \\
IO4 &= A\ tt20\int\ \xi^{2-k_1}d\xi = -A\ tt20\left[\tfrac{\xi^{3-k_1}}{k_1-3}\right]_{\xi_R}^{\xi} \\
IO5 &= -B\tfrac{\partial Z_0}{\partial\sigma}\int\ \xi^{3-k_1}\log\xi\ d\xi = \\
&+B\tfrac{\partial Z_0}{\partial\sigma}\left[\left[(k_1-4)\log\xi+1\right]\tfrac{\xi^{4-k_1}}{(k_1-4)^2}\right]_{\xi_R}^{\xi} \\
IO6 &= B\ tt20\int\ \xi^{3-k_1}d\xi = -B\ tt20\left[\tfrac{\xi^{4-k_1}}{k_1-4}\right]_{\xi_R}^{\xi} \\
IO7 &= -tt5\tfrac{\partial Z_0}{\partial\sigma}\int\ \xi^{-k_1}\log\xi\ d\xi = \\
&+tt5\tfrac{\partial Z_0}{\partial\sigma}\left[\left[(k_1-1)\log\xi+1\right]\tfrac{\xi^{1-k_1}}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IO8 &= tt5\ tt20\int\ \xi^{-k_1}\ d\xi = \\
&-tt5\ tt20\left[\tfrac{\xi^{1-k_1}}{(k_1-1)}\right]_{\xi_R}^{\xi}
\end{aligned}
\tag{A.10}
$$

where:

$$tt20 = \frac{\partial Z_0}{\partial \sigma} \log \xi_R + \frac{Z_0}{\xi_R} \frac{\partial \xi_R}{\partial \sigma} \tag{A.11}$$

$$tt5 = -\log \xi_0 - A\xi_0^2 - B\xi_0^3 \tag{A.12}$$

$\boxed{IOi}$

$$\begin{aligned}
IO9 &= \int \xi^{k_1-1} IO1 \, d\xi = \tfrac{\partial Z_0}{\partial \sigma} \left[\tfrac{\xi\left[(\log \xi)^2 - 2\log \xi + 2\right]}{k_1 - 1} + \tfrac{2\xi(\log \xi - 1)}{(k_1-1)^2} + \tfrac{2}{(k_1-1)^3} \right]_{\xi_R}^{\xi} \\
IO10 &= \int \xi^{k_1-1} IO2 \, d\xi = -\tfrac{tt20}{(k_1-1)^2} \left[\xi \left[(k_1 - 1) \log \xi - k_1 + 2 \right] \right]_{\xi_R}^{\xi} \\
IO11 &= \int \xi^{k_1-1} IO3 \, d\xi = \tfrac{\partial Z_0}{\partial \sigma} \tfrac{A}{9(k_1-3)^2} \left[\xi^3 \left[3(k_1 - 3) \log \xi - k_1 + 6 \right] \right]_{\xi_R}^{\xi} \\
IO12 &= \int \xi^{k_1-1} IO4 \, d\xi = -\tfrac{A \; tt20}{3(k_1-3)} \left[\xi^3 \right]_{\xi_R}^{\xi} \\
IO13 &= \int \xi^{k_1-1} IO5 \, d\xi = \tfrac{\partial Z_0}{\partial \sigma} \tfrac{B}{16(k_1-4)^2} \left[\left[4(k_1 - 4) \log \xi - k_1 + 8 \right] \right]_{\xi_R}^{\xi} \\
IO14 &= \int \xi^{k_1-1} IO6 \, d\xi = \tfrac{B \; tt20}{4(k_1-4)} \left[\xi^4 \right]_{\xi_R}^{\xi} \\
IO15 &= \int \xi^{k_1-1} IO7 \, d\xi = \tfrac{\partial Z_0}{\partial \sigma} \tfrac{tt5}{(k_1-1)^2} \left[\xi \left[(k_1 - 1) \log \xi - k_1 + 2 \right] \right]_{\xi_R}^{\xi} \\
IO16 &= \int \xi^{k_1-1} IO8 \, d\xi = -tt20 \; tt5 \tfrac{\xi}{k_1 - 1} \\
IO17 &= \int \xi^{k_1-1} (-tt21) \, d\xi = -tt21 \tfrac{\xi^{k_1}}{k_1}
\end{aligned} \tag{A.13}$$

$\boxed{INi}$

$$
\begin{aligned}
IN1 &= -\tfrac{\partial Z_0}{\partial n}\int \ \xi^{-k_1}\ (\log\xi)^2\ d\xi = \\
&+\tfrac{\partial Z_0}{\partial n}\left[\left[(k_1-1)^2\ (\log\xi)^2 + 2(k_1-1)\log\xi + 2\right]\tfrac{\xi^{1-k_1}}{(k_1-1)^3}\right]_{\xi_R}^{\xi} \\
IN2 &= tt24\int \ \xi^{-k_1}\log\xi d\xi = \\
&-tt24\left[\left[(k_1-1)\log\xi+1\right]\tfrac{\xi^{1-k_1}}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IN3 &= -A\tfrac{\partial Z_0}{\partial n}\int \ \xi^{2-k_1}\log\xi\ d\xi = \\
&+A\tfrac{\partial Z_0}{\partial n}\left[\left[(k_1-3)\log\xi+1\right]\tfrac{\xi^{3-k_1}}{(k_1-3)^2}\right]_{\xi_R}^{\xi} \\
IN4 &= A\ tt24\int \ \xi^{2-k_1}d\xi = -A\ tt24\left[\tfrac{\xi^{3-k_1}}{k_1-3}\right]_{\xi_R}^{\xi} \\
IN5 &= -B\tfrac{\partial Z_0}{\partial n}\int \ \xi^{3-k_1}\log\xi\ d\xi = \\
&+B\tfrac{\partial Z_0}{\partial n}\left[\left[(k_1-4)\log\xi+1\right]\tfrac{\xi^{4-k_1}}{(k_1-4)^2}\right]_{\xi_R}^{\xi} \\
IN6 &= B\ tt20\int \ \xi^{3-k_1}d\xi = -B\ tt20\left[\tfrac{\xi^{4-k_1}}{k_1-4}\right]_{\xi_R}^{\xi} \\
IN7 &= -tt5\tfrac{\partial Z_0}{\partial n}\int \ \xi^{-k_1}\log\xi\ d\xi = \\
&+tt5\tfrac{\partial Z_0}{\partial n}\left[\left[(k_1-1)\log\xi+1\right]\tfrac{\xi^{1-k_1}}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IN8 &= tt5\ tt20\int \ \xi^{-k_1}\ d\xi = -tt5\ tt20\left[\tfrac{\xi^{1-k_1}}{(k_1-1)}\right]_{\xi_R}^{\xi}
\end{aligned}
\tag{A.14}
$$

where:

$$tt24 = \frac{\partial Z_0}{\partial n} \log \xi_R + \frac{Z_0}{\xi_R} \frac{\partial \xi_R}{\partial n} \tag{A.15}$$

$\boxed{INi}$

$$\begin{aligned}
IN9 &= \int \xi^{k_1-1} IN1 \, d\xi = \tfrac{\partial Z_0}{\partial n} \left[\tfrac{\xi\left[(\log \xi)^2 - 2\log \xi + 2\right]}{k_1 - 1} + \tfrac{2\xi(\log \xi - 1)}{(k_1-1)^2} + \tfrac{2}{(k_1-1)^3} \right]_{\xi_R}^{\xi} \\
IN10 &= \int \xi^{k_1-1} IN2 \, d\xi = -\tfrac{tt20}{(k_1-1)^2} \left[\xi \left[(k_1 - 1) \log \xi - k_1 + 2 \right] \right]_{\xi_R}^{\xi} \\
IN11 &= \int \xi^{k_1-1} IN3 \, d\xi = \tfrac{\partial Z_0}{\partial n} \tfrac{A}{9(k_1-3)^2} \left[\xi^3 \left[3(k_1 - 3) \log \xi - k_1 + 6 \right] \right]_{\xi_R}^{\xi} \\
IN12 &= \int \xi^{k_1-1} IN4 \, d\xi = -\tfrac{A \; tt20}{3(k_1-3)} \left[\xi^3 \right]_{\xi_R}^{\xi} \\
IN13 &= \int \xi^{k_1-1} IN5 \, d\xi = \tfrac{\partial Z_0}{\partial n} \tfrac{B}{16(k_1-4)^2} \left[\left[4(k_1 - 4) \log \xi - k_1 + 8 \right] \right]_{\xi_R}^{\xi} \\
IN14 &= \int \xi^{k_1-1} IN6 \, d\xi = \tfrac{B \; tt20}{4(k_1-4)} \left[\xi^4 \right]_{\xi_R}^{\xi} \\
IN15 &= \int \xi^{k_1-1} IN7 \, d\xi = \tfrac{\partial Z_0}{\partial n} \tfrac{tt5}{(k_1-1)^2} \left[\xi \left[(k_1 - 1) \log \xi - k_1 + 2 \right] \right]_{\xi_R}^{\xi} \\
IN16 &= \int \xi^{k_1-1} IN8 \, d\xi = -tt20 \; tt5 \tfrac{\xi}{k_1 - 1} \\
IN17 &= \int \xi^{k_1-1} (-tt21) \, d\xi = -tt21 \tfrac{\xi^{k_1}}{k_1}
\end{aligned} \tag{A.16}$$

$\boxed{ICi}$

$$\begin{aligned}
IC1 &= \int \xi^{-k_1-1}\xi \log\xi \; d\xi = \int \xi^{-k_1} \log\xi \; d\xi = \left[-\tfrac{[(k_1-1)\log\xi+1]\xi^{1-k_1}}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IC2 &= \int \tfrac{A}{3}\xi^{-k_1-1}\xi^3 \; d\xi = \tfrac{A}{3}\int \xi^{2-k_1} d\xi = \left[-\tfrac{A\xi^{3-k_1}}{3(k_1-3)}\right]_{\xi_R}^{\xi} \\
IC3 &= \int \tfrac{B}{4}\xi^{-k_1-1}\xi^4 \; d\xi = \tfrac{B}{4}\int \xi^{3-k_1} d\xi = \left[-\tfrac{B\xi^{4-k_1}}{4(k_1-4)}\right]_{\xi_R}^{\xi} \\
IC4 &= \int \xi^{-k_1-1}\xi(tt5-1) \; d\xi = (tt5-1)\int \xi^{-k_1} d\xi = \left[\tfrac{(1-tt5)\xi^{1-k_1}}{k_1-1}\right]_{\xi_R}^{\xi} \\
IC5 &= \int \xi^{-k_1-1}\xi(-tt6) \; d\xi = -tt6\left[\tfrac{\xi^{-k_1}}{k_1}\right]_{\xi_R}^{\xi}
\end{aligned} \tag{A.17}$$

where:

$$tt6 = \xi_0\left[\log\xi_0 + \frac{A}{3}\xi_0^3 + \frac{B}{4}\xi_0^4 + tt5 - 1\right]$$

$\boxed{ICi}$

$$
\begin{aligned}
IC6 &= \int \xi^{k_1-1} IC1 d\xi = \left[-\tfrac{\xi[(k_1-1)\log\xi - k_1+2]}{(k_1-1)^2}\right]_{\xi_R}^{\xi} \\
IC7 &= 3\int \xi^{k_1-1} IC2 d\xi = \left[-\tfrac{A\xi^3}{9(k_1-3)}\right]_{\xi_R}^{\xi} \\
IC8 &= 4\int \xi^{k_1-1} IC3 d\xi = \left[-\tfrac{B\xi^4}{16(k_1-4)}\right]_{\xi_R}^{\xi} \\
IC9 &= \int \xi^{k_1-1} IC4 d\xi = \left[\tfrac{(1-tt5)\xi}{k_1-1}\right]_{\xi_R}^{\xi} \\
IC10 &= \int \xi^{k_1-1} IC5 d\xi = \left[\tfrac{(tt6)\log\xi}{k_1}\right]_{\xi_R}^{\xi} \\
IC11 &= \int \xi^{k_1-1}(-tt7) d\xi = -tt7\tfrac{\xi^{k_1}}{k_1}
\end{aligned}
\qquad (A.18)
$$

Separately, we write the solution of the function $G12$. It's important to note that the functions $G_j \quad [j = 1, 2]$ will be always evaluated in the form $\mathcal{C}G_j \quad [j = 1, 2]$: in fact G_j appear ever coupled with curvature $\mathcal{C}$ in the secondary flow (2.131) and (2.228). The latter is a very important aspect from numerical point of view, avoiding numerical instability eventually due to the presence of $\mathcal{C}$ (which becomes zero at the inflection points) at denominator of equation (2.134) and (2.232).

$\boxed{IIG_{12}}$

$$\begin{aligned}
I04 &= \int \xi^{-k_1}\; G_{12}\; d\xi \\
II05 &= \int x^{k_1-1} I04\; d\xi \\
I06 &= \int G_{12} d\xi \\
II07 &= \int \xi^{-k_1-1}\; I06\; d\xi \\
III08 &= \int \xi^{k_1-1}\; II07\; d\xi \\
I13 &= \int \xi^{-k_1}\; [-\tfrac{\partial Z_0}{\partial} \log \xi + tt24] G_{12}\; d\xi \\
II14 &= \int \xi^{k_1-1}\; I13\; d\xi
\end{aligned} \tag{A.19}$$

A.2 Solution for sediment concentration between 0.314 and ξ

Analytical integrals used to find concentration value between 0.314 and ξ are here reported. For sake of clarity we will abbreviate the quantity $\int_{0.314}^{\xi}$ through the notation $\int$.

$\boxed{ITi}$

$$
\begin{aligned}
IT1 &= \int \exp\left(-k_2\xi\right)\log\xi\ d\xi = -\tfrac{1}{k_2}\Big[\exp\left(-k_2\xi\right)\log\xi - E_i(k_2\xi)\Big]_{0.314}^{\xi} \\
IT2 &= \int A\exp\left(-k_2\xi\right)\xi^2\ d\xi = -\tfrac{A}{k_2^3}\Big[\exp\left(-k_2\xi\right)(k_2^3\xi^2 + 2k_2\xi + 2)\Big]_{0.314}^{\xi} \\
IT3 &= \int B\exp\left(-k_2\xi\right)\xi^3\ d\xi = -\tfrac{B}{k_2^4}\Big[\exp\left(-k_2\xi\right)(k_2^3\xi^3 + 3k_2^2\xi^2 + 6k_2\xi + 6)\Big]_{0.314}^{\xi} \\
IT4 &= \int \exp -k_2\xi tt5\ d\xi = -\tfrac{tt5}{k_2}\Big[\exp\left(-k_2\xi\right)\Big]_{0.314}^{\xi}
\end{aligned}
\tag{A.20}
$$

where the exponential integral E_i is defined as follows:

$$
Ei(t*y) = Ei(1, t*y) = \int_1^{\infty}\frac{\exp\left(-t\right)}{t}\ dt
$$

$\boxed{ITi}$

$$
\begin{aligned}
IT5 &= \int \exp(-k_2\xi) IT1 \; d\xi = \\
&= \tfrac{1}{k_2}\left[\xi \log\xi - \xi + \tfrac{1}{k_2}\left(\exp(-k_2\xi)E_i(k_2\xi) - \log\xi\right)\right]_{0.314}^{\xi} \\
IT6 &= \int \exp(-k_2\xi) IT2 \; d\xi = -\tfrac{A}{k_2^3}\left[k_2^2\xi^2 + 2k_2\xi + 2\right]_{0.314}^{\xi} \\
IT7 &= \int \exp(-k_2\xi) IT3 \; d\xi = -\tfrac{B}{k_2^4}\left[\tfrac{k_2^3\xi^4}{4} + k_2^2\xi^3 + 3k_2\xi^2 + 6\xi\right]_{0.314}^{\xi} \\
IT8 &= \int \exp(-k_2\xi) IT4 \; d\xi = -\tfrac{tt5}{k_2}\left[\xi\right]_{0.314}^{\xi} \\
IT9 &= \int \exp(-k_2\xi)(-tt002) \; d\xi = -\tfrac{tt002}{k_2}\left[\exp(k_2\xi)\right]_{0.314}^{\xi}
\end{aligned}
\tag{A.21}
$$

$\boxed{ILi}$

$$
\begin{aligned}
IL1 &= \int \exp\left(-k_2\xi\right)\log\xi\xi\ d\xi = \\
&-\tfrac{1}{k_2^2}\left[\exp\left(-k_2\xi\right)(k_2\xi\log\xi+\log\xi+1)+E_i(k_2\xi)\right]_{0.314}^{\xi} \\
IL2 &= \int \exp\left(-k_2\xi\right)\tfrac{A}{3}\xi^3\ d\xi = \\
&-\tfrac{B}{3k_2^4}\left[\exp\left(-k_2\xi\right)(k_2^3\xi^3+3k_2^2\xi^2+6k_2\xi+6)\right]_{0.314}^{\xi} \\
IL3 &= \int \exp\left(-k_2\tfrac{B}{4}\xi\right)\xi^4\ d\xi = \\
&-\tfrac{B\ \ tt006}{k_2^5}\left[\exp\left(-k_2\xi\right)(k_2^4\xi^4+4k_2^3\xi^3+12k_2^2\xi^2+24k_2\xi+24)\right]_{0.314}^{\xi} \\
IL4 &= \int \exp\left(-k_2\xi\right)(tt5-1)\xi\ d\xi = \\
&-\tfrac{(tt5-1)}{k_2^2}\left[\exp\left(-k_2\xi\right)(k_2\xi+1)\right]_{0.314}^{\xi} \\
IL5 &= \int \exp\left(-k_2\xi\right)(-tt6)\xi\ d\xi = \\
&-\tfrac{(tt6)}{k_2}\left[\exp\left(-k_2\xi\right)\right]_{0.314}^{\xi}
\end{aligned}
\tag{A.22}
$$

$\boxed{ILi}$

$$
\begin{aligned}
IL6 &= \int \exp(k_2\xi) IL1 \; d\xi = \\
&-\tfrac{1}{k_2^2}\left[\tfrac{1}{4}(2k_2\xi\log\xi - k_2\xi + 4\log\xi) + \tfrac{1}{k_2}(\exp(-k_2\xi)E_i(k_2\xi) + \log\xi)\right]_{0.314}^{\xi} \\
IL7 &= \int \exp(k_2\xi) IL2 \; d\xi = \\
&-\tfrac{A}{3k_2^4}\left[\tfrac{k_2^3}{4}\xi^4 + k_2^2\xi^3 + 3k_2\xi^2 + 6\xi)\right]_{0.314}^{\xi} \\
IL8 &= \int \exp(k_2\xi) IL3 \; d\xi = \\
&-\tfrac{B}{4k_2^5}\left[\tfrac{k_2^4}{5}\xi^5 + k_2^3\xi^4 + 4k_2^2\xi^3 + 12k_2\xi^2 + 24\xi)\right]_{0.314}^{\xi} \\
IL9 &= \int \exp(k_2\xi) IL4 \; d\xi = -\tfrac{(tt5-1)}{k_2^2}\left[\left[\xi + \tfrac{k_2}{2}\xi^2\right]\right]_{0.314}^{\xi} \\
IL10 &= \int \exp(k_2\xi) IL5 \; d\xi = +\tfrac{tt6}{k_2}\left[\xi\right]_{0.314}^{\xi} \\
IL11 &= \int \exp(k_2\xi)(-tt012) \; d\xi = -\tfrac{tt012}{k_2}\left[\exp(k_2\xi)\right]_{0.314}^{\xi}
\end{aligned}
\tag{A.23}
$$

$\boxed{IRi}$

$$
\begin{aligned}
IR1 &= -tt003\ IT1 = -\frac{tt003}{k_2}\Big[\exp(-k_2\xi)\log\xi + E_i(k_2\xi)\Big]_{0.314}^{\xi} \\
IR2 &= tt006\ IL1\ d\xi = \\
&\quad -\frac{tt006}{k_2^2}\Big[\exp(-k_2\xi)(k_2\xi\log\xi + \log\xi + 1) + E_i(k_2\xi)\Big]_{0.314}^{\xi} \\
IR3 &= -tt003\ IT2 = -\frac{A\ tt003}{k_2^3}\Big[\exp(-k_2\xi)(k_2^3\xi^2 + 2k_2\xi + 2)\Big]_{0.314}^{\xi} \\
IR4 &= 3\ tt006\ IL2 = \\
&\quad -\frac{A\ tt006}{k_2^4}\Big[\exp(-k_2\xi)(k_2^3\xi^3 + 3k_2^2\xi^2 + 6k_2\xi + 6)\Big]_{0.314}^{\xi} \\
IR5 &= tt003\ IT3 = \\
&\quad -\frac{B\ tt003}{k_2^4}\Big[\exp(-k_2\xi)(k_2^3\xi^3 + 3k_2^2\xi^2 + 6k_2\xi + 6)\Big]_{0.314}^{\xi} \\
IR6 &= 4\ tt006\ IL3 = \\
&\quad -\frac{B\ tt006}{k_2^5}\Big[\exp(-k_2\xi)(k_2^4\xi^4 + 4k_2^3\xi^3 + 12k_2^2\xi^2 + 24k_2\xi + 24)\Big]_{0.314}^{\xi} \\
IR7 &= tt003\ IT4 = -\frac{tt5\ tt003}{k_2}\Big[\exp(-k_2\xi)\Big]_{0.314}^{\xi} \\
IR8 &= \int\ tt5\ tt006\exp(-k_2\xi)\xi\ d\xi = -\frac{tt5\ tt006}{k_2^2}\Big[\exp(-k_2\xi)(1 + k_2\xi)\Big]_{0.314}^{\xi}
\end{aligned}
\tag{A.24}
$$

$\boxed{IRi}$

$$
\begin{aligned}
IR9 &= \int \exp(-k_2\xi) IR1\ d\xi = -tt003\ IT6 \\
IR10 &= \int \exp(k_2\xi) IR2\ d\xi = tt006\ IL6 \\
IR11 &= \int \exp(k_2\xi) IR3\ d\xi = tt003\ IT6 \\
IR12 &= \int \exp(k_2\xi) IR4\ d\xi = 3\ tt006\ IL7 \\
IR13 &= \int \exp(k_2\xi) IR5\ d\xi = tt003\ IT7 \\
IR14 &= \int \exp(k_2\xi) IR6\ d\xi = 4\ tt006\ IL8 \\
IR15 &= \int \exp(k_2\xi) IR7\ d\xi = tt003\ IT8 \\
IR16 &= \int \exp(k_2\xi) IR8\ d\xi = -\tfrac{tt006\ \ tt5}{k_2^2}\left[\tfrac{k_2^2}{2}\xi^2 + \xi\right]_{0.314}^{\xi} \\
IR17 &= \int \exp(k_2\xi)(-tt008)\ d\xi = -\tfrac{tt008}{k_2}\left[\exp(-k_2\xi)\right]_{0.314}^{\xi}
\end{aligned}
\tag{A.25}
$$

where:

$$tt003 = \left(\log\frac{\xi_R}{0.314} + 1\right)\frac{\partial Z_0}{\partial\sigma} + \frac{Z_0}{\xi_R}\frac{\partial \xi_R}{\partial\sigma}$$

$$tt006 = -\frac{1}{0.314}\frac{\partial Z_0}{\partial\sigma}$$

$$tt008 = \sum_{i=1}^{8} IRi|_{\xi_R}$$

$\boxed{IIG_{12}}$

By analogy with the range between ξ_R and 0.314, we write solution for the terms with G_{12}:

$$
\begin{aligned}
I01 &= \int \exp -(k_{2\xi)} G_{12}\ d\xi \\
II02 &= \exp(k_2\xi) I01\ d\xi = \left[\frac{\exp(k_2\xi)}{k_2} I01\right]_{0.314}^{\xi} - \frac{1}{k_2}\int G_{12}\ d\xi \\
I15 &= \int \exp -(k_2\xi)\ (tt003 + tt006\xi)\ G_{12}\ d\xi \\
II16 &= \int \exp(k_2\xi)\ II15\ d\xi \\
I10 &= \int G_{12}\ d\xi \\
II11 &= \int \exp -(k_2\xi)\ G_{12}\ d\xi \\
III12 &= \int \exp(k_2\xi)\ II11\ d\xi = \left[\frac{\exp(k_2\xi)}{k_2} II11\right]_{0.314}^{\xi} - \frac{1}{k_2}\int I10\ d\xi
\end{aligned}
\tag{A.26}
$$

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