Notes on Fluid Dynamics

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- Acheson (1990),
- Aris (1962),
- Barenblatt (2003),
- Batchelor (1967),
- Ockendon and Ockendon (1995),
- Pozrikidis (2010).

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The state of stress in fluids

The continuum approach I

Definition of a simple fluid

The characteristic property of fluids (both liquids and gases) consists in the ease with which they can be deformed.

A proper definition of a fluid is not easy to state as, in many circumstances, it is not obvious to distinguish a fluid from a solid.

In this course we will deal with **"simple fluids"**, which Batchelor (1967) defines as follows.

"A simple fluid is a material such that the relative positions of elements of the material change by an amount which is not small when suitable chosen forces, however small in magnitude, are applied to the material. ... In particular a simple fluid cannot withstand any tendency by applied forces to deform it in a way which leaves the volume unchanged."

Note: the above definition does not imply that there will not be resistance to deformation. Rather, it implies that this resistance goes to zero as the rate of deformation vanishes.

Microscopic structure of fluids

The macroscopic properties of solids and fluids are related to their molecular nature and to the forces acting between molecules. In the figure a qualitative diagram of the force between two molecules as a function of their distance d is shown.

$d < d_0 \rightarrow$ **repulsion**; $d > d_0 \rightarrow$ **attraction**, where $d_0 \approx 10^{-10}$ m.
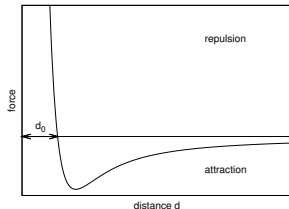
The continuum approach II

Let $\overline{d}$ be the average distance between molecules. We have

- **gases** $\rightarrow \overline{d} >> d_0$;
- **solids and liquids** $\rightarrow \overline{d} \approx d_0$.

In solids the relative position of particles is fixed, in fluids (liquids and gases) it can be freely rearranged.

Continuum assumption

Molecules are separated by voids and the percentage of volume occupied by molecules is very small compared to the total volume.

In most applications of fluid mechanics the typical spatial scale L under consideration is much larger than the spacing between molecules $\overline{d}$. We can then suppose that the behaviour of the fluid is the same as if the fluid was perfectly continuous in structure. This means that any physical property of the fluid, say f, can be regarded as a continuous function of space $\mathbf{x}$ (and possibly time t)

$$f = f(\mathbf{x}, t).$$

In order for the continuum approach to be valid it has to be possible to find a length scale L^* which is much smaller than the smallest spatial scale at which macroscopic changes take place and much larger than the microscopic (molecular) scale.

For instance in fluid mechanics normally a length scale $L^* = 10^{-5}$ m is much smaller than the scale of macroscopic changes but still we have $L^* \gg \overline{d}$.

Forces on a continuum I

Two kind of forces can act on a continuum body:

- **long distance forces;**
- **short distance forces.**

Long distance forces

Such forces are slowly varying in space. This means that if we consider a small volume δV the force is approximately constant over it. Therefore, we may write

$$\delta \mathbf{F} = \hat{\mathbf{f}} \delta V.$$

As long distance forces are proportional to the volume of fluid they act on, they are referred to as **volume or body forces**. In most cases of interest for this course $\delta \mathbf{F}$ will be proportional to the mass of the element

$$\delta \mathbf{F} = \rho \mathbf{f} \delta V,$$

where ρ denotes density, i.e. mass per unit volume. The dimensions of ρ are $[\rho] = ML^{-3}$ (with M mass and L length), and in the International System (SI) it is measured in kg m^{-3}.
The vector field $\mathbf{f}$ is denominated **body force field**. $\mathbf{f}$ has the dimension of an acceleration, or force per unit mass $[\mathbf{f}] = LT^{-2}$ (with T time), and in the SI it is measured in m/s^2.
In general $\mathbf{f}$ depends on space and time: $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$. If we want to compute the force $\mathbf{F}$ on a finite volume V we need to integrate $\mathbf{f}$ over V

$$\mathbf{F} = \iiint\limits_{V} \rho \mathbf{f} \, dV.$$

Forces on a continuum II

Short distance forces

Such forces are extremely rapidly variable in space and they act on very short distances. This means that short distance forces are only felt on the surface of contact between adjacent portions of fluid. Therefore, we may write

$$\delta \mathbf{\Sigma} = \mathbf{t} \delta S.$$

As short distance forces are proportional to the surface they act on, they are referred to as **surface forces**. The vector $\mathbf{t}$ is denominated **tension**. The tension $\mathbf{t}$ has the dimension of a force per unit surface $[\mathbf{t}] = FL^{-2} = ML^{-1}T^{-2}$, and in the SI it is measured in Pa=N m^{-2}.

The vector $\mathbf{t}$ depends on space $\mathbf{x}$, time t and on the unit vector $\mathbf{n}$ normal to the surface on which the stress acts: $\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n})$.

Convention: we assume that $\mathbf{t}$ is the force per unit surface that the fluid on the side of the surface towards which $\mathbf{n}$ points exerts on the fluid on the other side.

Important note: $\mathbf{t}(-\mathbf{n}) = -\mathbf{t}(\mathbf{n})$.

If we want to compute the force $\mathbf{\Sigma}$ on a finite surface S we need integrating $\mathbf{t}$ over S:

$$\mathbf{\Sigma} = \iint\limits_{S} \mathbf{t} \, dS.$$

Note that, if S is a closed surface, $\mathbf{\Sigma}$ represents the force that the fluid outside of S exerts on the fluid inside.

The stress tensor I

Cauchy's stress principle

We now wish to characterise the state of stress at a point P of a continuum. To this end we consider a small tetrahedron of volume δV centred in P. In the figure on the right $\mathbf{e}_i$ denotes the unit vector in the direction of the axis x_i ($i = 1, 2, 3$).

The total surface force acting on the tetrahedron is

$$\mathbf{t}(\mathbf{n})\delta S + \mathbf{t}(-\mathbf{e}_1)\delta S_1 + \mathbf{t}(-\mathbf{e}_2)\delta S_2 + \mathbf{t}(-\mathbf{e}_3)\delta S_3 = 0.$$
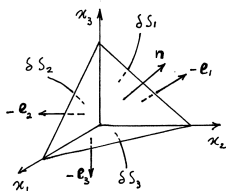
In the above expression we have not displayed the dependence of $\mathbf{t}$ on $\mathbf{x}$, as the value of $\mathbf{x}$ is approximately constant over the small tetrahedron. Moreover, t is fixed.

Note that if we wrote the momentum balance for the tetrahedron, volume forces would vanish more rapidly than surface forces as the volume tends to zero. Therefore, at leading order, only surface forces contribute to the balance.

We note that

$$\delta S_i = \mathbf{e}_i \cdot \mathbf{n}\delta S.$$

Therefore

$$\delta S \left[\mathbf{t}(\mathbf{n}) - \mathbf{t}(\mathbf{e}_1)\mathbf{e}_1 \cdot \mathbf{n} - \mathbf{t}(\mathbf{e}_2)\mathbf{e}_2 \cdot \mathbf{n} - \mathbf{t}(\mathbf{e}_3)\mathbf{e}_3 \cdot \mathbf{n} \right] = 0,$$

The stress tensor II

or, in index notation,

$$\delta S \left[t_i(\mathbf{n}) - t_i(\mathbf{e}_1)e_{1j}n_j - t_i(\mathbf{e}_2)e_{2j}n_j - t_i(\mathbf{e}_3)e_{3j}n_j \right] = 0.$$

Note that, throughout the course we will adopt **Einstein notation** or **Einstein summation convention**. According to this convention, when an index variable appears twice in a single term of a mathematical expression, it implies that we are summing over all possible values of the index (typically 1, 2, 3). Thus, for instance

$$f_j g_j = f_1 g_1 + f_2 g_2 + f_3 g_3 \qquad \text{or} \qquad f_j \frac{\partial f_i}{\partial x_j} = f_1 \frac{\partial f_i}{\partial x_1} + f_2 \frac{\partial f_i}{\partial x_2} + f_3 \frac{\partial f_i}{\partial x_3}.$$

We can now write

$$t_i(\mathbf{n}) = \left[t_i(\mathbf{e}_1)e_{1j} + t_i(\mathbf{e}_2)e_{2j} + t_i(\mathbf{e}_3)e_{3j} \right] n_j.$$

Since neither the vector $\mathbf{t}$ nor $\mathbf{n}$ depend on the coordinate system, the term in square brackets in the above equation is also independent of it. Thus it represents a second order tensor, say $\boldsymbol{\sigma}$ (or in index notation σ_{ij}).
We can thus write

$$t_i(\mathbf{n}) = \sigma_{ij}n_j, \quad \text{or, in vector notation,} \quad \mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}. \tag{1}$$

σ_{ij} is named the **Cauchy stress tensor**, or simply **stress tensor**. σ_{ij} represents the i component of the stress on the plane orthogonal to the unit vector $\mathbf{e}_j$.

The stress tensor III

Equation (1) implies that to characterise the stress in a point of a continuum we need a second order tensor, i.e. (given a coordinate system) 9 scalar quantities. We will show in the following (section 5) that σ_{ij} is symmetric ($\sigma_{ij} = \sigma_{ji}$), and therefore such scalar quantities reduce to 6.

The terms appearing in the principal diagonal of the matrix σ_{ij} represent the so called **normal stresses**, those out of the principal diagonal are named **tangential or shear stresses**.

It is always possible to choose Cartesian coordinates such that $\boldsymbol{\sigma}$ takes a diagonal form

$$
\left(\begin{array}{ccc}
\sigma_I & 0 & 0 \\
0 & \sigma_{II} & 0 \\
0 & 0 & \sigma_{III}
\end{array} \right),
$$

and σ_I, σ_{II}, σ_{III} are named **principal stresses** and they are the eigenvalues of the matrix representing σ_{ij}. The corresponding directions are called **principal directions**.

Obviously, the components of σ_{ij} depend on the coordinate system but the stress tensor does not as it is a quantity with a precise physical meaning.

For any second order tensor it is possible to define 3 invariants, i.e. 3 quantities that do not depend on the choice of the coordinate system. A commonly used set of invariants is given by

$$
\mathcal{I}_1 = \sigma_I + \sigma_{II} + \sigma_{III} = \operatorname{tr} \boldsymbol{\sigma} = \sigma_{jj}, \quad \mathcal{I}_2 = \sigma_I \sigma_{II} + \sigma_{II} \sigma_{III} + \sigma_{III} \sigma_I, \quad \mathcal{I}_3 = \sigma_I \sigma_{II} \sigma_{III} = \det \boldsymbol{\sigma}.
$$

Tension in a fluid at rest I

The structure of $\boldsymbol{\sigma}$ in a fluid at rest is a consequence of the definition of **simple fluid** put forward. We consider a small spherical domain in a fluid at rest. Since the sphere is very small $\boldsymbol{\sigma}$ must be approximately constant at all points within the sphere. We locally choose the principal axes so that we can write $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = \left(\begin{array}{ccc} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{array} \right).$$

We can now write $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$, where

$$\boldsymbol{\sigma}_1 = \left(\begin{array}{ccc} 1/3\sigma_{jj} & 0 & 0 \\ 0 & 1/3\sigma_{jj} & 0 \\ 0 & 0 & 1/3\sigma_{jj} \end{array} \right), \qquad \boldsymbol{\sigma}_2 = \left(\begin{array}{ccc} \sigma_I - 1/3\sigma_{jj} & 0 & 0 \\ 0 & \sigma_{II} - 1/3\sigma_{jj} & 0 \\ 0 & 0 & \sigma_{III} - 1/3\sigma_{jj} \end{array} \right).$$

The tensor $\boldsymbol{\sigma}_1$ is spherical. It represents a normal compression on the sphere (see figure (a) below). In fact on any portion δS of normal $\mathbf{n}$ the force is given by $\delta S \boldsymbol{\sigma}_1 \mathbf{n} = 1/3\delta S \sigma_{jj} \mathbf{n}$.

The second tensor $\boldsymbol{\sigma}_2$ is diagonal and the sum of the terms on the diagonal is zero. This means that, excluding the trivial case in which all terms are zero, at least one term is positive and one is negative.

Referring to the figure on the right this implies that this state of stress necessarily tend to change the shape of the small volume we are considering. This is not compatible with the definition of simple fluid given before, according to which such fluid is not able to withstand a system of forces that tends to change its shape.

(a)   (b)
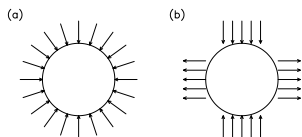
Tension in a fluid at rest II

Therefore, $\boldsymbol{\sigma}_2$ must be equal to zero in a fluid at rest. Since fluids are normally in a state of compression we set

$$\sigma_{ij} = -p\delta_{ij}, \qquad \text{or, in vector form,} \qquad \boldsymbol{\sigma} = -p\mathbf{I}, \qquad (2)$$

where the scalar quantity p is called **pressure**, and $\mathbf{I}$ is the identity matrix. Note that, due to the minus sign in the above equation, $p > 0$ implies compression. In general the pressure is a function of space and time $p(\mathbf{x}, t)$. p has the dimension of a force per unit area ($[p] = FL^{-2} = ML^{-1}T^{-2}$) and in the SI is measured in Pa.

Equation (2) implies that at a given point P of a fluid at rest the force acting on a small surface passing from P is equal to $-p\mathbf{n}$, i.e. it is always normal to the surface and its magnitude does not depend on the orientation of the surface.

Note: in some textbooks (2) is assumed as an indirect definition of a simple fluid **(Euler assumption)**.

Statics of fluids

The equation of statics I

Equation of statics in integral form

Let V be a volume of fluid within a body of fluid at rest and let S be its bounding surface. We wish to write the equilibrium equation for this volume. From the equilibrium of forces we have

$$\iiint\limits_V \rho \mathbf{f} dV + \iint\limits_S \mathbf{t} dS = 0. \tag{3}$$

Equation (2) allows to rewrite the above expression as

$$\iiint\limits_V \rho \mathbf{f} dV + \iint\limits_S -p\mathbf{n} dS = 0, \tag{4}$$

which represents the **integral form of the equation of statics**. The above equation is often conveniently written in compact form as

$$\mathbf{F} + \mathbf{\Sigma} = 0, \tag{5}$$

with $\mathbf{F}$ resultant of all body forces acting on V and $\mathbf{\Sigma}$ resultant of surface forces acting on S.

The equation of statics II

Equation of statics in differential form

Using Gauss theorem equation (4) can be written as

$$\iiint\limits_{V} \rho \mathbf{f} - \nabla p \, dV = 0.$$

Since V is arbitrary the following differential equation must hold

$$\rho \mathbf{f} - \nabla p = 0, \qquad \text{or, in index notation,} \qquad \rho f_i - \frac{\partial p}{\partial x_i} = 0, \tag{6}$$

which is the **equation of statics in differential form**.

Equilibrium to rotation

In principle, the above equation alone is not sufficient to ensure equilibrium as we also have to impose an equilibrium balance to rotation. This can be written as

$$\iiint\limits_{V} \rho \mathbf{x} \times \mathbf{f} \, dV + \iint\limits_{S} -p \mathbf{x} \times \mathbf{n} \, dS = 0, \tag{7}$$

The equation of statics III

or, in index notation,

$$\iiint\limits_V \rho\epsilon_{ijk}x_j f_k \, dV + \iint\limits_S -p\epsilon_{ijk}x_j n_k \, dS = 0. \tag{8}$$

Note: ϵ_{ijk} is the **alternating tensor**. Its terms are all equal to zero unless when i, j and k are different from each other, in which case ϵ_{ijk} takes the values 1 or -1 depending if i, j and k are or not in cyclic order. Thus, we have

i	j	k	ϵ_{ijk}
1	2	3	1
3	1	2	1
2	3	1	1
2	1	3	-1
1	3	2	-1
3	2	1	-1

Applying Gauss theorem to equation (8) we have:

$$\iiint\limits_V \epsilon_{ijk}\left[\rho x_j f_k - \frac{\partial}{\partial x_k}(px_j)\right] dV = 0.$$

The equation of statics IV

Carrying on the calculations:

$$
\begin{aligned}
&\iiint\limits_V \epsilon_{ijk} \left(\rho x_j f_k - \frac{\partial}{\partial x_k}(px_j) \right) dV = \\
&\iiint\limits_V \epsilon_{ijk} \left(\rho x_j f_k - x_j \frac{\partial p}{\partial x_k} - p \frac{\partial x_j}{\partial x_k} \right) dV = \\
&\iiint\limits_V \epsilon_{ijk} \left(\rho x_j f_k - x_j \frac{\partial p}{\partial x_k} - p \delta_{jk} \right) dV = \\
&\iiint\limits_V \epsilon_{ijk} \left(\rho x_j f_k - x_j \frac{\partial p}{\partial x_k} \right) dV - \iiint\limits_V p \epsilon_{ijk} \delta_{jk} dV = 0,
\end{aligned}
\tag{9}
$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$). The above equation is automatically satisfied as the first integral vanishes due to equation (6) and $\epsilon_{ijk}\delta_{ij} = 0$ by definition.

Implications of the equation of statics

Let us now consider the equation of statics (6). In order to integrate this equation we need an equation of state for the fluid, stating how the density ρ depends on the other physical properties of the fluid, and in particular p.

However, some general conclusions can be drawn by simple inspection of the equation.

- As a first consideration we note that not all $\mathbf{f}(\mathbf{x})$ and $p(\mathbf{x})$ allow for a fluid to be at rest. In particular the relationship $\rho \mathbf{f}(\mathbf{x}) = \nabla p$ implies that $\rho \mathbf{f}(\mathbf{x})$ admits a potential W, so that

$$\rho \mathbf{f}(\mathbf{x}) = -\nabla W.$$

- In the particular case in which $\rho = const$, $\mathbf{f}$ has to be **conservative**.
- If $\mathbf{f}$ is conservative we have that $\mathbf{f} = -\nabla \phi$. In this case we have

$$-\rho \nabla \phi = \nabla p.$$

Applying the curl to the above expression we find

$$-\nabla \times (\rho \nabla \phi) = \nabla \times \nabla p \quad \Rightarrow \quad -\nabla \rho \times \nabla \phi - \rho \cancel{\nabla \times \nabla \phi} = \cancel{\nabla \times \nabla p}.$$

The above relationship implies that level surfaces of ρ and ϕ must coincide.

Statics of incompressible fluids in the gravitational field

We assume

- $\rho = const$. In this case we say that the fluid behaves as if it was **incompressible**.
- **f** is the gravitational body force field.

We consider a system of Cartesian coordinates (x_1, x_2, x_3), with x_3 vertical upward directed axis. The gravitational field can therefore be written as $\mathbf{f} = (0, 0, -g)$.

With the above assumptions equation (6) can be easily solved to get

$$p = -\rho g x_3 + const,$$

and, after rearrangement, we obtain **Stevin law**

$$x_3 + \frac{p}{\gamma} = const, \tag{10}$$

where γ is the **specific weight** of the fluid ($[\gamma] = FL^{-3}$, measured in N m^{-3} in the SI).
The quantity $h = x_3 + p/\gamma$ is called **piezometric or hydraulic head**. Stevin law implies that, in an incompressible fluid at rest, h is constant.

Equilibrium conditions at interfaces I

Surface tension

The fact that small liquid drops form in air and gas bubbles form in liquids can be explained by assuming that a **surface tension** acts at the interface between the two fluids.

If we draw a curve across the interface we assume that a **force per unit length** of magnitude κ exists, acting on the surface containing the interface and in the direction orthogonal to the curve. The dimension of κ is $[\kappa] = FL^{-1} = MT^{-2}$ ans in the SI is measured in N m^{-1}.

Drop of water on a leaf.

The existence of such a force can be explained considering what happens at molecular level, close to the interface: due to the existence of the interface, there is no balance of molecular forces acting on particles very close to the interface.

κ can be positive (traction force on the surface) or negative (compression force on the surface), depending on the two fluids in contact. In particular we have:

- $\kappa > 0$ **immiscible fluids**;
- $\kappa < 0$ **miscible fluids**.

Equilibrium conditions at interfaces II

Pressure jump across a curved surface

We consider an equilibrium interface between two fluids. This implies that $\kappa = const$ on the surface.

We consider a curved surface. Let O be a point on the surface and let us adopt a system of coordinates centred in O and such that the $(x - y)$ plane is tangent to the surface. The equation of the surface is

$$F(x, y, z) = z - \zeta(x, y) = 0. \tag{11}$$

Note that ζ and its first derivatives are zero at $(x, y) = (0, 0)$. Close to O the approximate expression of the normal vector $\mathbf{n}$ is

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} \approx \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1 \right),$$

correct to the first order in the small quantities $\partial \zeta / \partial x$, $\partial \zeta / \partial y$.

The resultant of the tensile force on a small portion of the surface S containing O is given by

$$-\kappa \oint_C \mathbf{n} \times d\mathbf{x},$$

with $\mathbf{n}$ normal to the surface and $d\mathbf{x}$ a line element of the closed curve C bounding the surface S. Recalling the equation of the surface (11) we can write $d\mathbf{x} = (dx, dy, \frac{\partial \zeta}{\partial x} dx + \frac{\partial \zeta}{\partial y} dy)$.

Equilibrium conditions at interfaces III

If the surface is flat, $\mathbf{n}$ is uniform and the above integral is zero. If the surface is curved the resultant is directed, at leading order, along z and has magnitude

$$-\kappa \oint_C -\frac{\partial \zeta}{\partial x} dy + \frac{\partial \zeta}{\partial y} dx.$$

Green's theorem states that

$$\iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \oint_C f dx + g dy.$$

In the present case the above equation can be specified so that

$$f = -\frac{\partial \zeta}{\partial y}, \qquad g = \frac{\partial \zeta}{\partial x}.$$

Therefore we get:

$$\kappa \oint_C \frac{\partial \zeta}{\partial x} dy - \frac{\partial \zeta}{\partial y} dx = \kappa \iint_S \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) dS \approx \kappa \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)_O S.$$

We finally find

$$\kappa \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) = \kappa \left(\frac{1}{R_1} + \frac{1}{R_2} \right),$$

Equilibrium conditions at interfaces IV

where R_1 and R_2 are radii of curvature of the surface along two orthogonal directions. Note that it can be shown that $\frac{1}{R_1} + \frac{1}{R_2}$ is independent on the orientation chosen.

The above equation implies that, in order for a curved interface between two fluids to be in equilibrium, a pressure jump Δp must exist across the surface so that

$$\Delta p = \kappa \left(\frac{1}{R_1} + \frac{1}{R_2} \right). \tag{12}$$

Hydrostatic forces on flat surfaces I

We adopt in this section the following assumptions:

- $\rho = const$;
- $\mathbf{f} = (0, 0, -g)$ gravitational field.

We wish to compute the force on a flat solid surface.

Magnitude of the force
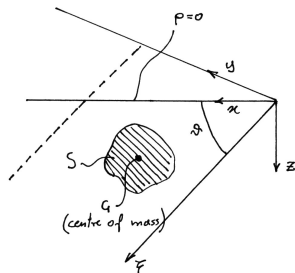
The magnitude of this force is given by

$$|\mathbf{\Sigma}| = \left| \iint\limits_{S} -p\mathbf{n}dS \right| = \iint\limits_{S} pdS.$$

Note that, by definition, $\mathbf{\Sigma}$ is the force that the surface exerts on the fluid. Thus the force of the fluid on the surface is equal to $-\mathbf{\Sigma}$.

We consider a plane inclined by an angle ϑ with respect to a horizontal plane and introduce a coordinate ζ, with origin on the horizontal plane where $p = 0$, laying on the surface and oriented along the line of maximum slope on the surface. We therefore can write, using equation (10),

$$p = \zeta\gamma \sin\vartheta.$$

Hydrostatic forces on flat surfaces II

Substituting in the definition of $|\boldsymbol{\Sigma}|$ we obtain

$$|\boldsymbol{\Sigma}| = \iint\limits_{S} \zeta \gamma \sin \vartheta dS = \gamma \sin \vartheta \iint\limits_{S} \zeta dS = \gamma \sin \vartheta \mathcal{S}, \tag{13}$$

where $\mathcal{S}$ is the **static moment** of the surface S with respect to the axis y, defined as

$$\mathcal{S} = \iint\limits_{S} \zeta dS. \tag{14}$$

$\mathcal{S}$ can be written as $\mathcal{S} = \zeta_G S$, with ζ_G being the ζ coordinate of the **centre of mass** of S. Thus we can write

$$|\boldsymbol{\Sigma}| = \gamma \sin \vartheta \zeta_G S = \gamma z_G S = p_G S, \tag{15}$$

where z is a vertical coordinate directed downwards and with origin on the horizontal plane $p = 0$ (see the figure of the previous page), and p_G is the pressure in the centre of mass of S.

Equation (15) states that **the magnitude of the force exerted by an incompressible fluid at rest in the gravitational field on a flat surface is given by the product of the pressure p_G at the centre of mass of the surface and the area of the surface S.**

Hydrostatic forces on flat surfaces III

Application point of the force

We now wish to determine where the force $\boldsymbol{\Sigma}$ is applied. To this end we impose the equilibrium to rotation with respect to the y axis, given by the intersection of the planes $\zeta = 0$ and $z = 0$,

$$\zeta_C |\boldsymbol{\Sigma}| = \iint\limits_S \zeta p dS = \iint\limits_S \gamma \sin \vartheta \zeta^2 dS = \gamma \sin \vartheta \iint\limits_S \zeta^2 dS = \gamma \sin \vartheta \mathcal{I}, \tag{16}$$

where we have introduced the **moment of inertia** $\mathcal{I}$ of the surface with respect to the axis y

$$\mathcal{I} = \iint\limits_S \zeta^2 dS. \tag{17}$$

Finally, we recall that

$$\mathcal{I} = \mathcal{I}_0 + \zeta_G^2 S, \tag{18}$$

where $\mathcal{I}_0$ is the moment of inertia of the surface with respect to an axis parallel to y and passing through the centre of mass of the surface. Thus, substituting (18) into (16) and recalling equations (13) and (14) we obtain

$$\zeta_C = \zeta_G + \frac{\mathcal{I}_0}{\zeta_G S} = \zeta_G + \frac{\mathcal{I}_0}{S}, \tag{19}$$

which is often more convenient to use than (16).

Hydrostatic forces of curved surfaces

In the case of forces on curved surfaces it is not possible to take the normal vector $\mathbf{n}$ out of the following integral

$$\mathbf{\Sigma} = -\iint\limits_{S} p\mathbf{n}dS,$$

as $\mathbf{n}$ changes from point to point on S. In this case it is necessary to specify explicitly $\mathbf{n}$ and solve the integral.

An alternative, often more convenient, method consists of selecting a closed control volume, bounded by the curved surface and by a suitable number of flat surfaces. In this case the calculation of the forces on the flat surfaces is straightforward and the force on the curved surface can be determined employing the integral form of the statics equation (4), provided it is possible to compute the volume of the control volume.

Kinematics of fluids

Spatial and material coordinates I

The **kinematics of fluids** studies fluid motion per se, with no concern to the forces which generate the motion. All kinematic notions that will be introduced in the present chapter are valid for any fluid described as a continuum. A very good reference for kinematics of fluid is Aris (1962); the present section is largely based on this textbook.

Understanding how to study fluid motion from the kinematic point of view is a prerequisite to study the dynamics of fluids, which will be considered in the following chapter.

The basic mathematical idea is that, within the continuum approach, fluid motion can be described by a **point transformation**.

Material coordinates

Let us consider a fluid particle which at time t_0 is located in the position $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. The same particle at time t is at position $\mathbf{x} = (x_1, x_2, x_3)$. Without loss of generality we can set $t_0 = 0$. The motion of the particle in the time interval $[0, t]$ is described by the function

$$\mathbf{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}, t), \qquad \text{or, in index notation,} \qquad x_i = \varphi_i(\xi_1, \xi_2, \xi_3, t), \qquad (20)$$

which, at any time t, tells us the position in space of the particle that was in $\boldsymbol{\xi}$ at $t = 0$.

- If in the equation (20) we fix $\boldsymbol{\xi}$ we obtain the **particle trajectory**.
- $\boldsymbol{\xi}$ are named **material or Lagrangian coordinates** as a particular value of $\boldsymbol{\xi}$ identifies the material particle that at $t = 0$ was in $\boldsymbol{\xi}$.

Spatial and material coordinates II

Particle velocity

We can define the particle velocity as

$$\mathbf{u} = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{\xi}, t)}{\partial t}.$$

Particle acceleration

We can define the particle acceleration as

$$\mathbf{a} = \frac{\partial \mathbf{u}(\boldsymbol{\xi}, t)}{\partial t} = \frac{\partial^2 \boldsymbol{\varphi}(\boldsymbol{\xi}, t)}{\partial t^2}.$$

Spatial and material coordinates III

Spatial coordinates

We assume that the motion is continuous, that at a given time a single particle cannot occupy two different positions and, conversely, that a single point in space cannot be occupied simultaneously by two particles. This implies that equation (20) can be inverted to obtain

$$\boldsymbol{\xi} = \boldsymbol{\Phi}(\mathbf{x}, t), \qquad \text{or, in index notation,} \qquad \xi_i = \Phi_i(x_1, x_2, x_3, t). \tag{21}$$

Equation (21) gives the initial position (at $t = 0$) of a material particle that at time t is in $\mathbf{x}$. Mathematically, the condition of invertibility of (20) can be expressed as $J > 0$ (see Aris, 1962), where the Jacobian J is defined as

$$J = \det\left[\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}\right]. \tag{22}$$

Knowledge of equation (20) or (21) is enough to completely describe the flow. The flow, however, can also be studied by describing how any fluid property, say $\mathcal{F}$ (e.g. density, pressure, velocity, . . .) changes in time at any position in space.

$$\mathcal{F} = f_S(\mathbf{x}, t).$$

This approach is referred to as **spatial approach** or **Eulerian approach**.

Spatial and material coordinates IV

Alternatively, we can describe the evolution of a fluid property $\mathcal{F}$ associated with a given fluid particle. In this case we write

$$\mathcal{F} = f_M(\boldsymbol{\xi}, t).$$

Note that a given value of $\boldsymbol{\xi}$ identifies the particle that in $t = 0$ was in $\boldsymbol{\xi}$. This approach is referred to as **material approach** or **Lagrangian approach**.

The material derivative I

The time derivative of a generic physical property of the fluid $\mathcal{F}$ has a different meaning in Eulerian and Lagrangian coordinates.

- Eulerian coordinates:

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{\partial f_S(\mathbf{x}, t)}{\partial t} \qquad \textbf{local derivative}.$$

  This represents the variation in time of $\mathcal{F}$ at a given point in space. Such a point can, in general, be occupied by different particles at different times.

- Lagrangian coordinates:

$$\frac{D \mathcal{F}}{D t} = \frac{\partial f_M(\boldsymbol{\xi}, t)}{\partial t} \qquad \textbf{material derivative}.$$

  This represents the time evolution of $\mathcal{F}$ associated with a given material particle.

The material derivative II

Since the physical meaning of the two derivatives is different it is customary in fluid mechanics to denote them with different symbols.

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} \equiv \text{time derivative at constant } \mathbf{x},$$

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{\xi}} \equiv \text{time derivative at constant } \boldsymbol{\xi}.$$

Note: If the fluid property $\mathcal{F}$ is the position of a material particle ($\mathcal{F} = \varphi_i$) we have

$$u_i = \frac{D\varphi_i}{Dt}, \quad \text{or in vector notation} \quad \mathbf{u} = \frac{D\boldsymbol{\varphi}}{Dt}, \tag{23}$$

which is the velocity of the fluid particle.

In general it is more convenient in fluid mechanics to adopt a spatial (Eulerian) description of the flow. However, for the definition of some physical quantities the material derivative is required. For instance the acceleration $\mathbf{a}$ is defined as

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt},$$

while $\partial \mathbf{u}/\partial t \neq \mathbf{a}$, as it represents the rate of change of velocity at a fixed point in space, i.e. it is not referred to a material particle.

The material derivative III

It is then often necessary to define the material derivative in terms of spatial coordinates. Using (20) we can write

$$f_S(\mathbf{x}, t) = f_S[\boldsymbol{\varphi}(\boldsymbol{\xi}, t), t] = f_M(\boldsymbol{\xi}, t),$$

from which it follows

$$\frac{D\mathcal{F}}{Dt} = \frac{\partial f_M(\boldsymbol{\xi}, t)}{\partial t} = \frac{\partial f_S}{\partial t} + \frac{\partial f_S}{\partial x_i}\frac{\partial \varphi_i}{\partial t} = \frac{\partial f_S}{\partial t} + u_i \frac{\partial f_S}{\partial x_i}.$$

Thus we find

$$\frac{D\mathcal{F}}{Dt} = \frac{\partial \mathcal{F}}{\partial t} + u_i \frac{\partial \mathcal{F}}{\partial x_i}, \qquad \text{or, in vector form,} \qquad \frac{D\mathcal{F}}{Dt} = \frac{\partial \mathcal{F}}{\partial t} + \mathbf{u} \cdot \nabla \mathcal{F}. \qquad (24)$$

If $\mathcal{F}$ is a vector quantity we obtain

$$\frac{D\mathcal{F}_i}{Dt} = \frac{\partial \mathcal{F}_i}{\partial t} + u_j \frac{\partial \mathcal{F}_i}{\partial x_j}, \qquad \text{or in vector form,} \qquad \frac{D\boldsymbol{\mathcal{F}}}{Dt} = \frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\mathcal{F}}. \qquad (25)$$

Definition of some kinematic quantities I

Trajectories or particle paths

Equation (20) can be seen as a parametric equation of a curve in space, with t as parameter. The curve goes through the point $\boldsymbol{\xi}$ when $t = 0$. It represents the **particle path** or **pathline** or **particle trajectory**.

Particle trajectories can be obtained from a spatial description of the flow by integration of the spatial velocity field

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \qquad \mathbf{x}(0) = \boldsymbol{\xi}. \qquad (26)$$

Steady flow

The velocity field in spatial coordinates is described by the vector field $\mathbf{u}(\mathbf{x}, t)$. If $\mathbf{u}$ does not depend on time the flow is said to be **steady**. Note that steadiness of flow does not imply that each material particle has a constant velocity in time as $\mathbf{u}(\boldsymbol{\xi}, t)$ might still depend on time.

Streamlines

Given a spatial description of a velocity field $\mathbf{u}(\mathbf{x}, t)$, streamlines are curves which are at all points in space parallel to the velocity vector. Mathematically, they are therefore defined as

$$d\mathbf{x} \times \mathbf{u} = 0, \qquad (27)$$

Definition of some kinematic quantities II

with $d\mathbf{x}$ an infinitesimal segment along the streamline. The above expression can also be written as

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3}. \tag{28}$$

The unit vector $d\mathbf{x}/|d\mathbf{x}|$ can be written as $d\mathbf{x}/ds$, where the curve parameter s is the arc length measured from an initial point $\mathbf{x}_0 = \mathbf{x}(s = 0)$. Equation (27) then implies that

$$\frac{d\mathbf{x}}{ds} = \frac{\mathbf{u}}{|\mathbf{u}|}. \tag{29}$$

Particle paths and streamlines are not in general coincident. However they are in the following cases.

- **Steady flow.** In this case the equation for a pathline is $\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x})$. The element of the arc length along the pathline is $ds = |\mathbf{u}|dt$, which, substituted in the above expression, yields

$$\frac{d\mathbf{x}}{ds} = \frac{\mathbf{u}}{|\mathbf{u}|},$$

  which shows that, for a steady flow, the differential equation for pathlines and streamlines are the same.

Definition of some kinematic quantities III

- **Unsteady flow the direction of which does not change with time.** In this case we can write $\mathbf{u}(\mathbf{x}, t) = f(\mathbf{x}, t)\mathbf{u_0}(\mathbf{x})$ with $\mathbf{u_0}$ the velocity field at the initial time. In this case the argument used for steady flows still holds.

Uniform flow

A flow is said to be **uniform** if $\mathbf{u}$ does not depend on $\mathbf{x}$.

$$\mathbf{u} = \mathbf{u}(t).$$

This is a very strong requirement. Sometimes the flow is called uniform if $\mathbf{u}$ does not change along the streamlines.

Plane flow

A flow is said to be **plane** or **two-dimensional** if it is everywhere orthogonal to one direction and independent of translations along such direction.

In a plane flow it is therefore possible to choose a system of Cartesian coordinates (x_1, x_2, x_3) so that $\mathbf{u}$ has the form

$$\mathbf{u} = (u_1, u_2, 0),$$

and u_1 and u_2 do not depend on x_3.

Axisymmetric flow

A flow is said to be **axisymmetric** if, chosen a proper system of cylindrical coordinates (z, r, φ) the velocity $\mathbf{u} = (u_z, u_r, u_\varphi)$ is independent of the azimuthal coordinate φ, and $u_\varphi = 0$.

Reynolds transport theorem I

Let $\mathcal{F}(\mathbf{x}, t)$ be a property, either a scalar or a vector, of the fluid and $V(t)$ a **material volume** entirely occupied by the fluid. A material volume is a volume which is always constituted by the same particles. We can define the integral

$$F(t) = \iiint\limits_{V(t)} \mathcal{F}(\mathbf{x}, t) dV. \tag{30}$$

We wish to evaluate the material derivative of F. Since $V(t)$ depends on time the derivative D/Dt can not be taken into the integral. However, if we work with material coordinates $\boldsymbol{\xi}$, the volume remains unchanged in time and equal to the value V_0 it had at the initial time. We can thus write

$$\frac{D}{Dt} \iiint\limits_{V(t)} \mathcal{F}(\mathbf{x}, t) dV = \frac{D}{Dt} \iiint\limits_{V_0} \mathcal{F}(\boldsymbol{\xi}, t) J dV_0,$$

where $dV = J dV_0$, with J Jacobian of the transformation, defined by (22). We can now write

$$\iiint\limits_{V_0} J \frac{D\mathcal{F}}{Dt} + \mathcal{F} \frac{DJ}{Dt} dV_0.$$

It can be shown (see section 12) that

$$\frac{DJ}{Dt} = (\nabla \cdot \mathbf{u}) J. \tag{31}$$

Reynolds transport theorem II

The above integral can then be written as

$$\iiint\limits_{V_0} \left[\frac{D\mathcal{F}}{Dt} + \mathcal{F}(\nabla \cdot \mathbf{u}) \right] J dV_0,$$

and going back to the spatial coordinates $\mathbf{x}$, we find

$$\iiint\limits_{V(t)} \left[\frac{D\mathcal{F}}{Dt} + \mathcal{F}(\nabla \cdot \mathbf{u}) \right] dV.$$

Finally, recalling (24), we have

$$\frac{D}{Dt} \iiint\limits_{V(t)} \mathcal{F}(\mathbf{x}, t) dV = \iiint\limits_{V(t)} \left[\frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot (\mathcal{F}\mathbf{u}) \right] dV. \tag{32}$$

This result is known as **Reynolds transport theorem**. The above expression can be also be written as

$$\frac{D}{Dt} \iiint\limits_{V(t)} \mathcal{F}(\mathbf{x}, t) dV = \iiint\limits_{V(t)} \frac{\partial \mathcal{F}}{\partial t} dV + \iint\limits_{S(t)} \mathcal{F}\mathbf{u} \cdot \mathbf{n} dA, \tag{33}$$

Reynolds transport theorem III

with $S(t)$ being the bounding surface of the volume $V(t)$ and **n** the outer normal to this surface. Equation (33) shows that the material derivative of a variable $\mathcal{F}$ integrated over a material volume $V(t)$ can be written as the integral of $\partial \mathcal{F}/\partial t$ over the volume $V(t)$ plus the flux of $\mathcal{F}$ through the surface $S(t)$ of this volume.

Principle of conservation of mass

Let us consider a material volume V with bounding surface S. The principle of conservation of mass imposes that: **the material derivative of the mass of fluid in V is equal to zero.**

The mass of the fluid in V is given by

$$\iiint\limits_V \rho dV.$$

Therefore we have

$$\frac{D}{Dt} \iiint\limits_V \rho dV = 0.$$

Recalling (32) we have:

$$\iiint\limits_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) dV = 0. \tag{34}$$

since the volume V is arbitrary the following differential equation holds

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \text{or, in index notation,} \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0. \tag{35}$$

This equation is known in fluid mechanics as **continuity equation**.
In the particular case in which the fluid is incompressible, i.e. the density ρ is constant, the above equation reduces to

$$\nabla \cdot \mathbf{u} = 0, \qquad \text{or, in index notation,} \qquad \frac{\partial u_j}{\partial x_j} = 0. \tag{36}$$

This implies that the velocity field of an incompressible fluid is divergence free.

The streamfunction I

A differential form

$$df = p(x, y)dx + q(x, y)dy,$$

is said to be an **exact differential** if $\int df$ is path independent. This happens when

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Therefore, in this case

$$p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y},$$

and this implies

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \tag{37}$$

Plane flow of an incompressible fluid

Let us consider a plane flow on the (x_1, x_2) plane so that the velocity has only two components u_1 and u_2. Let us also assume that the fluid is incompressible. The continuity equation (36) reduces to

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0.$$

The streamfunction II

The above expression implies that the following differential

$$d\psi = -u_2 dx_1 + u_1 dx_2,$$

is exact, as the condition (37) is satisfied. Then we have

$$u_1 = \frac{\partial \psi}{\partial x_2}, \qquad u_2 = -\frac{\partial \psi}{\partial x_1}, \tag{38}$$

and the scalar function $\psi(x_1, x_2, t)$ is defined as

$$\psi - \psi_0 = \int (-u_2 dx_1 + u_1 dx_2). \tag{39}$$

In the above expression ψ_0 is a constant and the line integral is taken on an arbitrary path joining the reference point O to a point P with coordinates (x_1, x_2). We know that, as $d\psi$ is an exact differential, the value of $\psi - \psi_0$ does not depend on the path of integration but only on the initial and finals points.

The function ψ has a very important physical meaning. The flux of fluid volume across the line joining the points O and P (taken positive if the flux is in the anti-clockwise direction about P) is given by the integral

$$\int (-u_2 dx_1 + u_1 dx_2).$$

This means that the flux through any curve joining two points is equal to the difference of the value of ψ at these points.
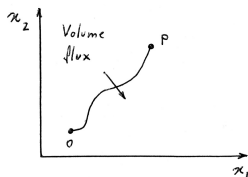
The streamfunction III

Therefore the value of ψ is constant along streamlines as, by definition, the flux across any streamline is zero. For this reason the function ψ is named **streamfunction**.
The advantage of having introduced the streamfunction is that we can describe the flow using a scalar function rather than the vector function **u**.

Axisymmetric flow of an incompressible fluid

Let us now consider an axisymmetric flow of an incompressible fluid.
Let us assume a system of **cylindrical coordinates** (z, r, φ). The corresponding velocity components are (u_z, u_r, u_φ). Due to the axisymmetry of the flow we know that $u_\varphi = 0$ and that u_r and u_z do not depend on φ. In this case the continuity equation (36) reads

$$\nabla \cdot \mathbf{u} = \frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{\partial r} = 0.$$

We can again define a streamfunction as

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad \psi - \psi_0 = \int r(u_z dr - u_r dz). \qquad (40)$$

The streamfunction for an incompressible axisymmetric flow can also be expressed in terms of other orthogonal systems of coordinates, e.g. spherical polar coordinates, see equation (121).

The velocity gradient tensor I

Let us consider two nearby points P and Q with material coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$. At time t their position is $\mathbf{x}(\boldsymbol{\xi}, t)$ and $\mathbf{x}(\boldsymbol{\xi} + d\boldsymbol{\xi}, t)$. We can relate the position of the two particles with the following relationship

$$x_i(\boldsymbol{\xi} + d\boldsymbol{\xi}, t) = x_i(\boldsymbol{\xi}, t) + \frac{\partial x_i}{\partial \xi_j} d\xi_j + O(d^2),$$

where $O(d^2)$ represents terms of order $d\xi^2$ or smaller that will be neglected. The small displacement vector $d\boldsymbol{\xi}$ at the time t has become $d\mathbf{x} = \mathbf{x}(\boldsymbol{\xi} + d\boldsymbol{\xi}, t) - \mathbf{x}(\boldsymbol{\xi}, t)$ and it takes the expression

$$dx_i = \frac{\partial x_i}{\partial \xi_j} d\xi_j. \tag{41}$$

Definition: the quantity $\frac{\partial x_i}{\partial \xi_j}$ is a tensor which is named **displacement gradient tensor**. This tensor is fundamental in the theory of elasticity.

In fluid mechanics it is more significant to reason in terms of velocities ($\mathbf{u} = D\mathbf{x}/Dt$). The relative velocity of two particles with material coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$ can be written as

$$du_i = \frac{\partial u_i}{\partial \xi_j} d\xi_j = \frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) d\xi_j. \tag{42}$$

The velocity gradient tensor II

Inverting (41) we can rewrite the above expression as

$$du_i = \frac{\partial u_i}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j} dx_j = \frac{\partial u_i}{\partial x_j} dx_j. \tag{43}$$

The above equation expresses the relative velocity in terms of the current relative position.

Definition: the quantity $\frac{\partial u_i}{\partial x_j}$ (or $\nabla \mathbf{u}$ in vector form) is a tensor that is named **velocity gradient tensor**.

In general $\nabla \mathbf{u}$ is non symmetric. Any tensor can be decomposed into a symmetric and an antisymmetric part. In particular we can write

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right), \quad \text{or in vector form} \quad \nabla \mathbf{u} = \mathbf{D} + \mathbf{\Omega}. \tag{44}$$

Above we have defined

$$D_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad \text{rate of deformation tensor}, \tag{45}$$

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right), \qquad \text{rate of rotation tensor}. \tag{46}$$

Both the above tensors play a vey important role in fluid mechanics. Their physical meaning is explained in the two following sections.

Physical interpretation of the rate of deformation tensor I

We now wish to interpret the physical meaning of the rate of deformation tensor **D**. Let us consider how a small material element of fluid deforms during motion. Let P and Q be two close material particles with coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$, whose positions at time t are $\mathbf{x}(\boldsymbol{\xi}, t)$ and $\mathbf{x}(\boldsymbol{\xi} + d\boldsymbol{\xi}, t)$. Let the length of the small segment connecting P and Q at time t be ds. Recalling (41) we can write

$$ds^2 = dx_i dx_i = \frac{\partial x_i}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_k} d\xi_j d\xi_k.$$

Let us now take the material derivative of ds^2.

$$\frac{D}{Dt} ds^2 = \left(\frac{\partial u_i}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_k} + \frac{\partial x_i}{\partial \xi_j} \frac{\partial u_i}{\partial \xi_k} \right) d\xi_j d\xi_k = 2 \frac{\partial u_i}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_k} d\xi_j d\xi_k.$$

Note that we have used the fact that $d\xi_j$ and $d\xi_k$ do not change in time as they are material segments. Moreover, we could swap j and k as they both are dummy indexes.
Recalling (41), (42) and (43) we know that

$$\frac{\partial u_i}{\partial \xi_j} d\xi_j = \frac{\partial u_i}{\partial x_j} dx_j, \qquad \frac{\partial x_i}{\partial \xi_k} d\xi_k = dx_i.$$

Therefore, we can write

$$\frac{1}{2} \frac{D}{Dt} ds^2 = ds \frac{D}{Dt} ds = \frac{\partial u_i}{\partial x_j} dx_i dx_j = D_{ij} dx_i dx_j.$$

Physical interpretation of the rate of deformation tensor II

In the above expression we have used the fact the antisymmetric terms in $\partial u_i / \partial x_j$ vanish upon summation and, therefore, only the symmetric part of the velocity gradient tensor (i.e. D_{ij}) survives.
The above expression can also be rewritten as

$$\frac{1}{ds}\frac{D}{Dt}ds = D_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}. \tag{47}$$

The term dx_i/ds is the ith component of a unit vector in the direction of the segment PQ.
Therefore equation (47) states that the rate of change of the length of the segment (as a fraction of its length) is related to its direction through the deformation tensor **D**.
We can also observe that if **D** = 0 the segment PQ remains of constant length. Therefore we can state that **if D = 0 the motion is locally and instantaneously rigid**. The tensor **D** is therefore related to **deformation of material elements**.

Meaning of the terms on the main diagonal of D

Let PQ be parallel to the coordinate axis x_1. In this case $\dfrac{d\mathbf{x}}{ds} = \mathbf{e}_1$, with $\mathbf{e}_1$ unit vector in the direction of x_1. Then equation (47) simplifies to

$$\frac{1}{dx_1}\frac{D}{Dt}dx_1 = D_{11}.$$

Physical interpretation of the rate of deformation tensor III

Thus the element D_{11} represents the rate of longitudinal strain of an element parallel to x_1. Obviously, the same interpretation applies to the other two terms on the main diagonal of **D**, i.e. D_{22} and D_{33}.

Meaning of the terms out of the main diagonal of D

We now consider two segments PQ and PR, where R is a material particle with material coordinates $\boldsymbol{\xi} + d\boldsymbol{\xi}'$. Let ds' be the length of the segment PR and θ the angle between the segments PQ and PR. We then have

$$ds\ ds' \cos\theta = dx_i dx_i'.$$
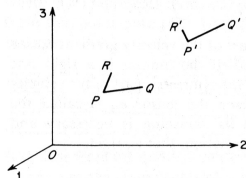
Taking the material derivative of the above expression, using again (43), we have

$$\frac{D}{Dt}(ds\ ds' \cos\theta) = du_i dx_i' + dx_i du_i' = \frac{\partial u_i}{\partial x_j} dx_j dx_i' + dx_i \frac{\partial u_i}{\partial x_j} dx_j'.$$

As i and j are dummy indexes they can be interchanged, and we can then write

$$\cos\theta \left(\frac{1}{ds}\frac{D}{Dt}ds + \frac{1}{ds'}\frac{D}{Dt}ds' \right) - \sin\theta \frac{D\theta}{Dt} = \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \frac{dx_j}{ds}\frac{dx_i'}{ds'} = 2D_{ij}\frac{dx_i'}{ds'}\frac{dx_j}{ds}.$$

Physical interpretation of the rate of deformation tensor IV

Now suppose, as an example, that $d\mathbf{x}'$ is parallel to the axis x_1 and $d\mathbf{x}$ to the axis x_2. This implies that $dx_i'/ds' = \delta_{i1}$, $dx_i/ds = \delta_{j2}$ and $\theta = \theta_{12} = \pi/2$. Then we have

$$-\frac{D\theta_{12}}{Dt} = 2D_{12}.$$

This implies that the term D_{ij} (with $i \neq j$) can be interpreted as one half of the rate of decrease of the angle between two segments parallel to the x_i and x_j axes, respectively.

Physical interpretation of the rate of rotation tensor I

We now consider the tensor $\boldsymbol{\Omega}$ defined by equation (46). We first note that an anti-symmetric tensor $\boldsymbol{\Omega}$ can be related to a vector $\boldsymbol{\omega}$ by the following relationship

$$\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k, \tag{48}$$

where the coefficient $-1/2$ has been introduced for convenience. $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ have the following forms

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\begin{array}{ccc} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{array}\right), \qquad \boldsymbol{\omega} = \left(\begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \end{array}\right).$$

Comparing the above equation with the definition of $\boldsymbol{\Omega}$ given in (46) we obtain

$$\omega_1 = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \qquad \omega_2 = \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \qquad \omega_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

Thus $\boldsymbol{\omega}$ is the curl of the velocity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}, \qquad \text{or in index form} \qquad \omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{49}$$

In fluid mechanics the vector $\boldsymbol{\omega}$ is known as **vorticity**.

Physical interpretation of the rate of rotation tensor II

To show the physical meaning of vorticity let us recall Stokes theorem

$$\iint\limits_{S} (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \iint\limits_{S} \boldsymbol{\omega} \cdot \mathbf{n} dS = \oint\limits_{l} \mathbf{u} \cdot d\mathbf{l},$$

which holds for any open surface S bounded by a closed curve l.

We now choose a plane surface S with normal $\mathbf{n}$, bounded by a small circle l of radius r centred at $\mathbf{x}$. Let $\mathbf{r}$ be a unit vector connecting the point $\mathbf{x}$ to any point on the circle l. Let moreover $\mathbf{l}$ be a unit vector tangential to the circle. We thus have $\mathbf{l} = \mathbf{n} \times \mathbf{r}$. The average of the projection of the angular velocity of points on l in the normal direction $\mathbf{n}$ is

$$\frac{1}{2\pi r^2} \oint\limits_{l} \mathbf{n} \cdot (\mathbf{r} \times \mathbf{u}) dl = \frac{1}{2\pi r^2} \oint\limits_{l} \mathbf{u} \cdot (\mathbf{n} \times \mathbf{r}) dl = \frac{1}{2\pi r^2} \oint\limits_{l} \mathbf{u} \cdot \mathbf{l} dl = \frac{1}{2S} \iint\limits_{S} \boldsymbol{\omega} \cdot \mathbf{n} dS \approx \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{n}.$$

As this result is valid for any $\mathbf{n}$, this shows that the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ can be interpreted as **twice the angular velocity of the fluid**.

Dynamics of fluids

Momentum equation in integral form

Let us consider a material volume V with bounding surface S. Newton's first principle states that: **the material derivative of the momentum of the fluid in V is equal to the resultant of all external forces acting on the volume.**

The momentum of the fluid in V is given by

$$\iiint\limits_V \rho \mathbf{u} dV.$$

Therefore we have (in index notation):

$$\frac{D}{Dt} \iiint\limits_V \rho u_i dV = \iiint\limits_V \rho f_i dV + \iint\limits_S t_i dS. \tag{50}$$

Recalling (33) we have:

$$\iiint\limits_V \frac{\partial}{\partial t}(\rho u_i) dV + \iint\limits_S \rho u_i u_j n_j dS = \iiint\limits_V \rho f_i dV + \iint\limits_S t_i dS. \tag{51}$$

This is the **integral form of the momentum equation** and is often written in compact form as

$$\mathbf{I} + \mathbf{W} = \mathbf{F} + \mathbf{\Sigma}, \tag{52}$$

with $\mathbf{I}$ named local inertia and $\mathbf{W}$ being the flux of momentum across S.

Momentum equation in differential form I

Let us now consider the expression

$$\frac{D}{Dt}\iiint\limits_{V}\rho\mathcal{F}dV = \iiint\limits_{V}\frac{\partial}{\partial t}(\rho\mathcal{F})+\frac{\partial}{\partial x_j}(\rho\mathcal{F}u_j)dV = \iiint\limits_{V}\mathcal{F}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\mathcal{F}}{\partial t}+\mathcal{F}\frac{\partial}{\partial x_j}(\rho u_j)+\rho u_j\frac{\partial\mathcal{F}}{\partial x_j}dV,$$

with $\mathcal{F}$ any function of space and time. Recalling (35) this simplifies to

$$\frac{D}{Dt}\iiint\limits_{V}\rho\mathcal{F}dV = \iiint\limits_{V}\rho\frac{\partial\mathcal{F}}{\partial t}+\rho u_j\frac{\partial\mathcal{F}}{\partial x_j}dV = \iiint\limits_{V}\rho\frac{D}{Dt}\mathcal{F}dV. \qquad (53)$$

In the particular case in which the generic function $\mathcal{F}$ is the velocity $\mathbf{u}$ we have

$$\frac{D}{Dt}\iiint\limits_{V}\rho\mathbf{u}dV = \iiint\limits_{V}\rho\frac{D}{Dt}\mathbf{u}dV. \qquad (54)$$

Using equations (1) and (54), equation (50) can be written as

$$\iiint\limits_{V}\rho\frac{Du_i}{Dt}dV = \iiint\limits_{V}\rho f_i dV + \iint\limits_{S}\sigma_{ij}n_j dS.$$

Momentum equation in differential form II

Using Gauss theorem we get

$$\iiint\limits_{V} \rho \frac{Du_i}{Dt} - \rho f_i - \frac{\partial}{\partial x_j}\sigma_{ij}\, dV = 0.$$

Since V is arbitrary the following differential equation must hold

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) - \rho f_i - \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \text{or, in vector form,} \quad \rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{f} - \nabla\cdot\boldsymbol{\sigma} = 0. \tag{55}$$

This is known as **Cauchy equation**. This equation holds for any continuum body. In order to specify the nature of the continuum a further relationship is needed, describing how the stress tensor $\boldsymbol{\sigma}$ depends on the kinematic state of the continuum. This relationship is called **constitutive law** and will be discussed in section 6.

Principle of conservation of the moment of momentum I

Given a material volume V, the material derivative of the moment of momentum of the fluid in V is equal to the resultant of all external moments acting on V.

The above principle is expressed mathematically as follows.

$$\frac{D}{Dt} \iiint\limits_V \rho \mathbf{x} \times \mathbf{u} dV = \iiint\limits_V \rho \mathbf{x} \times \mathbf{f} dV + \iint\limits_S \mathbf{x} \times \mathbf{t} dS, \tag{56}$$

or, in index notation,

$$\epsilon_{ijk} \left(\frac{D}{Dt} \iiint\limits_V \rho x_j u_k dV - \iiint\limits_V \rho x_j f_k dV - \iint\limits_S x_j t_k dS \right) = 0. \tag{57}$$

We use again equation (54) and note that $Dx_j/Dt = u_j$. Moreover, the definition of the operator ϵ_{ijk} implies that

$$\epsilon_{ijk} \iiint\limits_V \rho u_j u_k dV = 0.$$

Thus we have, also using Gauss theorem and equation (1),

$$\epsilon_{ijk} \left[\iiint\limits_V \rho x_j \left(\frac{Du_k}{Dt} - f_k \right) - x_j \frac{\partial \sigma_{kl}}{\partial x_l} - \sigma_{kl} \frac{\partial x_j}{\partial x_l} dV \right] = 0,$$

Principle of conservation of the moment of momentum II

and after rearrangement

$$\epsilon_{ijk}\left[\iiint\limits_V x_j\left(\rho\frac{Du_k}{Dt}-\rho f_k-\frac{\partial\sigma_{kl}}{\partial x_l}\right)-\delta_{jl}\sigma_{kl}dV\right]=0,$$

The term in brackets in the above equation is zero for equation (55). Therefore we obtain

$$\epsilon_{ijk}\iiint\limits_V \delta_{jl}\sigma_{kl}dV=0.$$

Since in the above expression V is arbitrary the following differential equation must hold:

$$\epsilon_{ijk}\delta_{jl}\sigma_{kl}=0,$$

or

$$\epsilon_{ijk}\sigma_{kj}=0.$$

The above equation implies:

$$\sigma_{kj}=\sigma_{jk}, \tag{58}$$

which imposes that **the stress tensor must be symmetrical**.

Equation for the mechanical energy I

Let us now consider Cauchy equation (55) and multiply it by u_i. Since i is now a repeated index we obtain the following scalar equation

$$\rho u_i \frac{Du_i}{Dt} - \rho u_i f_i - u_i \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \Rightarrow \quad \frac{1}{2}\rho \frac{Du_i^2}{Dt} - \rho u_i f_i - \frac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right) + \sigma_{ij}\frac{\partial u_i}{\partial x_j} = 0.$$

Reorganising the above expression and using the fact that the tensor σ_{ij} is symmetric, we have

$$\frac{1}{2}\rho \frac{Du_i^2}{Dt} = \rho u_i f_i + \frac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right) - \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\sigma_{ij},$$

or, recalling the definition of the rate of deformation tensor D_{ij}, given by (45),

$$\frac{1}{2}\rho \frac{Du_i^2}{Dt} = \rho u_i f_i + \frac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right) - D_{ij}\sigma_{ij}.$$

Integrating the above equation over an arbitrary volume V and using (53) we get

$$\frac{D}{Dt}\iiint\limits_V \frac{1}{2}\rho u_i^2 dV = \iiint\limits_V \rho u_i f_i dV + \iint\limits_S u_i \sigma_{ij} n_j dS - \iiint\limits_V D_{ij}\sigma_{ij} dV.$$

Equation for the mechanical energy II

Note that we can define the kinetic energy $\mathcal{E}_k$ associated with the fluid in V as

$$\mathcal{E}_k = \iiint\limits_V \frac{1}{2} \rho u_i^2 \, dV.$$

Thus we obtain

$$\frac{D}{Dt} \mathcal{E}_k = \iiint\limits_V \rho u_i f_i \, dV + \iint\limits_S u_i t_i \, dS - \iiint\limits_V D_{ij} \sigma_{ij} \, dV, \qquad (59)$$

or in vector form

$$\frac{D}{Dt} \mathcal{E}_k = \iiint\limits_V \rho \mathbf{u} \cdot \mathbf{f} \, dV + \iint\limits_S \mathbf{u} \cdot \mathbf{t} \, dS - \iiint\limits_V \mathbf{D} : \boldsymbol{\sigma} \, dV. \qquad (60)$$

The above equation states that **the rate of change of the kinetic energy of the fluid in the material volume V is equal to the power associated to the resultant of all external forces minus the internal power used to deform the fluid within V.** The last term in equation (60) is therefore associated with internal energy dissipation.

The equations of motion for Newtonian incompressible fluids

Definition of pressure in a moving fluid I

In section 2 we showed that, in a fluid at rest, the stress tensor takes the simple form

$$\sigma_{ij} = -p\delta_{ij},$$

where the scalar quantity p is the static pressure.

In the case of a moving fluid the situation is more complicated. In particular:

- the tangential stresses are not necessarily equal to zero;
- the normal stresses might depend on the orientation of the surface they act on.

This implies that the simple notion of pressure as a normal stress acting equally in all directions is lost. We wish now to find a proper definition for the pressure in the case of a moving fluid.

A natural choice is to consider $\frac{1}{3}\sigma_{ii} = \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}$, which we know to be an invariant under rotation of the axes. A simple physical interpretation of $\frac{1}{3}\sigma_{ii}$ is available. Let us consider a small cube with side dl centred in $\mathbf{x}$. As the cube is small we can assume that $\boldsymbol{\sigma}$ is constant within it. Taking a system of Cartesian coordinates (x_1, x_2, x_3) with axes parallel to the sides of the cube the **average value of the normal component of the stress over the surface** of the cube is

$$\frac{1}{6dl^2}\left(2\sigma_{11} + 2\sigma_{22} + 2\sigma_{33}\right) dl^2 = \frac{1}{3}\sigma_{ii}.$$

As the σ_{ii} is an invariant of $\boldsymbol{\sigma}$, the numerical value of $\frac{1}{3}\sigma_{ii}$ is independent of the orientation of the cube.

Definition of pressure in a moving fluid II

The quantity $\frac{1}{3}\sigma_{ii}$ reduces to the static fluid pressure when the fluid is at rest, and its mechanical significance makes it an appropriate generalisation of the elementary notion of pressure.
Therefore, we adopt the following **definition of pressure**

$$p = -\frac{1}{3}\sigma_{ii}, \qquad \text{or,} \qquad p = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}. \tag{61}$$

Important note

- **Incompressible fluids**
  For an incompressible fluid the pressure p is an independent, purely dynamical variable. In the rest of this course we will deal exclusively with incompressible fluids.

- **Compressible fluids**
  In the case of compressible fluids we know from classical thermodynamics that we can define the pressure of the fluid as a **parameter of state**, making use of an **equation of state**.
  Thermodynamical relations refer to equilibrium conditions, so we can denote the thermodynamic pressure as p_e.
  The connection between p and p_e is not trivial as p refers to dynamic conditions, in which elements of fluid in relative motion might not be in thermodynamic equilibrium. A thorough discussion of this subject can be found in Batchelor (1967). Here it suffices to say that, for most applications, is it reasonably correct to assume $p = p_e$.

Definition of pressure in a moving fluid III

For the discussion to follow it is convenient to split to the stress tensor σ_{ij} into an **isotropic part** $-p\delta_{ij}$, and a **deviatoric part** d_{ij} which is entirely due to fluid motion. We thus write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}. \tag{62}$$

The tensor d_{ij} accounts for tangential stresses and also normal stresses whose sum is zero.

Constitutive relationship for Newtonian fluids I

We derive the constitutive relationship under the following assumptions.

1. The tensor $\mathbf{d}$ is a continuous function of $\nabla \mathbf{u}$.
2. If $\nabla \mathbf{u} = 0$ then $\mathbf{d} = 0$, so that $\boldsymbol{\sigma} = -p\mathbf{I}$, i.e. the stress reduces to the stress in static conditions.
3. The fluid is homogeneous, i.e. $\boldsymbol{\sigma}$ does not depend explicitly on $\mathbf{x}$.
4. The fluid is isotropic, i.e. there is no preferred direction.
5. The relationship between $\mathbf{d}$ and $\nabla \mathbf{u}$ is linear.

Both the tensors $\mathbf{d}$ and $\nabla \mathbf{u}$ have nine scalar components. The linear assumption means that each component of $\mathbf{d}$ is proportional to the nine components of $\nabla \mathbf{u}$. Hence, in the most general case there are 81 scalar coefficients that relate the two tensors, in the form

$$d_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}, \tag{63}$$

where A_{ijkl} is a fourth-order tensor which depends on the local state of the fluid but not directly on the velocity distribution. Note that since d_{ij} is symmetrical so it must be A_{ijkl} in the indices i and j.

It is convenient at this stage to recall the decomposition of the velocity gradient tensor (44) into a symmetric and an anti-symmetric part

$$\frac{\partial u_i}{\partial x_j} = D_{ij} + \Omega_{ij}.$$

Constitutive relationship for Newtonian fluids II

The assumption of isotropy of the fluid implies that the tensor A_{ijkl} has to be isotropic. A tensor is said to be isotropic when its components are unchanged by rotation of the frame of reference. It is known from books on Cartesian tensors (e.g. Aris, 1962) that all isotropic tensors of even order can be written as the sum of products of δ tensors, with δ being the Kronecker tensor. In the case of a fourth-order tensor we can write

$$A_{ijkl} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk} + \mu''\delta_{ij}\delta_{kl},$$

where μ, μ' and μ'' are scalar coefficients. Since A_{ijkl} is symmetrical in i and j it must be

$$\mu = \mu'.$$

If $\mu = \mu'$ the tensor A_{ijkl} is also symmetrical in the indices k and l. This implies that

$$A_{ijkl}\Omega_{kl} = 0,$$

as Ω_{kl} is anti-symmetric. The fact that d_{ij} can not depend on Ω_{kl} is reasonable as, on the ground of intuition, we do not expect that a motion locally consisting of a rigid body rotation induces stress in the fluid. Note that this also implies that the assumption 2 has to be rewritten as $\mathbf{D} = 0 \Rightarrow \mathbf{d} = 0$.
We now have that equation (63) reduces to

$$d_{ij} = \mu\delta_{ik}\delta_{jl}D_{kl} + \mu\delta_{il}\delta_{jk}D_{kl} + \mu''\delta_{ij}\delta_{kl}D_{kl} = \mu D_{ij} + \mu D_{ji} + \mu''\delta_{ij}D_{kk}.$$

Constitutive relationship for Newtonian fluids III

Recalling that $D_{kk} = \frac{\partial u_k}{\partial x_k} = \nabla \cdot \mathbf{u}$, the above expression takes the form

$$d_{ij} = 2\mu D_{ij} + \mu'' \nabla \cdot \mathbf{u} \ \delta_{ij}. \tag{64}$$

Finally, we recall that, by definition, d_{ij} makes no contribution to the mean normal stress, therefore

$$d_{ii} = (2\mu + 3\mu'')\nabla \cdot \mathbf{u} = 0,$$

and, since this expression holds for any $\mathbf{u}$, we find

$$2\mu + 3\mu'' = 0. \tag{65}$$

From (62), (64) and (65) we finally obtain the constitutive equation for a Newtonian fluid in the form

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\left(D_{ij} - \frac{1}{3}\nabla \cdot \mathbf{u} \ \delta_{ij}\right), \quad \text{or, in vector form,} \quad \boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\left(\mathbf{D} - \frac{1}{3}(\nabla \cdot \mathbf{u})\mathbf{I}\right). \tag{66}$$

Notice that for an incompressible fluid we have $\nabla \cdot \mathbf{u} = 0$ by the continuity equation (36), therefore the constitutive law simplifies in this case to

$$\sigma_{ij} = -p\delta_{ij} + 2\mu D_{ij}, \quad \text{or, in vector form,} \quad \boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{D}. \tag{67}$$

Constitutive relationship for Newtonian fluids IV

Definitions

- μ is named **dynamic viscosity**. It has dimensions $[\mu] = ML^{-1}T^{-1}$, and in the IS it is measured in N s m^{-2}.

- It is often convenient to define a **kinematic viscosity** as

$$\nu = \frac{\mu}{\rho}. \tag{68}$$

The kinematic viscosity has dimensions $[\nu] = L^2T^{-1}$, and in the IS is measured in m^{2}s^{-1}.

Inviscid fluids

A fluid is said to be **inviscid** or **ideal** if $\mu = 0$. For an inviscid fluid the stress tensor reads

$$\sigma_{ij} = -p\delta_{ij}, \quad \text{or, in vector form,} \quad \boldsymbol{\sigma} = -p\mathbf{I}, \tag{69}$$

i.e. it takes the same form as for a fluid at rest. Note that inviscid fluids do not exist in nature. However, in some cases, real fluids can behave similarly to ideal fluids. This happens in flows in which viscosity plays a negligible effect.

The Navier-Stokes equations

We now wish to derive the equations of motions for an incompressible Newtonian fluid. We consider the Cauchy equation (55) and substitute into it the constitutive relationship (67). We obtain

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) - \rho f_i - \frac{\partial}{\partial x_j} \left(-p\delta_{ij} + 2\mu D_{ij} \right) = 0. \tag{70}$$

Let us consider the last term of the above expression. We can write it as

$$\frac{\partial}{\partial x_j} \left(2\mu D_{ij} \right) = 2\mu \frac{\partial}{\partial x_j} \left[\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] = \mu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right).$$

For the continuity equation, we have $\frac{\partial u_j}{\partial x_j} = 0 \Rightarrow \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} = 0$. We can then write equation (70) as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - f_i + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} = 0, \quad \text{or, in vector form,} \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{f} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} = 0. \tag{71}$$

Recalling the definition of material derivative (25) the above equation can also be written as

$$\frac{Du_i}{Dt} - f_i + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} = 0, \quad \text{or, in vector form,} \quad \frac{D\mathbf{u}}{Dt} - \mathbf{f} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} = 0. \tag{72}$$

These are called the **Navier-Stokes equations** and are of fundamental importance in fluid mechanics. They govern the motion of a Newtonian incompressible fluid and have to be solved together with the continuity equation (36).

The dynamic pressure

We now assume that the body force acting on the fluid is gravity, therefore we set in the Navier-Stokes equation (71) $\mathbf{f} = \mathbf{g}$. When ρ is constant the pressure p in a point $\mathbf{x}$ of the fluid can be written as

$$p = p_0 + \rho \mathbf{g} \cdot \mathbf{x} + P, \tag{73}$$

where p_0 is a constant and $p_0 + \rho \mathbf{g} \cdot \mathbf{x}$ is the pressure that would exist in the fluid if it was at rest. Finally, P is the part of the pressure which is associated to fluid motion and can be named **dynamic pressure**. This is in fact the departure of pressure from the hydrostatic distribution. Therefore, in the Navier-Stokes equations, the term $\rho \mathbf{g} - \nabla p$ can be replaced with $-\nabla P$. Thus we have:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla P - \nu \nabla^2 \mathbf{u} = 0. \tag{74}$$

If the Navier-Stokes equations are written in terms of the dynamic pressure gravity does not explicitly appear in the equations.
In the following whenever gravity will not be included in the Navier-Stokes this will be done with the understanding that the pressure is the dynamic pressure (even if p will sometimes be used instead of P).

Initial and boundary conditions

Initial and boundary conditions for the Navier-Stokes equations

We know from the previous section that the motion of an incompressible Newtonian fluid is governed by the Navier-Stokes equations (71) and the continuity equation (36), namely

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{f} + \frac{1}{\rho}\nabla p - \nu \nabla^2 \mathbf{u} = 0,$$

$$\nabla \cdot \mathbf{u} = 0.$$

Initial conditions

To find an unsteady solution of the above equations, we need to prescribe **initial conditions**, i.e. the initial (at time $t = 0$) spatial distribution within the domain of pressure and velocity

$$p(\mathbf{x}, 0), \qquad \mathbf{u}(\mathbf{x}, 0). \tag{75}$$

Boundary conditions

Equations (71) and (36) have also to be solved subjected to suitable boundary conditions.
We will discuss in the following the boundary conditions that have to be imposed at the interface between two continuum media.
We will then specify these conditions to the following, particularly relevant, cases:

- **solid impermeable walls**;
- **free surfaces**, e.g. interfaces between a liquid and a gas.

Kinematic boundary condition I

The **kinematic boundary condition** imposes that at a boundary of the domain the normal velocity of the surface $v_n = \mathbf{v} \cdot \mathbf{n}$ (with $\mathbf{v}$ velocity of the boundary and $\mathbf{n}$ unit vector normal to the surface) is equal to the normal velocity of fluid particles on the surface $u_n = \mathbf{u} \cdot \mathbf{n}$. Thus we have

$$u_n = v_n \qquad \text{at the boundary.} \tag{76}$$

Let us determine v_n. Let $F(\mathbf{x}, t) = 0$ be the equation of the surface and $\mathbf{n}$ the normal to this surface, defined as

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|}. \tag{77}$$

Let us consider a small displacement of the surface in the time interval dt. The differential dF taken along the direction normal to $F = 0$ in the time interval dt has to be equal to zero for $F = 0$ to still represent the equation of the surface. Thus

$$dF = \frac{\partial F}{\partial n} dn + \frac{\partial F}{\partial t} dt = 0. \tag{78}$$

In the above expression dn represents the displacement of the interface along the normal direction in the time interval dt. The normal component of the velocity of the surface is

$$v_n = \frac{dn}{dt}. \tag{79}$$

Kinematic boundary condition II

Comparing (79) and (78) we obtain

$$v_n = -\frac{\partial F/\partial t}{\partial F/\partial n}.$$

Equation (77) implies $\mathbf{n} \cdot \mathbf{n}|\nabla F| = \nabla F \cdot \mathbf{n} \Rightarrow |\nabla F| = \frac{\partial F}{\partial n}$. Therefore the above equation can be written as

$$v_n = -\frac{\partial F/\partial t}{|\nabla F|}. \tag{80}$$

Substituting (80) into (76) we find

$$-\frac{\partial F/\partial t}{|\nabla F|} = \mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot \frac{\nabla F}{|\nabla F|},$$

from which, recalling (24)

$$\frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = \frac{DF}{Dt} = 0. \tag{81}$$

The above equation states that the $F = 0$ **is a material surface**, i.e. it is always constituted by the same fluid particles.

Continuity of the tangential component of the velocity

Given a boundary surface between two continuum media experience shows that **the tangential component of the velocity is continuous across the interface**. Let us denote with subscripts a and b the two continuum media. We thus have

$$u_{a\ t} = u_{b\ t} \qquad \text{at the boundary,} \tag{82}$$

where subscript t indicates the tangential components of $\mathbf{u}$.

This condition can be justified by the observation that a discontinuity of the tangential velocity would give rise to the generation of intense (infinite) stress on the surface, which would tend to smooth out the discontinuity itself.

Dynamic boundary conditions

Let us now consider an interface between two fluids. Since the boundary is immaterial, i.e. it has no mass, the elements that constitute the interface have to be in equilibrium to each other. This implies that:

- the tangential component of the stress has to be continuous across the interface;
- a jump in the normal component of the stress is admissible, which has to be balanced by the surface tension, according to equation (12).

Thus, recalling (67) we can write

$$(-p_a \mathbf{I} + 2\mu_a \mathbf{D}_a)\mathbf{n} - (-p_b \mathbf{I} + 2\mu_b \mathbf{D}_b)\mathbf{n} = \kappa \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \mathbf{n}, \qquad \text{at the interface} \qquad (83)$$

where subscripts a and b denote the fluid at the two sides of the interface, and the normal unit vector $\mathbf{n}$ points from the fluid a to b.

Two relevant cases

Let us now consider two cases of particular relevance in fluid mechanics.

Fluid in contact with a solid impermeable wall

When a fluid is in contact with a solid the boundary conditions described above take a very simple form.

- The kinematic boundary condition (81) and the conditions imposing the continuity of the tangential component of the velocity (82), imply that the velocity of the fluid at the wall $\mathbf{u}$ has to be equal to the velocity of the wall $\mathbf{u}_w$. Thus we have

$$\mathbf{u} = \mathbf{u}_w \qquad \text{at the wall.} \tag{84}$$

  This is named **no-slip boundary condition**.
  In the particular case in which the solid is not moving we obtain

$$\mathbf{u} = 0 \qquad \text{at the wall.} \tag{85}$$

- There is no need to impose the dynamic boundary conditions (83), unless the problem for the solid deformation is also solved, i.e. it is assumed that the solid is deformable.

Interface between a liquid and a gas (free surface)

Typically there is no need to solve the problem for the gas motion. This has the following consequences.

- Conditions (82) are no longer needed.
- In equation (83) the stress on the gas side (b) reduces to the contribution of the pressure $-p_{\text{gas}}\mathbf{n}$.

Scaling and dimensional analysis

Units of measurement and systems of units I

A very comprehensive book on scaling and dimensional analysis, which pays particular attention to problems in fluid mechanics, is Barenblatt (2003). This section is based on this book.
Measurement of a physical quantity is a comparison of a certain quantity with an appropriate standard, or **unit of measurement**.
We can divide the units for measuring physical quantities into two categories:

- **fundamental units;**
- **derived units.**

This has the following meaning. Let us consider a certain class of phenomena (e.g. mechanics). Let us list a number of quantities of interest and let us adopt reference values for these quantities as fundamental units. For instance we can choose mass, length and time standards as fundamental units. Once fundamental units have been decided upon it is possible to obtain derived units using the definition of the physical quantities. For instance, we know that density is mass per unit volume. We can therefore measure the density of a certain body by comparing it with the density of a body that contains a unit of mass in a volume equal to the cube of a unit of length.

Important note. Given a certain class of phenomena there is a **minimum number of fundamental units** necessary to measure all quantities within that class. However, a system of units needs not be **minimal**, i.e. we may choose as fundamental units more units than we strictly need.

Definition. A set of fundamental units that is sufficient for measuring all physical properties of the class of phenomena under consideration is called a **system of units**.

Units of measurement and systems of units II

- A system of units consisting of one fundamental unit (e.g. the metre) is sufficient to describe **geometric objects**.
- Two fundamental units (e.g. the metre and the second) are sufficient to describe **kinematic phenomena**.
- Three fundamental units (e.g. the metre, the second and the kilogram) are sufficient to describe **dynamic phenomena**.
- . . .

In the International System of Units SI the fundamental units for studying dynamic phenomena are:

- the kilogram kg for mass (equal to the mass of the International Prototype Kilogram, preserved at the Bureau of Weights and measures in Paris);
- the metre m for length (the length of the path travelled by light in vacuum during a time interval of $1/299,792,458$ s);
- the second s for time (the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom).

Units of measurement and systems of units III

Definition. Two systems of units are said to belong to the same **class of systems of units** if they differ only in the magnitude of the fundamental units, but not in their physical nature.

For instance if we choose to describe a mechanical problem adopting as fundamental units one kilometre ($= 10^3$ m), one metric ton ($= 10^3$ kg) and one hour ($= 3600$ s) we have a system of units in the same class as the SI (metre-kilogram-second).

If we regard the metre-kilogram-second as the **original system** in its class, then the corresponding units on an arbitrary system in the same class are obtained as follows

$$\text{unit of length} = m/L, \qquad \text{unit of mass} = kg/M, \qquad \text{unit of time} = s/T, \qquad (86)$$

where L, M and T are **positive numbers that indicate the factor by which the fundamental units change by passing from the original system to another system in the same class**. The class of systems of units based on length, mass and time is called LMT class.

Dimension of a physical quantity I

Definition of dimension

The function that determines the factor by which the numerical value of a physical quantity changes upon passage from the original system of units to an arbitrary system within a given class is called **dimension function or dimension** of that quantity.

The dimension of a quantity $\mathcal{F}$ is denoted by $[\mathcal{F}]$.

For example, if the units of length and time are changed by factors L and T, respectively, then the unit of velocity changes by a factor LT^{-1}. According to the above definition we can say that LT^{-1} is the dimension of velocity.

Definition. Quantities whose numerical value is independent of the choice of the fundamental units within a given class of systems of units are called **dimensionless**.

Important principle. In any equation with physical meaning all terms must have the same dimensions. If this was not the case an equality in one system of units would not be an equality in another system of units within the same class.

Thus, from Newton law we find that the dimension of force ($[F]$) in the LMT class is

$$[F] = [m][a] = LMT^{-2},$$

with m mass and a acceleration.

If, on the other hand, we adopt the LFT class (length-force-time), then the dimension of mass is $[m] = L^{-1}FT^2$.

Dimension of a physical quantity II

The dimension function is a power-law monomial

We will prove this using the LMT class of systems of units. We know that the dimension of a physical quantity a within this class depends on L, M and T only.

$$[a] = \phi(L, M, T).$$

Suppose we have chosen an original system (e.g. metre-kilogram-second). Moreover, we choose two further systems in the same class, say 1 and 2, so that, upon passage from the original system to these new systems, the fundamental units decrease by factors L_1, M_1, T_1 and L_2, M_2, T_3, respectively.

Let a be the numerical value of the quantity in the original system. This value will become, by definition of dimension, $a_1 = a\phi(L_1, M_1, T_1)$ in the first new system and $a_2 = a\phi(L_2, M_2, T_2)$ in the second one. Thus we have

$$\frac{a_2}{a_1} = \frac{\phi(L_2, M_2, T_2)}{\phi(L_1, M_1, T_1)}. \tag{87}$$

All systems of units within a given class are **equivalent**, i.e. there are no preferred systems. This implies that we may assume system 1 as the original system of the class. System 2 can then be obtained by decreasing the fundamental units by L_2/L_1, M_2/M_1 and T_2/T_1. This implies that the numerical value a_2 of the considered quantity can now be written as

$$a_2 = a_1 \phi\left(\frac{L_2}{L_1}, \frac{M_2}{M_1}, \frac{T_2}{T_1}\right).$$

Dimension of a physical quantity III

Therefore we have

$$\frac{a_2}{a_1} = \phi\left(\frac{L_2}{L_1}, \frac{M_2}{M_1}, \frac{T_2}{T_1}\right). \tag{88}$$

Comparing equations (87) and (88) we obtain the following functional equation for ϕ

$$\frac{\phi(L_2, M_2, T_2)}{\phi(L_1, M_1, T_1)} = \phi\left(\frac{L_2}{L_1}, \frac{M_2}{M_1}, \frac{T_2}{T_1}\right). \tag{89}$$

To solve this equation we proceed as follows. We first differentiate both sides of (89) with respect to L_2 and then set $L_2 = L_1 = L$, $M_2 = M_1 = M$ and $T_2 = T_1 = T$, finding

$$\frac{\frac{\partial}{\partial L}\phi(L, M, T)}{\phi(L, M, T)} = \frac{1}{L}\frac{\partial}{\partial L}\phi(1, 1, 1) = \frac{\alpha}{L},$$

where $\alpha = \frac{\partial}{\partial L}\phi(1, 1, 1)$ is a constant. The solution of the above equation is

$$\phi(L, M, T) = L^\alpha C_1(M, T).$$

Substituting this expression into (89), we find the following functional equation for C_1

$$\frac{C_1(M_2, T_2)}{C_1(M_1, T_1)} = C_1\left(\frac{M_2}{M_1}, \frac{T_2}{T_1}\right).$$

Dimension of a physical quantity IV

We now differentiate this equation with respect to M_2 and then set $M_2 = M_1 = M$ and $T_2 = T_1 = T$.

$$\frac{\frac{\partial}{\partial M} C_1(M, T)}{C_1(M, T)} = \frac{1}{M} \frac{\partial}{\partial M} C_1(1, 1) = \frac{\beta}{M},$$

where, again, $\beta = \frac{\partial}{\partial M} C_1(1, 1)$ is a constant. Solving for C_1 we obtain

$$C_1 = M^\beta C_2(T).$$

Proceeding in a similar way we finally find

$$C_2(T) = C_3 T^\gamma.$$

Thus the solution is

$$\phi = C_3 L^\alpha M^\beta T^\gamma.$$

The constant C_3 has to be equal to 1 as $L = M = T = 1$ means that the fundamental units remain unchanged, so that the value of the quantity a also has to remain unchanged and, therefore, it must be $\phi(1, 1, 1) = 1$. We then finally have

$$\phi = L^\alpha M^\beta T^\gamma. \tag{90}$$

and we can easily verify that this is actually a solution of our original functional equation (89).

Quantities with independent dimensions I

Definition

The quantities $a_1, a_2, \ldots, a_k$ are said to have **independent dimensions** if the monomial $a_1^\alpha a_2^\beta \ldots a_k^\omega$ has a dimension function equal to 1 (i.e. it is dimensionless) only for $\alpha = \beta = \cdots = \omega = 0$.

Example

Let us consider, for example, the quantities density ($[\rho] = ML^{-3}$), velocity ($[u] = LT^{-1}$) and force ($[f] = MLT^{-2}$). Let us now construct the monomial $\Gamma = \rho^\alpha u^\beta f^\gamma$. We require this monomial to be dimensionless, thus

$$[\Gamma] = [\rho]^\alpha [u]^\beta [f]^\gamma = M^\alpha L^{-3\alpha} L^\beta T^{-\beta} M^\gamma L^\gamma T^{-2\gamma} =$$
$$= M^{\alpha+\gamma} L^{-3\alpha+\beta+\gamma} T^{-\beta-2\gamma} = 1.$$

The above equation implies

$$\alpha + \gamma = 0,$$
$$-3\alpha + \beta + \gamma = 0,$$
$$-\beta - 2\gamma = 0.$$

This above system has no solution unless $\alpha = \beta = \gamma = 0$. This means that the quantities ρ, u and f have independent dimensions.

Quantities with independent dimensions II

Theorem

Within a certain class of systems of units, it is always possible to pass from an original system of units to another system, such that any quantity, say a_1, in the set of quantities $a_1, \ldots, a_k$ with independent dimensions, changes its numerical value while all the others remain unchanged.

Proof

Let us consider a system of units $PQ \ldots$. Let us consider a set of quantities with independent dimensions, whose values in a chosen original system of units are $a_1, \ldots, a_k$. Upon change of the system of units to an arbitrary one, their numerical value becomes $a'_1, \ldots, a'_k$, such that

$$a'_1 = a_1 P^{\alpha_1} Q^{\beta_1} \ldots, \quad \ldots \quad a'_k = a_k P^{\alpha_k} Q^{\beta_k} \ldots,$$

where the powers $\alpha_i, \beta_i, \ldots$ $(i = 1, \ldots, k)$ are determined by the dimensions of each quantity. We want to find a system of units such that

$$a'_1 = a_1 P^{\alpha_1} Q^{\beta_1} \ldots, \quad a'_2 = a_2, \quad \ldots \quad a'_k = a_k.$$

We thus have a system of equations

$$P^{\alpha_1} Q^{\beta_1} \cdots = A_1, \quad P^{\alpha_2} Q^{\beta_2} \cdots = 1, \quad \ldots, \quad P^{\alpha_k} Q^{\beta_k} \doteq 1.$$

Taking the logarithm of the above equations we obtain

$$\alpha_1 \ln P + \beta_1 \ln Q + \cdots = \ln A_1, \quad \alpha_2 \ln P + \beta_2 \ln Q + \cdots = 0, \quad \ldots, \quad \alpha_k \ln P + \beta_k \ln Q + \cdots = 0.$$

Quantities with independent dimensions III

This system has a solution unless the left-hand side of the first equation is a linear combination of the remaining ones, so that

$$\alpha_1 \ln P + \beta_1 \ln Q + \cdots = c_2(\alpha_2 \ln P_2 + \beta_2 \ln Q + \dots) + \cdots + c_k(\alpha_k \ln P_k + \beta_k \ln Q + \dots),$$

with $c_2, \dots, c_k$ constants. However, this implies, going back to the exponent form, that

$$P^{\alpha_1} Q^{\beta_1} \cdots = \left(P^{\alpha_2} Q^{\beta_2} \right)^{c_2} \dots \left(P^{\alpha_k} Q^{\beta_k} \right)^{c_k},$$

or

$$[a_1] = [a_2]^{c_2} \dots [a_k]^{c_k}.$$

This contradicts the fact that $a_1, \dots, a_k$ have independent dimensions and the theorem is therefore proved.

Buckingham's Π theorem I

Any physical study (experimental or theoretical) consists in finding one or several relationships between physical quantities in the form

$$a = f(a_1, \ldots, a_k, b_1, \ldots, b_m). \tag{91}$$

In the above expression a denotes the quantity of interest. On the right-hand side of the above equation we have separated the physical quantities into two groups.

- The k quantities $a_1, \ldots, a_k$ have independent dimensions;
- the m quantities $b_1, \ldots, b_m$ can be expressed in terms of the dimensions of $a_1, \ldots, a_k$.

Thus we can write

$$[b_1] = [a_1]^{\alpha_1} \ldots [a_k]^{\omega_1}, \quad \ldots \quad [b_m] = [a_1]^{\alpha_m} \ldots [a_k]^{\omega_m}.$$

Note: it must be that the dimension of a is dependent on the dimensions of $a_1, \ldots, a_k$, so that

$$[a] = [a_1]^{\alpha} \ldots [a_k]^{\omega}.$$

Indeed, if a had dimensions independent from the dimensions of the variables $a_1, \ldots, a_k$, for the theorem proved above it would be possible to pass from the original system of units to another system, such that the numerical value of a would change and the numerical values of $a_1, \ldots, a_k$ and $b_1, \ldots, b_m$ would remain unchanged. This would indicate the need to include further quantities on the right-hand side of equation (91).

Buckingham's Π theorem II

We now introduce

$$\Pi = \frac{a}{a_1^{\alpha} \dots a_k^{\omega}},$$

$$\Pi_1 = \frac{b_1}{a_1^{\alpha_1} \dots a_k^{\omega_1}}, \quad \dots \quad \Pi_m = \frac{b_m}{a_1^{\alpha_m} \dots a_k^{\omega_m}}.$$

We can thus write equation (91) as

$$\Pi = \frac{f(a_1, \dots, a_k, b_1, \dots, b_m)}{a_1^{\alpha} \dots a_k^{\omega}} = \frac{f(a_1, \dots, a_k, \Pi_1 a_1^{\alpha_1} \dots a_k^{\omega_1}, \dots, \Pi_m a_1^{\alpha_m} \dots a_k^{\omega_m})}{a_1^{\alpha} \dots a_k^{\omega}},$$

or

$$\Pi = \mathcal{F}(a_1, \dots, a_k, \Pi_1, \dots, \Pi_m).$$

Π and Π_i ($i = 1, \dots, m$) are dimensionless, therefore they don't change their numerical value upon changing of the system of units. Now, suppose that we change the system of units so that a_1 changes its value and $a_2, \dots, a_k$ remain unchanged. In the above equation a_1 would be the only variable to change and this indicates that the function $\mathcal{F}$ can not depend of a_1. The same argument holds for all the $a_1, \dots, a_k$ variables. Therefore the above equation can be written as

$$\Pi = \mathcal{F}(\Pi_1, \dots, \Pi_m). \tag{92}$$

Buckingham's Π theorem III

We have therefore proved that equation (91) is equivalent to equation (92), which involves only dimensionless variables. Note, moreover, that (92) **involves a smaller number of variables than (91)**. In particular, the number of variables involved has decreased by k, i.e. by the number of variables involved in (91) that have independent dimensions. This fact is of fundamental importance and it is one of the main reasons for which working with dimensionless quantities is typically desirable.

Dimensionless Navier-Stokes equations I

When dealing with theoretical modelling of physical phenomena it is convenient to work with dimensionless equations. The main reasons for that are:

- according to the Π theorem the number of parameters involved in the problem decreases if one passes from a dimensional to a dimensionless formulation;
- in dimensionless form (if proper scalings are adopted) it is much easier to evaluate the relative importance of different terms appearing in one equation.

Let us consider the Navier-Stokes equation and assume that the body force is gravity. Equations (71) and the continuity equation (36), can then be written as

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}}_{①} \underbrace{-\mathbf{g}}_{②} + \underbrace{\frac{1}{\rho}\nabla p}_{③} \underbrace{-\nu\nabla^2 \mathbf{u}}_{④} = 0, \tag{93}$$

where the vector $\mathbf{g}$, representing the gravitational field, has magnitude g and is directed vertically downwards. We recall the physical meaning of all terms:

- ①: convective terms;
- ②: gravity;
- ③: pressure gradient;
- ④: viscous term.

Dimensionless Navier-Stokes equations II

We now wish to scale the above equation. Suppose that L is a characteristic length scale of the domain under consideration and U a characteristic velocity. We can then introduce the following dimensionless coordinates and variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \qquad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \qquad t^* = \frac{t}{L/U}.$$

Above and in what follows superscript stars indicate dimensionless quantities. We still have to scale the pressure. We might consider two different situations:

① In equation (93) ③ balances with ④. In this case we can write

$$p^* = \frac{p}{\rho \nu U/L}.$$

② If, on the other hand, in (93) the pressure gradient ③ balances with the convective terms ①, we can scale the pressure as follows

$$p^* = \frac{p}{\rho U^2}.$$

Dimensionless Navier-Stokes equations III

Low Reynolds number flows

Let us consider the first case. Making equation (93) dimensionless using the above scales we obtain

$$Re\left[\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right] + \frac{Re}{Fr^2}\mathbf{z} + \nabla^* p^* - \nabla^{*2}\mathbf{u}^* = 0, \tag{94}$$

where $\mathbf{z}$ is the upward directed vertical unit vector.
In the above equation we have defined two dimensionless parameters.

- $Re = \dfrac{UL}{\nu}$: **Reynolds number**. It represents the ratio between the magnitude of inertial (convective) terms and viscous terms. It plays a fundamental role in fluid mechanics.

- $Fr = \dfrac{U}{\sqrt{gL}}$: **Froude number**. It represents the square root of the ratio between the magnitude of inertial (convective) terms and gravitational terms. It plays a fundamental role when gravity is important, e.g. in free surface flows.

If we now consider the limit $Re \to 0$ the dimensionless Navier-Stokes equation (94) reduces to the so called **Stokes equation**, i.e.

$$\nabla^* p^* - \nabla^{*2}\mathbf{u}^* = 0. \tag{95}$$

This equation is much simpler than the Navier-Stokes equation as it is linear. In section 10 we will derive some analytical solutions of equation (95).

Dimensionless Navier-Stokes equations IV

Large Reynolds number flows

Let us now consider the case in which the pressure gradient balances the convective terms. The dimensionless Navier-Stokes equation takes the form

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* + \frac{1}{Fr^2}\mathbf{z} + \nabla^* p^* - \frac{1}{Re}\nabla^{*2}\mathbf{u}^* = 0. \tag{96}$$

In the limit $Re \to \infty$ the viscous term in equation (96) tends to zero. Thus we are led to think that, at large values of Re, the fluid behaves as an ideal or inviscid fluid.
This argument, however, has to be used with care as dropping off the viscous term from (96) means to neglect the term containing the highest order derivatives in the equation. Therefore, if the viscous term in (96) is neglected it is not possible to impose all boundary conditions. To resolve this contradiction we need to assume that at the boundaries thin **boundary layers** form, within which viscous terms in the Navier-Stokes equations have the same magnitude as convective terms.
If boundary layers keep very thin everywhere in the flow domain the fluid out of the boundary layers, in the core of the domain, actually behaves as if it was inviscid. This point will be discussed in more detail in section 11.

Unidirectional flows

Introduction to unidirectional flows

We consider the flow of an incompressible Newtonian fluid in the gravitational field. We thus have equations (93) and the continuity equation (36), namely

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{g} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u} = 0,$$

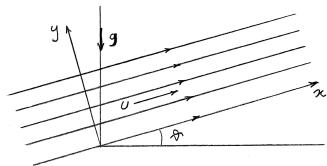$$\nabla \cdot \mathbf{u} = 0.$$

We consider a **unidirectional flow**, i.e. a flow in which the velocity has everywhere the same direction (say the direction of the axis x) and it is independent of x. Thus, assuming Cartesian coordinates (x, y, z), we have

$$\mathbf{u} = [u(y, z, t), 0, 0]. \tag{97}$$

It is easy to check that with the velocity field (97) **all the non linear terms in the Navier-Stokes equations vanish**. Thus the governing equations in this case are linear and therefore much more amenable for analytical treatment.

Some examples of unidirectional flows I

Let us consider the unidirectional flow shown in the figure. The direction of flow is inclined by an angle ϑ with respect to a horizontal plane. Referring to the figure we consider the system of Cartesian coordinates (x, y, z), with x direction of flow. The corresponding velocity components are $[u(y, z), 0, 0]$. In this case the Navier-Stokes equations take the form

$$\frac{\partial u}{\partial t} + g \sin \vartheta + \frac{1}{\rho} \frac{\partial p}{\partial x} - \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0, \qquad (98)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial y} + g \cos \vartheta = 0, \qquad (99)$$

$$\frac{\partial p}{\partial z} = 0. \qquad (100)$$

Equations (99) and (100) simply impose that the pressure distribution is hydrostatic on the cross-section of the flow (planes with $x = const$). This also implies that, as in hydrostatics (10), the piezometric (or hydraulic) head h is constant on such planes and is thus a function of x only. We can thus write

$$h(x) = x \sin \vartheta + y \cos \vartheta + \frac{p(x, y)}{\gamma},$$

Some examples of unidirectional flows II

from which

$$\frac{\partial h}{\partial x} = \sin \vartheta + \frac{1}{\gamma} \frac{\partial p}{\partial x}.$$

We can then rewrite equation (98) as

$$\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} - \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0. \tag{101}$$

Since u does not depend on x it follows that also $\partial h/\partial x$ is independent of x. Hence, we can write

$$\frac{\partial h}{\partial x} = -j(t),$$

where j is function of time only. Upon substitution of j, equation (101) takes the form

$$\frac{\partial u}{\partial t} - gj - \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0. \tag{102}$$

In the particular case of steady flow this simplifies to

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{g}{\nu} j. \tag{103}$$

Some examples of unidirectional flows III

Couette-Poiseuille flow

Let us now consider a particular case of the flow described above, i.e. the flow within a gap formed by two flat parallel walls, each one of which is moving in the x direction with a given velocity, say u_1 (lower wall) and u_2 (upper wall). Moreover, we assume that j has a constant prescribed value. We wish to study the motion of a fluid within this gap. In this case $\mathbf{u} = [u(y), 0, 0]$.
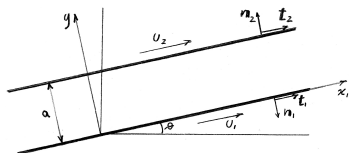
Equation (103) can be written as

$$\frac{d^2 u}{dy^2} = -\frac{g}{\nu} j,$$

The solution of the above equation is

$$u = -\frac{gj}{2\nu} y^2 + c_1 y + c_2.$$

The constants c_1 and c_2 can be determined imposing the no-slip boundary conditions at the walls, i.e.

$$u(0) = u_1, \qquad u(a) = u_2.$$

Some examples of unidirectional flows IV

We finally find

$$u = \frac{gj}{2\nu}(a-y)y + \frac{u_2 - u_1}{a}y + u_1. \tag{104}$$

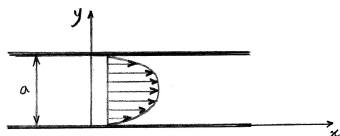From the above solution we can easily compute the volume flux per unit length q as

$$q = \int_0^a u\,dy = \frac{gj}{12\nu}a^3 + \frac{u_2 a}{2} + \frac{u_1 a}{2}. \tag{105}$$

We now consider a few particular cases.

- **Poiseuille flow:** $j \neq 0$, $u_1 = u_2 = 0$.

  In this case the flow is driven by a hydraulic head gradient alone. The velocity distribution (104) reduces to

  $$u = \frac{gj}{2\nu}(a-y)y, \qquad (106)$$

  i.e. the velocity profile is parabolic. The maximum velocity is located at the centre of the channel ($y = a/2$) and is equal to

  $$u_{\max} = \frac{gj}{8\nu}a^2,$$

Some examples of unidirectional flows V

and the volume flux per unit length is

$$q = \frac{gj}{12\nu}a^3 = \frac{2}{3}u_{\max}a.$$

The average velocity $\overline{u}$ is equal to $\frac{2}{3}u_{\max}$.

Let us now compute the shear stress on the wall. The stress tensor has the form

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \mu\dfrac{du}{dy} & 0 \\ \mu\dfrac{du}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - p\mathbf{I}.$$

Thus we easily find that the tangential stress τ exerted by each wall is given by

$$\tau = -\frac{\gamma j}{2}a.$$

Quite surprisingly the shear stress is not dependent on the viscosity of the fluid.

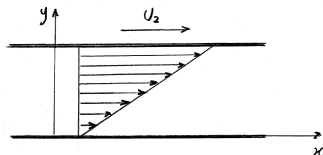## Some examples of unidirectional flows VI

- **Couette flow:** $j = 0$, $u_1 = 0$, $u_2 \neq 0$.

  In this case the flow is driven by the
  movement of the upper wall and the
  hydraulic head gradient is zero. The
  velocity distribution (104) reduces to

  $$u = \frac{u_2}{a} y. \qquad (107)$$

  Moreover, we find

  $$q = \frac{u_2 a}{2}, \qquad \tau = \mu \frac{u_2}{a}.$$

Some examples of unidirectional flows VII

Unidirectional free-surface flow

We now consider a steady free-surface flow over an inclined plane, as shown in the figure. The velocity vector can be written as $\mathbf{u} = [u(y), 0, 0]$ and equation (103) reduces to

$$\frac{d^2 u}{dy^2} = -\frac{g}{\nu} j,$$

subjected to the no-slip condition at $y = 0$ and the dynamic boundary conditions at $y = h$, i.e.
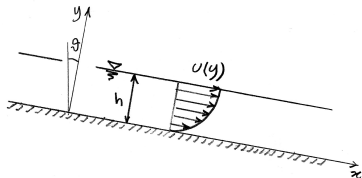
$$u(0) = 0, \qquad \frac{du}{dy}\bigg|_{y=h} = 0.$$

The corresponding velocity distribution and flux per unit length are

$$u = \frac{gj}{2\nu}(2h - y)y, \qquad q = \int_0^h u\,dy = \frac{gj}{3\nu}h^3.$$

If, as in fact is normally the case, the flux q and the slope j are fixed (rather than h) we obtain the following expression for h

$$h = \sqrt[3]{\frac{3\nu q}{gj}}.$$

Some examples of unidirectional flows VIII

Axisymmetric Poiseuille flow

Let us now consider a steady, completely developed flow in a straight pipe with circular cross-section of radius R. Let the pipe axis be in the z direction and let the flow be axisymmetric. In cylindrical coordinates (z, r, φ) the velocity vector takes the form $\mathbf{u} = [u(r), 0, 0]$, with u velocity component in the z direction. With these coordinates equation (103) takes the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) u = -\frac{gj}{\nu}, \quad \Rightarrow \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = -\frac{gj}{\nu}.$$
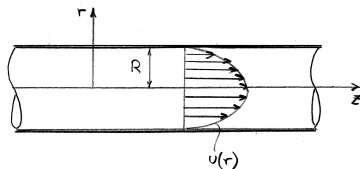
The above equation has to be solved subjected to the no-slip boundary condition at $r = R$ and a regularity condition in $r = 0$.
We then have

$$r \frac{du}{dr} = -\frac{gj}{2\nu} r^2 + c_1, \quad \Rightarrow \quad u = -\frac{gj}{4\nu} r^2 + c_1 \log r + c_2.$$

Regularity at $r = 0$ imposes $c_1 = 0$. Moreover, enforcing the no-slip boundary condition yields $c_2 = \frac{gj}{4\nu} R^2$. The solution is

$$u = \frac{gj}{4\nu} \left(R^2 - r^2 \right). \tag{108}$$

Some examples of unidirectional flows IX

This is known as **Poiseuille flow**. The velocity profile is a paraboloid. The volume flux Q is given by

$$Q = \int_0^R \int_0^{2\pi} u r d\varphi dr = \frac{gj\pi}{8\nu} R^4. \tag{109}$$

Written in cylindrical coordinates (z, r, φ) the stress tensor $\boldsymbol{\sigma}$ for this flow field takes the form

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \mu\dfrac{du}{dr} & 0 \\ \mu\dfrac{du}{dr} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - p\mathbf{I}.$$

We then easily compute the tangential stress τ on the wall, which reads

$$\tau = -\frac{\gamma j}{2} R.$$

Unsteady unidirectional flows I

Flow over a periodically oscillating plate

Let us now consider one example of unsteady unidirectional flow. In this case we need solving equation (102).

We consider the flow in the region $y > 0$ induced by periodic motion along the x axis of a rigid flat wall located at $y = 0$. The velocity of the wall u_w can be written as

$$u_w = u_0 \cos(\omega t) = \frac{u_0}{2} \left(e^{i\omega t} + c.c. \right),$$

where $c.c.$ denotes the complex conjugate.

Since there is no imposed pressure gradient equation (102) reduces to

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0. \tag{110}$$

We seek a separable variable solution in the form

$$u = f(y) e^{i\omega t} + c.c. \tag{111}$$

where $f(y)$ is a complex function. Substituting (111) into (110) we obtain

$$\frac{i\omega}{\nu} f - \frac{d^2 f}{dy^2} = 0.$$

Unsteady unidirectional flows II

Remembering that $\sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$, the solution of the problem is

$$u(y, t) = \left[c_1 \exp\left(-\frac{y(1 + i)}{\sqrt{2\frac{\nu}{\omega}}} \right) + c_2 \exp\left(\frac{y(1 + i)}{\sqrt{2\frac{\nu}{\omega}}} \right) \right] e^{i\omega t} + c.c.$$

As the solution should not be divergent for $y \to \infty$ we require $c_2 = 0$. Moreover, the no-slip boundary condition at the wall imposes
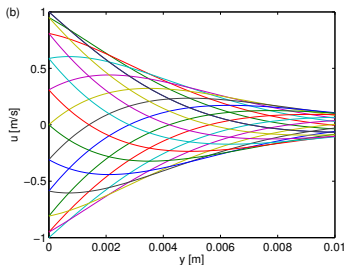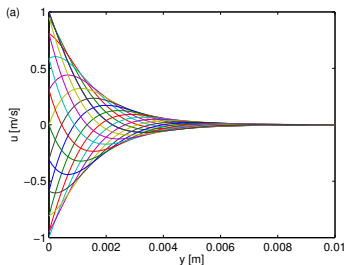
$$c_1 = \frac{u_0}{2}.$$

Thus the solution of (110) is

$$u(y, t) = \frac{u_0}{2} \exp\left(-\frac{y(1 + i)}{\sqrt{2\frac{\nu}{\omega}}} \right) e^{i\omega t} + c.c. = u_0 \exp\left(-\frac{y}{\sqrt{2\frac{\nu}{\omega}}} \right) \cos\left(\omega t - \frac{y}{\sqrt{2\frac{\nu}{\omega}}} \right). \quad (112)$$

Unsteady unidirectional flows III

The solution is sketched in the figure for $u_0 = 1$ m/s, $\nu = 10^{-6}$ m^2/s (water) and for two different values of ω, (a) $\omega = 1$ s^{-1}, (b) $\omega = 0.1$ s^{-1}.

It is important to notice that the velocity does not spread to infinity in the y direction for long times. The solution (112) suggests that a characteristic length scale l of the layer of fluid interested by motion is given by

$$l \approx \sqrt{\frac{\nu}{\omega}}.$$

Axisymmetric flow with circular streamlines I

We present here another case in which the Navier-Stokes equations take a linear form. Let us consider a flow such that all streamlines are circles centred on a common axis of symmetry. Moreover, adopting cylindrical coordinates (z, r, φ), we assume that the velocity, which is purely azimuthal, only depends of the radial coordinate r and, possibly, on time t. Thus we have $\mathbf{u} = [0, 0, w(r, t)]$. We finally assume axisymmetry, so that $\frac{\partial}{\partial \varphi} \equiv 0$. This flow is strictly related with unidirectional flows.

The continuity equation and the and Navier-Stokes equations in cylindrical coordinates are reported in the appendix 13 (equations (166), (167), (168) and (169)). It is immediate to verify that, when the velocity field takes the form $\mathbf{u} = [0, 0, w(r, t)]$, and the flow is axisymmetric the above equations reduce to

$$\frac{\partial p}{\partial z} = 0 \tag{113}$$

$$-\frac{w^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{114}$$

$$\frac{\partial w}{\partial t} - \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) - \frac{w}{r^2} \right] = 0, \tag{115}$$

and the continuity equation is automatically satisfied. Equation (113) implies that the pressure does not depend on z, and equation (114) that the radial variation of p supplies the force

Axisymmetric flow with circular streamlines II

necessary to keep the fluid element moving along a circular path. Finally, equation (115) is linear and it is the analogous of equation (102), for a unidirectional flow.

Steady flow between two concentric rotating cylinders

Let us consider two concentric cylinders with radius R_1 and R_2, respectively ($R_2 > R_1$). Each of the cylinders rotates with a given constant angular velocity (Ω_1 and Ω_2). The gap between the cylinders is filled with fluid. The flow is steady and equation (115) reduces to

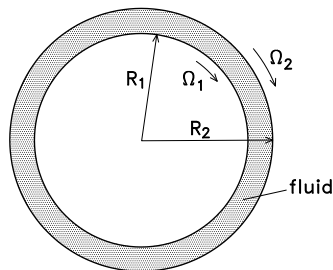$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) - \frac{w}{r} = 0,$$

with the boundary conditions

$$w = R_1\Omega_1 \quad (r = R_1), \qquad w = R_2\Omega_2 \quad (r = R_2).$$

The above equation can be rewritten as

$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{dw}{dr} - rw\right) = 0.$$

Axisymmetric flow with circular streamlines III

We then have

$$r^2 \frac{dw}{dr} - rw = c_1, \quad \Rightarrow \quad \frac{dw}{dr} = \frac{c_1}{r^2} + \frac{w}{r}, \quad \Rightarrow \quad w = -\frac{c_1}{2r} + c_2 r.$$

Enforcing the boundary conditions we finally find

$$w = \frac{1}{r} \left(\frac{\Omega_1 - \Omega_2}{R_1^{-2} - R_2^{-2}} \right) + r \left(\frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_2^2} \right). \tag{116}$$

We now consider a few particular cases.

- $\Omega_1 = 0$, **the inner cylinder is at rest.**
  In this case the solution (116) simplifies to

$$w = -\frac{\Omega_2 R_2^2 (r^2 - R_1^2)}{r(R_1^2 - R_2^2)}. \tag{117}$$

  The shear stress τ on the outer cylinder ($r = R_2$) is

$$\tau = \sigma_{r\varphi}|_{r=R_2} = \mu \left(\frac{dw}{dr} - \frac{w}{r} \right)_{r=R_2} = \frac{2\mu\Omega_2 R_1^2}{R_1^2 - R_2^2},$$

Axisymmetric flow with circular streamlines IV

and the couple per unit length of cylinder m necessary to keep the outer cylinder in rotation is

$$m = 2\pi R_2^2 \tau.$$

This device (rotational cylinder rheometer) is often used to measure the viscosity of a fluid, as, by rearrangement of the above formula, it is possible to obtain the value of the dynamic viscosity μ by measuring the couple m required to keep the outer cylinder in motion.

- $R_1 = 0$, $\Omega_1 = \Omega_2$, **flow inside a single rotating cylinder.**
  From (116) we immediately get

$$w = r\Omega_2,$$

which is a rigid body rotation.

- $R_2 \to \infty$, $\Omega_2 \to 0$, **flow around a single rotating cylinder.**
  From (116) we obtain

$$w = \frac{R_1^2 \Omega_1}{r}. \tag{118}$$

This is the so called "free vortex" velocity distribution. Notice that the vorticity associated with this velocity field is everywhere zero. In this case the couple per unit length m transmitted to the fluid by the rotating cylinder is

$$m = 4\pi R_1^2 \mu \Omega_1,$$

Axisymmetric flow with circular streamlines V

and it is not zero. This implies a continuum growth of the angular momentum of the fluid. This is not in contrast with the assumption of steady flow since the total angular momentum associated with the velocity distribution (118) is infinite.

- $\Omega_1 = 0$, $\dfrac{R_2 - R_1}{R_1} \ll 1$.

In this case we can write

$$\varepsilon = \frac{R_2 - R_1}{R_1}, \qquad R_2 = R_1(1 + \varepsilon),$$

with $\varepsilon \ll 1$. Let us now define a new coordinate ζ as

$$\zeta = \frac{r - R_1}{R_2 - R_1} = \frac{r - R_1}{\varepsilon R_1}, \quad \Rightarrow \quad r = R_1(1 + \varepsilon\zeta),$$

with $0 \leq \zeta \leq 1$. Substituting the above expression into (117) and expanding in terms of ε we find

$$w = \Omega_2 R_1 \zeta + \Omega_2 R_1 \zeta \left(\frac{3}{2} - \frac{1}{2}\zeta \right) \varepsilon + \mathcal{O}(\varepsilon^2)$$

It appears that in the limit of small gap (compared with the radius) the velocity tends to a linear distribution, i.e. to the Couette flow (107).

Low Reynolds number flows

Introduction to low Reynolds number flows

In section 8 we have shown (page 98) that for low values of the Reynolds number, the equations of motion reduce, at leading order, to the following linear equations

$$\nabla^2 \mathbf{u} = \nabla p, \tag{119a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{119b}$$

The above equations, being linear, are much more amenable to analytical treatment than the original Navier-Stokes equations.

In the following of this section we consider the classical solution obtained by Stokes in 1851 for the slow flow past a sphere.

Slow flow past a sphere I

For the flow around a sphere of radius a a sensible definition for the Reynolds number is

$$Re = \frac{aU}{\nu},$$

where U is the magnitude of the velocity far from the sphere. We consider a flow such that $Re \ll 1$. Moreover, let the pressure far from the sphere be equal to p_0.
We make our problem dimensionless using the following scales

$$\mathbf{x}^* = \frac{\mathbf{x}}{a}, \qquad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \qquad (p^*, p_0^*) = \frac{(p, p_0)}{\rho \nu U / a},$$

where the symbol $*$ denotes dimensionless variables. In the following we adopt a dimensionless approach but skip the $*$ to simplify the notation.
Let $\mathbf{i}$ be the unit vector in the direction of the flow very far from the sphere (see the figure below). The flow is axisymmetrical about the direction $\mathbf{i}$. We consider a system of polar spherical coordinates (r, ϑ, φ), centred in the centre of the sphere, with ϑ the zenithal and φ the azimuthal coordinates, respectively. The corresponding velocity components are $\mathbf{u} = (u_r, u_\vartheta, u_\varphi)$. The direction $\mathbf{i}$ coincides with the axis $\vartheta = 0, \pi$. Our dimensionless problem can be written as
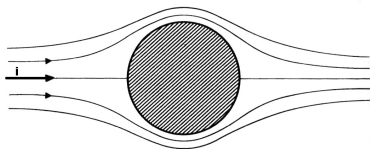
$$\nabla^2 \mathbf{u} = \nabla p, \qquad\qquad (120a)$$
$$\nabla \cdot \mathbf{u} = 0, \qquad\qquad (120b)$$
$$\mathbf{u} = 0 \qquad (r = 1), \qquad (120c)$$
$$\mathbf{u} \to \mathbf{i} \qquad (r \to \infty) \qquad (120d)$$
$$p \to p_0 \qquad (r \to \infty) \qquad (120e)$$

Slow flow past a sphere II

As a consequence of the axisymmetry of the flow we have

$$\frac{\partial}{\partial \varphi} = 0, \qquad u_\varphi = 0,$$

and the continuity equation takes the form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta u_\vartheta \right) = 0.$$

This allows us to introduce the so called **Stokes streamfunction** ψ, defined as

$$u_r = \frac{1}{r^2 \sin \vartheta} \frac{\partial \psi}{\partial \vartheta}, \qquad u_\vartheta = -\frac{1}{r \sin \vartheta} \frac{\partial \psi}{\partial r}. \tag{121}$$

Given a vector $\mathbf{b} = (b_r, b_\vartheta, b_\varphi)$ we have

$$\nabla \times \mathbf{b} = \frac{\mathbf{r}}{r \sin \vartheta} \left[\frac{\partial}{\partial \vartheta} \left(b_\varphi \sin \vartheta \right) - \frac{\partial b_\vartheta}{\partial \varphi} \right] + \frac{\boldsymbol{\vartheta}}{r} \left[\frac{1}{\sin \vartheta} \frac{\partial b_r}{\partial \varphi} - \frac{\partial}{\partial r} \left(r b_\varphi \right) \right] + \frac{\boldsymbol{\varphi}}{r} \left[\frac{\partial}{\partial r} \left(r b_\vartheta \right) - \frac{\partial b_r}{\partial \vartheta} \right],$$

with $\mathbf{r}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\varphi}$ unit vectors along the three coordinate directions.
It is then easy to show that

$$\mathbf{u} = \nabla \times \left(0, 0, \frac{\psi}{r \sin \vartheta} \right) = \text{curl} \left(0, 0, \frac{\psi}{r \sin \vartheta} \right). \tag{122}$$

Slow flow past a sphere III

For future convenience we use the notation (curl) rather than ($\nabla \times$) for the curl operator.
Recalling the vector identity

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \text{curl curl } \mathbf{u}, \tag{123}$$

equation (120a) can be written as

$$\text{curl curl } \mathbf{u} = -\nabla p.$$

Further taking the curl of the above expression we can eliminate the pressure, to get

$$\text{curl}^3 \mathbf{u} = 0.$$

Using (122) the above expression can written in terms of the streamfunction as

$$\text{curl}^4 \left(0, 0, \frac{\psi}{r \sin \vartheta} \right) = 0. \tag{124}$$

It is not difficult to show that

$$\text{curl}^2 \left(0, 0, \frac{\psi}{r \sin \vartheta} \right) = \left(0, 0, \frac{-D^2 \psi}{r \sin \vartheta} \right),$$

where the operator D^2 is defined as

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} - \frac{\cot \vartheta}{r^2} \frac{\partial}{\partial \vartheta}.$$

Slow flow past a sphere IV

Therefore, equation (124) reduces to

$$D^4\psi = 0. \tag{125}$$

The boundary conditions, written in terms of ψ, take the following form.

- **Condition on the sphere surface** $(r = 1)$

  Using the definition of the streamfunction (121) the condition (120c) can be written as

  $$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial \vartheta} = 0 \qquad (r = 1). \tag{126}$$

- **Condition at infinity** $(r \to \infty)$

  To write the condition at infinity (120d) as a function of ψ we note that

  $$u_r = \cos\vartheta = \frac{1}{r^2 \sin\vartheta} \frac{\partial \psi}{\partial \vartheta} \quad \Rightarrow \quad \frac{\partial \psi}{\partial \vartheta} = r^2 \cos\vartheta \sin\vartheta \quad \Rightarrow \quad \psi = \frac{r^2 \sin^2\vartheta}{2} + g_1(r),$$

  $$u_\vartheta = -\sin\vartheta = -\frac{1}{r\sin\vartheta} \frac{\partial \psi}{\partial r} \quad \Rightarrow \quad \frac{\partial \psi}{\partial r} = r\sin^2\vartheta \quad \Rightarrow \quad \psi = \frac{r^2 \sin^2\vartheta}{2} + g_2(\vartheta).$$

  Comparing the above expressions we find

  $$\psi = \frac{r^2 \sin^2\vartheta}{2} + C. \tag{127}$$

Slow flow past a sphere V

We now seek a separable variable solution, thus writing $\psi = f(r)g(\vartheta)$. The boundary condition (127) suggests to choose $g(\vartheta) = \sin^2 \vartheta$, so that

$$\psi(r, \vartheta) = f(r) \sin^2 \vartheta.$$

After some algebra it can be shown that

$$D^2 \psi = \left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right) f \sin^2 \vartheta.$$

and hence we finally have to solve the following equation

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right)^2 f = 0. \tag{128}$$

The general solution of this homogeneous equation is

$$f = ar^4 + br^2 + cr + dr^{-1}.$$

The boundary condition at infinity shows that it must be $a = 0$ and $b = \frac{1}{2}$. The condition at the sphere surface imposes $c = -\frac{3}{4}$ and $d = \frac{1}{4}$. Thus the **Stokes solution** for the flow past a sphere is

$$\psi = \left(\frac{1}{2} r^2 - \frac{3}{4} r + \frac{1}{4} r^{-1} \right) \sin^2 \vartheta. \tag{129}$$

Slow flow past a sphere VI

The two velocity components are immediately found from (121) and read

$$u_r = \left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\cos\vartheta, \qquad u_\vartheta = \left(-1 + \frac{3}{4r} + \frac{1}{4r^3}\right)\sin\vartheta. \tag{130}$$

Finally, the pressure can be calculated from equation (120a) and is found to be

$$p = p_0 - \frac{3\cos\vartheta}{2r^2}. \tag{131}$$

We can now compute the **drag force F** that the flow exerts on the sphere. This quantity is of particular practical interest. Obviously, **F** is in the **i** direction, so we just have to compute the following scalar quantity

$$F = \iint_{r=1} \left(\sigma_{rr}\cos\vartheta - \sigma_{r\vartheta}\sin\vartheta\right) dS,$$

which is the force magnitude. Note that the above expression is dimensionless; to find the dimensional force we have to multiply it by $\rho\nu a U$.
We have

$$\sigma_{rr}|_{r=1} = \left(-p + 2\frac{\partial u_r}{\partial r}\right)_{r=1} = -p_0 + \frac{3\cos\vartheta}{2},$$

$$\sigma_{r\vartheta}|_{r=1} = \left[r\frac{\partial}{\partial r}\left(\frac{u_\vartheta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \vartheta}\right]_{r=1} = -\frac{3}{2}\sin\vartheta.$$

Slow flow past a sphere VII

The contribution to the drag from the normal stress is given by

$$\int_0^{2\pi} \int_0^{\pi} \left(-p_0 + \frac{3\cos\vartheta}{2} \right) \cos\vartheta \sin\vartheta d\vartheta d\varphi = 2\pi,$$

and the contribution from the tangential stress is

$$\int_0^{2\pi} \int_0^{\pi} \frac{3\sin^3\vartheta}{2} d\vartheta d\varphi = 4\pi.$$

The dimensionless drag force on the sphere is then

$$F = 6\pi, \tag{132}$$

and, going back to dimensional quantities,

$$\textbf{Stokes drag force} = 6\pi\rho\nu a U. \tag{133}$$

Lubrication theory I

This technique provides a good approximation to the real solution when **the domain of the fluid is long and thin**.

For simplicity let us assume that the flow is two dimensional (all derivatives with respect to the third coordinate, say z, may be neglected) and that the height of the domain is $h(x)$ and a typical streamwise length is L.

The fluid velocity at the vessel walls is zero (no-slip condition) but the fluid velocity at the surface of the cell equals the cell velocity (U). Therefore changes in the x-velocity u are on the order of U, that is $|\Delta u| \sim U$, and $|\partial u/\partial y| \sim |\Delta u/\Delta y| \sim U/h_0$, where h_0 is a characteristic value of $h(x)$.

The change in fluid velocity as we move through a distance L in the x-direction is likely to be at most U, and therefore $|\partial u/\partial x| \sim U/L$. The continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

implies that $|\partial v/\partial y| \sim U/L$; hence $|\Delta v| \sim h_0 U/L$.

Scaling

We nondimensionalise

$$x = Lx^*, \quad y = h_0 y^*, \quad h(x) = h_0 h^*(x^*), \quad u = Uu^*, \quad v = h_0 Uv^*/L, \quad p = p_0 p^*,$$

where p_0 is an appropriate scale for the pressure (to be chosen). Note that x^*, y^*, u^*, v^* and p^* are all order 1. (Note also that the flow has a low Reynolds number, so we expect to scale the pressure gradient with the viscous terms.)

Lubrication theory II

Neglecting gravity and assuming a steady solution, the nondimensional governing equations are

$$\epsilon^2 Re \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) = - \frac{h_0^2 p_0}{\mu U L} \frac{\partial p^*}{\partial x^*} + \epsilon^2 \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{134}$$

$$\epsilon^3 Re \left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) = - \frac{h_0^2 p_0}{\epsilon \mu U L} \frac{\partial p^*}{\partial y^*} + \epsilon^3 \frac{\partial^2 v^*}{\partial x^{*2}} + \epsilon \frac{\partial^2 v^*}{\partial y^{*2}}, \tag{135}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{136}$$

where $\epsilon = h_0/L \ll 1$ and $Re = UL/\nu$.
We may immediately cancel the viscous terms that have a repeated x^*-derivative since they are much smaller than the viscous terms with a repeated y^*-derivative. Balancing the pressure derivative and viscous terms in the x-component equation (134) leads to the scaling $p_0 = \mu U L / h_0^2$.
Multiplying equation (135) by ϵ and simplifying, equations (134) and (135) can be written as

$$\epsilon^2 Re \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) = - \frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{137}$$

$$0 = - \frac{\partial p^*}{\partial y^*}, \tag{138}$$

where **we have neglected terms of order ϵ^2 and terms of order $\epsilon^3 Re$ relative to the leading-order terms.**

Lubrication theory III

Solution procedure

- The quantity $\epsilon^2 Re$ is called the **reduced Reynolds number**. We assume it is not too large, which places an upper bound on the possible flux.
- We may immediately solve (138) to find that the pressure is a function of x^* only, that is, **the pressure is constant over the height of the gap**.
- The governing equations are thus (137) and (136), where p^* is a function of x^* only and these must be solved subject to no-slip boundary conditions for u^* at the walls.

Lubrication theory IV

Series expansion for small reduced Reynolds number

In the case that the reduced Reynolds number is small, $\epsilon^2 Re \ll 1$ we can use a **series expansion** method to find the velocity, by setting

$$u^* = u_0^* + \epsilon^2 Re\, u_1^* + \left(\epsilon^2 Re\right)^2 u_2^* + \ldots,$$

$$v^* = v_0^* + \epsilon^2 Re\, v_1^* + \left(\epsilon^2 Re\right)^2 v_2^* + \ldots,$$

$$p^* = p_0^* + \epsilon^2 Re\, p_1^* + \left(\epsilon^2 Re\right)^2 p_2^* + \ldots.$$

noting that all the p_i^*'s are independent of y, and then solving for u_0^* (from equation (137)), v_0^* (from equation (136)), u_1^* (from equation (137)), v_1^* (from equation (136)), *etc* in that order. An equation for the pressure can be obtained by integrating the continuity equation over the gap height.

In many cases it is sufficiently accurate to find just the first terms u_0^* and v_0^* (or even just u_0^*).

Generalisation

Note that we could generalise this approach to include:

- dependence upon the third spatial dimension;
- time-dependence of the solution;
- gravity;
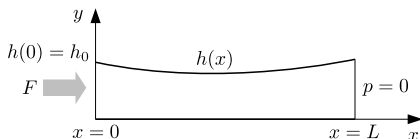- . . . .

Lubrication theory V

Example of solution

We consider the domain shown in the figure. For simplicity, we assume two-dimensional flow. We wish to solve the flow in the gap $0 \leq y \leq h(x)$, with $0 \leq x \leq L$.
The flow
is subject to the following boundary conditions:

- no-slip at $y = 0$ and $y = h(x)$;
- given
  flux per unit length $F = \int_0^{h_0} u dy$ at $x = 0$;
- given pressure $p = 0$ at $x = L$.

We assume that $h_0 = h(0)$ is a typical value of
the thickness of the domain in the y-direction and
assume that $\epsilon = h_0/L \ll 1$. We can, therefore, apply the lubrication theory.
We scale the variables as follows

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{h_0}, \qquad u^* = \frac{u}{U}, \qquad v^* = \frac{v}{\epsilon U},$$

with $U = F/h_0$.

Lubrication theory VI

Assuming that $\epsilon^2 Re \ll 1$, we need to solve the following dimensionless equations (see equations (137), (138) and (136))

$$\frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\partial p^*}{\partial x^*} = 0, \tag{139}$$

$$\frac{\partial p^*}{\partial y^*} = 0, \tag{140}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{141}$$

subject to the boundary conditions

$$u^* = v^* = 0 \qquad\qquad\qquad (y^* = 0), \tag{142}$$

$$u^* = v^* = 0 \qquad\qquad\qquad [y^* = h^*(x^*)], \tag{143}$$

$$\int_0^1 u^* dy^* = 1 \qquad\qquad\qquad (x^* = 0), \tag{144}$$

$$p^* = 0 \qquad\qquad\qquad (x^* = 1). \tag{145}$$

Lubrication theory VII

Equation (140) imposes that p^* cannot depend on y^*. As a consequence equation (139) can be integrated with respect to y^* and, also using the boundary conditions (142) and (143), we obtain

$$u^*(x^*, y^*) = \frac{1}{2} \frac{dp^*}{dx^*} \left(y^{*2} - h^* y^* \right). \tag{146}$$

In the above expression the term dp^*/dx^* is still an unknown function of x^*.
Using the boundary condition (144) and (146) we find that

$$\left. \frac{dp^*}{dx^*} \right|_{x^*=0} = -12. \tag{147}$$

We now integrate the continuity equation (141) with respect to y^*

$$\int_o^{h^*} \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \, dy^* = \cancel{v^*(h^*)} - \cancel{v^*(0)} + \int_o^{h^*} \frac{\partial u^*}{\partial x^*} \, dy^* = 0,$$

where we have used the no-slip boundary conditions (142) and (143).
Using Leibniz rule[1] and, again, the no-slip boundary conditions (142) and (143) we obtain the following second order equation for the pressure

$$\frac{d}{dx^*} \left(h^{*3} \frac{dp^*}{dx^*} \right) = 0.$$

Lubrication theory VIII

From the above equation and using (147) we obtain

$$\frac{dp^*}{dx^*} = -\frac{12}{h^{*3}},$$

which we can plug into equation (146) to obtain the following expression for the velocity in the x^*-direction

$$u^*(x^*, y^*) = -\frac{6}{h^{*3}} \left(y^{*2} - h^* y^* \right).$$

The y^*-component of the velocity can be obtained from the continuity equation (141) and reads

$$v^*(x^*, y^*) = -6 \left(-\frac{y^{*3}}{h^{*4}} + \frac{y^{*2}}{h^{*3}} \right) \frac{dh^*}{dx^*}.$$

Finally, the pressure distribution can be obtained by integrating (128) and using the boundary condition (145).
We note that we managed to obtain an analytical expression for the velocity without having to specify the shape of the domain $h^*(x^*)$.

---

[1]

$$\int\limits_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z} dx = \frac{\partial}{\partial z} \int\limits_{a(z)}^{b(z)} f(x, z) dx - f(b, z) \frac{\partial b(z)}{\partial z} + f(a, z) \frac{\partial a(z)}{\partial z}.$$

High Reynolds number flows

The Bernoulli theorem I

As a first tool to study high Reynolds number flows we introduce the **Bernoulli theorem**. As it will appear in the following, provided that some assumptions hold, this theorem is a very powerful tool to solve practical problems by very simple means.

Let us recall the following vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{148}$$

Plugging it into the Navier-Stokes equation, we can rewrite (71) as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} - \mathbf{f} + \nabla\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2\right) - \nu\nabla^2\mathbf{u} = 0, \tag{149}$$

where $\boldsymbol{\omega}$ is the vorticity defined by (49).

Let us now assume that $\mathbf{f}$ is conservative. We can then write $\mathbf{f} = -\nabla\Psi$, with Ψ scalar potential function. Let, moreover assume that the flow is steady, so that $\partial\mathbf{u}/\partial t = 0$. Under the above assumptions we can write (149) as

$$\nabla H' = \mathbf{u} \times \boldsymbol{\omega} + \nu\nabla^2\mathbf{u},$$

where we have defined

$$H' = \frac{1}{2}|\mathbf{u}|^2 + \Psi + \frac{p}{\rho}. \tag{150}$$

The Bernoulli theorem II

If the fluid is inviscid (or, more realistically, if viscosity plays a negligible role in the flow under consideration), we have

$$\nabla H' = \mathbf{u} \times \boldsymbol{\omega}.$$

Projecting the above equation in the direction of flow we obtain

$$\mathbf{u} \cdot \nabla H' = 0.$$

The above equation implies that H' **is constant along the streamlines**, which is a remarkably simple result.

Particular case: gravitational body force field

In the particular case in which the body force is gravity we have

$$\Psi = gz,$$

with z a vertical and upwards directed coordinate. In this case we then have

$$H' = \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + gz.$$

It is customary, in fluid mechanics and hydraulics to work with the quantity $H = H'/g$, so that

$$H = z + \frac{p}{\gamma} + \frac{|\mathbf{u}|^2}{2g}. \tag{151}$$

H is named **total head** and it represents the total mechanical energy per unit weight of the fluid. Note that if the fluid is at rest H reduces to the pressure head, defined in section 3.

The Bernoulli theorem III

Bernoulli theorem can be stated as follows. If the following conditions are satisfied:

- the fluid is incompressible,
- the body force is gravity (or more in general it is conservative),
- the flow is steady,
- the effects of viscosity are negligible,

then **the total head H is constant along streamlines**. Note that the value of H can differ from one streamline to another.

Vorticity equation and vorticity production I

Vorticity equation

We wish to determine an equation for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, see equation (49). The importance of the vorticity equation for studying large Reynolds number flows will become clear in the following.

We take the curl of the Navier-Stokes equation (71), in which we assume that the body force field is conservative, so that we can write $\mathbf{f} = -\nabla \Psi$. We then obtain

$$\nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) - \mathbf{u} \times \boldsymbol{\omega} + \nabla \Psi + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} \right] = 0. \tag{152}$$

In the above equation we have used the vector identity (148).

As the curl of a gradient is zero the second, fourth and fifth terms in equation (152) vanish. Therefore we can write, using the index notation,

$$\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial t} - \epsilon_{ijk} \left[\frac{\partial}{\partial x_j} \left(\epsilon_{klm} u_l \omega_m \right) \right] - \nu \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial^2 u_k}{\partial x_l^2} = 0,$$

or

$$\frac{\partial \omega_i}{\partial t} - \frac{\partial}{\partial x_j} (\epsilon_{ijk} \epsilon_{klm} u_l \omega_m) - \nu \frac{\partial^2 \omega_i}{\partial x_l^2} = 0.$$

We note that by definition of the alternating tensor we have $\epsilon_{ijk} = \epsilon_{kij}$. Moreover, the following formula holds

$$\epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}. \tag{153}$$

Vorticity equation and vorticity production II

Therefore, we can write

$$\frac{\partial \omega_i}{\partial t} - \frac{\partial}{\partial x_j}(u_i \omega_j - u_j \omega_i) - \nu \frac{\partial^2 \omega_i}{\partial x_j^2} = 0.$$

From the continuity equation we have that $\partial u_j / \partial x_j = 0$. Moreover, the divergence of a curl is zero, and therefore, $\partial \omega_j / \partial x_j = 0$. The above equation then simplifies to

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} - \omega_j \frac{\partial u_i}{\partial x_j} - \nu \frac{\partial^2 \omega_i}{\partial x_j^2} = 0, \tag{154}$$

or, in vector form,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \boldsymbol{\omega} = 0. \tag{155}$$

This equation is called **vorticity equation** and it is of fundamental importance in fluid mechanics.

- The first and second terms in equation (155) represent advective transport of vorticity.
- The last term represents viscous (diffusive) transport of vorticity.
- The third term does not have a counterpart in the Navier-Stokes equations. It accounts for changes of vorticity due to deformation of material elements of the fluid.
- Note, that pressure and body force do not appear in the vorticity equation.

Vorticity equation and vorticity production III

Changes of vorticity in a volume V

Let us now study how the amount of vorticity changes in a fluid volume V. A suitable measure of the amount of vorticity is the **enstrophy**, defined as $(\boldsymbol{\omega} \cdot \boldsymbol{\omega})/2$. Multiplying equation (154) by ω_i we obtain

$$\frac{\partial}{\partial t}\left(\frac{\omega_i^2}{2}\right) + u_j \frac{\partial}{\partial x_j}\left(\frac{\omega_i^2}{2}\right) - \omega_i \omega_j \frac{\partial u_i}{\partial x_j} - \nu \omega_i \frac{\partial^2 \omega_i}{\partial x_j^2} = 0.$$

After some algebraic manipulation this can be written as

$$\frac{\partial}{\partial t}\left(\frac{\omega_i^2}{2}\right) + u_j \frac{\partial}{\partial x_j}\left(\frac{\omega_i^2}{2}\right) - \omega_i \omega_j \frac{\partial u_i}{\partial x_j} - \nu \left[\frac{\partial^2}{\partial x_j^2}\left(\frac{\omega_i^2}{2}\right) - \left(\frac{\partial \omega_i}{\partial x_j}\right)^2\right] = 0.$$

Taking the integral of the above expression over a material volume V and applying the Reynolds transport theorem we obtain

$$\frac{D}{Dt}\iiint\limits_V \left(\frac{\omega_i^2}{2}\right) dV = \iiint\limits_V \omega_i \omega_j \frac{\partial u_i}{\partial x_j} dV + \iiint\limits_V \nu \frac{\partial^2}{\partial x_j^2}\left(\frac{\omega_i^2}{2}\right) dV - \iiint\limits_V \nu \left(\frac{\partial \omega_i}{\partial x_j}\right)^2 dV. \quad (156)$$

- The term on the left hand side represents the time variation of the enstrophy associated with the volume V.

Vorticity equation and vorticity production IV

- The first term on the right hand side induces changes of vorticity in V, as a response to the velocity distribution. Note, however, that this is not a source term: if at some time the vorticity within V is zero this term can not produce new vorticity.

- Making use of Gauss theorem the second term on the right hand side can be transformed into a flux term across the surface S bounding V as follows

$$\iiint\limits_V \nu \frac{\partial^2}{\partial x_j^2} \left(\frac{\omega_i^2}{2} \right) dV = \frac{\nu}{2} \iint\limits_S n_j \frac{\partial}{\partial x_j} \omega_i^2 \, dS.$$

Therefore it does not produce nor dissipate vorticity.

- Finally the last term represents viscous dissipation of vorticity and always contributes to decrease the amount of vorticity within the volume V.

Equation (156) shows that the **vorticity can not be generated within a body of fluid**. It then follows that **vorticity can only be generated at the boundary of the domain**.
A typical source of vorticity is, for instance, the no-slip condition (84), which holds in correspondence of solid walls.

Vorticity equation and vorticity production V

Generation of vorticity due to an impulsively started solid body

To understand the generation and transport of vorticity let us consider an example: a fluid occupying an infinite region and initially (at time $t = 0$) at rest is set in motion by a solid body immersed in the fluid that, at $t = 0$, impulsively starts moving with velocity U. Suppose that we study this flow in a frame moving with the solid body.

We can think that the development of motion in the fluid takes place in three different phases.

1. At the initial time ($t = 0$) the fluid starts moving and the flow is irrotational, i.e. the vorticity is zero everywhere. In fact the vorticity is initially confined in an infinitesimally thin layer at the wall and, within that layer the vorticity is theoretically infinite.

2. For $t > 0$ the vorticity starts be transported away from the wall. Transport occurs both for viscous diffusion and advection. If diffusion was the only transport mechanism the thickness of the boundary layer in which the flow is not irrotational would be of order $\sqrt{\nu t}$ at time t. At the very initial stage advection is expected to have a relatively small effect as, initially, the normal component of the relative velocity of the fluid with respect to the wall is expected to be small. Thus for small times the thickness of the boundary layer will be of the order of $\sqrt{\nu t}$.

Vorticity equation and vorticity production VI

❸ For larger times two different scenarios might occur.

The body is thin and oriented in the direction of motion

In this case the normal component of the relative velocity close to the wall will remain small even for large times. In this case a steady flow might be reached in which longitudinal advection and diffusion are balanced. If L is the longitudinal spatial dimension of the body the characteristic time for a fluid particle to travel in the region close to the body is of order L/U. In this case the vorticity keeps confined within a boundary layer at the wall with thickness δ of order $\sqrt{\nu L/U}$. It follows that

$$\frac{\delta}{L} \propto \frac{1}{\sqrt{UL/\nu}} = \frac{1}{\sqrt{Re}}.$$

In this case it is said that **boundary layer separation** does not occur.
The above considerations suggest that, **at large values of the Reynolds number, if no boundary layer separation occurs, the motion is irrotational in most of the domain**. We will see in the next subsection that the absence of vorticity allows for great simplification of the governing equations.

The body is thick or not oriented in the direction of motion

In this case advection in the direction normal to the body is strong and the region with vorticity grows rapidly. In this case it is said that boundary layer separation occurs.

Irrotational flows I

Potential function of the velocity

We have seen in the last section that at large values of the Reynolds number it might happen that in most of the flow domain motion remains irrotational. We now wish to study if the assumptions of

- **incompressible fluid**, and
- **irrotational flow**,

i.e.

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \times \mathbf{u} = 0. \tag{157}$$

allow for any simplifications of the problem. Note that the conditions (157) are of purely kinematic nature even if they are consequence of the dynamic behaviour of the fluid.

Let us consider a closed reducible curve C and let us take the line integral of the velocity along this curve. We have, by Stokes theorem,

$$\oint_C \mathbf{u} \cdot d\mathbf{x} = \iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \iint_S \boldsymbol{\omega} \cdot \mathbf{n} dS = 0.$$

We now consider any two points, say O and P, on C. They split C into two curves, C_1 and C_2, with both C_1 and C_2 joining O to P. We then have

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{x} = \int_{C_2} \mathbf{u} \cdot d\mathbf{x}.$$

Irrotational flows II

This implies that the integral between O and P does not depend on the path of integration but only on the starting and ending points. We can then define a **potential function** $\phi(\mathbf{x})$ of the velocity field, such that

$$\phi(\mathbf{x}) = \phi(\mathbf{x}_0) + \int_0^P \mathbf{u} \cdot d\mathbf{x}. \tag{158}$$

Equation (158) implies that we can write

$$\mathbf{u} = \nabla \phi. \tag{159}$$

If we recall the continuity equation for an incompressible fluid, i.e. $\nabla \cdot \mathbf{u} = 0$, and plug (159) into it we find

$$\nabla^2 \phi = 0, \tag{160}$$

which implies that the potential function ϕ has to be harmonic. In other words the velocity potential function satisfies the Laplace equation. If we solve the problem for the function ϕ we can then easily find the velocity $\mathbf{u}$ using equation (159).
It is clear that the mathematical problem for an irrotational flow is much easier than that for a rotational flow, for the following main reasons:

- equation (160) is linear, whereas the Navier-Stokes equations are nonlinear;
- in the case of an irrotational flow it is sufficient to solve the problem for a scalar function rather than a vector function;

Irrotational flows III

- the problem for the pressure is decoupled from the problem for the velocity field. How to compute the pressure in an irrotational flow will be discussed in the following.

Owing to the properties of equation (160) we can state that the velocity distribution has the following properties.

- As equation (160) is elliptic the solution for ϕ and all its derivatives is smooth except, at most, on the boundary.
- The function ϕ is single-valued if the considered domain is simply connected.

In the following we will only consider the case of simply connected regions.

Conditions for ϕ to be uniquely determined

Let us now consider the boundary conditions we need to impose for the solution for ϕ to be unique. We first note that the following vector identity holds

$$\nabla \cdot (\phi \mathbf{u}) = \phi \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \phi = \mathbf{u} \cdot \mathbf{u},$$

from which we can write, also using the divergence theorem,

$$\iiint\limits_{V} \mathbf{u} \cdot \mathbf{u} dV = \iiint\limits_{V} \nabla \cdot (\phi \mathbf{u}) dV = \iint\limits_{S} \phi \mathbf{u} \cdot \mathbf{n} dS. \tag{161}$$

Irrotational flows IV

Let $\mathbf{u}_1 = \nabla\phi_1$ and $\mathbf{u}_2 = \nabla\phi_2$ be two solutions of equation (160). The difference $(\mathbf{u}_1 - \mathbf{u}_2)$ is also a solution, owing to the linearity of the equation. Recalling equation (161) we can write

$$\iiint\limits_V |\mathbf{u}_1 - \mathbf{u}_2|^2 \, dV = \iint\limits_S (\phi_1 - \phi_2)(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} dS.$$

The above expression shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ coincide if

- $(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} = 0$ on the boundary S, i.e. if the normal components of the velocity are the same (Neumann conditions);
- or if $\phi_1 = \phi_2$ on S (Dirichlet conditions);
- of if $(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} = 0$ on a portion of S and $\phi_1 = \phi_2$ on the remaining part of S.

It is important to notice that for an irrotational flow it is not possible to impose the no-slip condition at solid walls as only the normal component of the velocity is required. However, close to rigid walls a boundary layer exists, in which the flow is rotational. To determine the flow in the boundary layer the Navier-Stokes equations have to be solved.

Bernoulli equation for irrotational flows I

We have shown that the potential function ϕ can be obtained by using the irrotationality of the flow and the continuity equation. We now consider the momentum equation (Navier-Stokes equation) and study how it simplifies in the case of irrotational flow. We anticipate that the use of the momentum equation will allow us to determine the pressure.

Recalling the vector identities (123) and (148) the Navier-Stokes equation (71) can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{|\mathbf{u}|^2}{2} \right) - \mathbf{u} \times \boldsymbol{\omega} + \frac{1}{\rho} \nabla p - \mathbf{f} = 0$$

Assuming that the the body force field is conservative we can write $\mathbf{f} = -\nabla \Psi$. If the flow is irrotational we have $\boldsymbol{\omega} = 0$ and $\mathbf{u} = \nabla \phi$. Thus we can write

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho} + \Psi \right) = 0.$$

This equation can be solved to get

$$\frac{\partial \phi}{\partial t} + \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho} + \Psi = F(t). \tag{162}$$

Notice that, without loss of generality, we can introduce a function $\tilde{\phi}$ such that

$$\tilde{\phi} = \phi - \int F dt,$$

Bernoulli equation for irrotational flows II

which allows to eliminate the unknown function $F(t)$ from (162) to finally obtain

$$\mathcal{H} = \frac{\partial \tilde{\phi}}{\partial t} + \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho} + \Psi = c, \tag{163}$$

with c arbitrary constant. This is the Bernoulli theorem for irrotational flows. Notice that the last three terms of $\mathcal{H}$ in the above equation represent H' as defined by equation (150).
It is important to stress that Bernoulli theorem holds in a stronger form in the case of irrotational flows. In particular

- $\mathcal{H}$ is constant also in unsteady flow conditions (this is not true for H' in rotational flows);
- $\mathcal{H}$ is constant in the whole domain and not only along streamlines as it is the case for H' in rotational flows;
- the theorem holds exactly for real viscous fluids, provided the vorticity is everywhere zero.

Important note: it is important to stress that even if in irrotational flows the viscous terms in the Navier-Stokes equation vanish, **the viscous stresses are not necessarily zero**. In fact it is the divergence of the stress tensor that vanishes, not the stress itself. In other words viscous stresses might exist in irrotational flows. However they do not contribute to the momentum equation.

Appendix A:
material derivative of the Jacobian

Determinants

Definition:
A permutation $i, j, \ldots, p$ of the first n integers $1, 2, \ldots n$ is called **even** or **odd** according as the natural order can be restored by an even or odd number of interchanges.

Definition: The **determinant** of a $n \times n$ matrix A with elements a_{ij} is

$$\det A = \sum \pm a_{1i} a_{2j} \ldots a_{np}, \tag{164}$$

where the summation is taken over all permutations $i, j, \ldots, p$ of the integer numbers $1, 2, \ldots, n$, and the sign is positive for even permutations and negative for odd ones.

Therefore, for instance, the determinant of a 3×3 matrix A is

$$\begin{aligned} \det A = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

Note that in each term of the sum there is only one element from each row and each column.

Derivative of a determinant
If the elements of a $n \times n$ matrix A are function of a variable s, so that $a_{ij}(s)$, the derivative with respect to s of $\det A$ is the sum of n determinants obtained by replacing one row of A by the derivatives of its elements.

Derivative of the Jacobian I

We consider the Jacobian

$$
J = \det \begin{pmatrix}
\dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_1}{\partial \xi_3} \\[2mm]
\dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\[2mm]
\dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3}
\end{pmatrix}.
$$

We wish to compute its material derivative DJ/Dt. Let us consider an element of the above matrix. We have

$$
\frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_j}\frac{Dx_j}{Dt} = \frac{\partial u_i}{\partial \xi_j}.
$$

In the above expression we could interchange the order of differentiation because D/Dt is differentiation with constant $\boldsymbol{\xi}$ by definition of material derivative. If we regard u_i as a function of (x_1, x_2, x_3) we can write

$$
\frac{\partial u_i}{\partial \xi_j} = \frac{\partial u_i}{\partial x_1}\frac{\partial x_1}{\partial \xi_j} + \frac{\partial u_i}{\partial x_2}\frac{\partial x_2}{\partial \xi_j} + \frac{\partial u_i}{\partial x_3}\frac{\partial x_3}{\partial \xi_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial \xi_j}.
$$

Derivative of the Jacobian II

We know that the derivative of a determinant of a 3×3 matrix is the sum of three determinants, each of a matrix in which one row is differentiated. Thus to compute DJ/Dt we have to sum up three terms the first of which is

$$\det \begin{pmatrix} \dfrac{\partial u_1}{\partial \xi_1} & \dfrac{\partial u_1}{\partial \xi_2} & \dfrac{\partial u_1}{\partial \xi_3} \\[2mm] \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\[2mm] \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{pmatrix} = \det \begin{pmatrix} \dfrac{\partial u_1}{\partial x_k}\dfrac{\partial x_k}{\partial \xi_1} & \dfrac{\partial u_1}{\partial x_k}\dfrac{\partial x_k}{\partial \xi_2} & \dfrac{\partial u_1}{\partial x_k}\dfrac{\partial x_k}{\partial \xi_3} \\[2mm] \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\[2mm] \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{pmatrix}.$$

With $k = 1$ we have

$$\det \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1}\dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial u_1}{\partial x_1}\dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial u_1}{\partial x_1}\dfrac{\partial x_1}{\partial \xi_3} \\[2mm] \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\[2mm] \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{pmatrix} = \dfrac{\partial u_1}{\partial x_1} J.$$

With $k = 2, 3$ we have $\partial u_1/\partial x_k$ times the determinant of a matrix with two identical rows, which is therefore equal to zero.

Derivative of the Jacobian III

Computing the other two term of the DJ/Dt we thus finally find

$$\frac{DJ}{Dt} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) J = (\nabla \cdot \mathbf{u})J. \tag{165}$$

Appendix B:
the equations of motion in different coordinates systems

Cylindrical coordinates

Let us consider cylindrical coordinates (z, r, φ), with corresponding velocity components (u_z, u_r, u_φ).

Continuity equation

$$\frac{\partial u_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} = 0 \tag{166}$$

Navier-Stokes equations

$$\frac{\partial u_z}{\partial t} + u_z \frac{\partial u_z}{\partial z} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r} \frac{\partial u_z}{\partial \varphi} + \frac{1}{\rho} \frac{\partial p}{\partial z} - \nu \left[\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \varphi^2} \right] = 0. \tag{167}$$

$$\frac{\partial u_r}{\partial t} + u_z \frac{\partial u_r}{\partial z} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r} \frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r} + \frac{1}{\rho} \frac{\partial p}{\partial r} +$$
$$- \nu \left[\frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \varphi^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\varphi}{\partial \varphi} \right] = 0. \tag{168}$$

$$\frac{\partial u_\varphi}{\partial t} + u_z \frac{\partial u_\varphi}{\partial z} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r u_\varphi}{r} + \frac{1}{\rho r} \frac{\partial p}{\partial \varphi} +$$
$$- \nu \left[\frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\varphi}{\partial \varphi^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r^2} \right] = 0. \tag{169}$$

Spherical polar coordinates I

Let us consider spherical polar coordinates (r, ϑ, φ) (radial, zenithal and azimuthal), with corresponding velocity components $(u_r, u_\vartheta, u_\varphi)$.

Continuity equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta u_\vartheta \right) + \frac{1}{r \sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} = 0. \tag{170}$$

Navier-Stokes equations

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\vartheta}{r} \frac{\partial u_r}{\partial \vartheta} + \frac{u_\varphi}{r \sin \vartheta} \frac{\partial u_r}{\partial \varphi} - \frac{u_\vartheta^2}{r} - \frac{u_\varphi^2}{r} + \frac{1}{\rho} \frac{\partial p}{\partial r} +$$

$$- \nu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial u_r}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 u_r}{\partial \varphi^2} +$$

$$- \frac{2 u_r}{r^2} - \frac{2}{r^2 \sin \vartheta} \frac{\partial (u_\vartheta \sin \vartheta)}{\partial \vartheta} - \frac{2}{r^2 \sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} \right] = 0. \tag{171}$$

$$\frac{\partial u_\vartheta}{\partial t} + u_r \frac{\partial u_\vartheta}{\partial r} + \frac{u_\vartheta}{r} \frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_\varphi}{r \sin \vartheta} \frac{\partial u_\vartheta}{\partial \varphi} + \frac{u_r u_\vartheta}{r} - \frac{u_\varphi^2 \cot \vartheta}{r} + \frac{1}{\rho r} \frac{\partial p}{\partial \vartheta} +$$

$$- \nu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_\vartheta}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial u_\vartheta}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 u_\vartheta}{\partial \varphi^2} +$$

Spherical polar coordinates II

$$+ \frac{2}{r^2} \frac{\partial u_r}{\partial \vartheta} - \frac{u_\vartheta}{r^2 \sin^2 \vartheta} - \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial u_\varphi}{\partial \varphi} \Bigg] = 0. \tag{172}$$

$$\frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_\vartheta}{r} \frac{\partial u_\varphi}{\partial \vartheta} + \frac{u_\varphi}{r \sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r u_\varphi}{r} + \frac{u_\vartheta u_\varphi \cot \vartheta}{r} + \frac{1}{\rho r \sin \vartheta} \frac{\partial p}{\partial \varphi} +$$

$$- \nu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_\varphi}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial u_\varphi}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 u_\varphi}{\partial \varphi^2} + \right.$$

$$\left. + \frac{2}{r^2 \sin \vartheta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial u_\vartheta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2 \vartheta} \right] = 0. \tag{173}$$

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